TSUKUBA J. MATH.
Vol. 7 No. 2 (1983). 205–214

DTr-INVARINT MODULES

By

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Throughout this paper, we shall work over a fixed basic artin algebra Λ and deal only with finitely generated right modules. Let X be an indecomposable module. We say that X is DTr-invariant if $DTrX\cong X$. In [7], with some other conditions, the author has shown that Λ is a local Nakayama algebra if there is a DTr-invariant module. The aim of this paper is to generalize this result.

Recall that an indecomposable module X is said to be DTr-periodic if, more generally, $DTr^{n}X\cong X$ for some positive integer n. In Riedtmann [8], Todorov [10] and Happel-Preiser-Ringel [6], they have completely determined the Cartan class of a component of the stable Auslander-Reiten quiver contaning a DTr-periodic module (see [6] for detales). In [6], they have also shown that a component of the Auslander-Reiten quiver containing a DTr-periodic module is a quasi-serial component (in the sence of [9]) if it contains neither projective nor injective modules. It seems, however, that there has not been given any characterization of a component of the (not stable) Auslander-Reiten quiver contaning a DTr-periodic module. In this paper, we shall investigate the case in which there is a DTr-invariant module and prove

THEOREM 1. *Suppose there is a DTr-invariant module A. Then either Λ is a local Nakayama algebra or the component of the Auslander-Reiten quiver containing A is a quasi-serial component* (*in the sence of* [9]) *consisting of only DTr-invariant modules.*

Let X be a DTr-invariant module and Y an indecomposable summand of the middle term of the Auslander-Reiten sequence ending in X. Then there are irreducible maps both from X to Y and from Y to X. The converse holds.

THEOREM 2. *Let X, Y be indecomposable modules. Suppose there are irreducible maps both from X to Y and from Y to X. Then either X or Y is DTr-invariant. Thus either Λ is a local Nakayama algebra or the component of the Auslander-Reiten quiver containing X, Y is a quasi-serial component* (*in the sence*

Received June 14, 1982

of [9]) *consisting of only* DTr*-invariant modules.*

Recently, the author learned that the similar result of Theorem 2 was obtained by K. Bautista and S. O. Smalø [4].

It is well known that there is a quasi-serial component consisting of only DTr-invariant modules if Λ is an hereditary algebra of tame representation type (see [5]).

The proof of Theorems 1, 2 will be performed by calculating composition lengths, and in that of Theorem 1 the work of Auslander [1, Theorem 6.5] will play an impotant roll (see also [10, Proposition 2.3]).

For an indecomposable module X, let $F(X)=\mathrm{End}(X)/\mathrm{Rad}(X, X)$, this is a division ring, and for two indecomposable modules X, Y, let $N(X, Y)=\mathrm{Rad}(X, Y)/\mathrm{Rad}^2(X, Y)$, this is an $F(Y)-F(X)$-bimodule called the bimodule of irreducible maps (see [8], [10] for details). The Auslander-Reiten quiver has as vertices the isomorphism classes of the indecomposable modules, and there is an arrow $[X]\to[Y]$ if $N(X, Y)\neq 0$, which is endowed with the valuation (d_{XY}, d'_{XY}) such that $d_{XY}=\dim_{F(Y)}N(X, Y)$ and $d'_{XY}=\dim N(X, Y)_{F(X)}$. Two indecomposable modules X, Y belong, by definition, to the same component if there is a sequence $X=X_0, X_1, \cdots, X_r=Y$ of indecomposable modules such that either $N(X_{i-1}, X_i)\neq 0$ or $N(X_i, X_{i-1})\neq 0$ for all i.

We refer to [2], [3] for DTr, Auslander-Reiten sequences and so on, and shall freely use results of [2], [3].

In what follows, we denote by τ (resp. τ^{-1}) DTr (resp. TrD) and by $|X|$ the composition length of a module X.

1. Proof of Theorem 1.

Let A be a τ-invariant module and $0\to A\to\bigoplus_{i=1}^{r}B_i^{a_i}\to A\to 0$ be the Auslander-Reiten sequence, where B_i's are non-isomorphic indecomposable modules and $a_i=\dim_{F(B_i)}N(A, B_i)$ for all i. By induction, it is sufficient to show that the possible cases are the following:

(1) Some B_i is projective-injective. We get $\mathrm{rad}\, B_i\cong A\cong B_i/\mathrm{soc}\, B_i$, thus $\mathrm{top}(\mathrm{rad}\, B_i)\cong\mathrm{top}\, B_i$, this means that Λ is a local Nakayama algebra.

(2) We have $r=1$, $a_1=1$, and B_1 is τ-invariant.

(3) We have $r=2$, $a_1=a_2=1$, and each B_i is τ-invariant.

We have to exclude the other cases. Note that $\tau B_i\cong B_j$, $a_i=a_j$ for some j if B_i is not projective, and that $\tau^{-1}B_i\cong B_k$, $a_i=a_k$ for some k if B_i is not injective.

(a) Consider, first, the case in which some B_i is not τ-periodic. Then $\tau^n B_i$ is projective for some non-negative integer n, and $\tau^m B_i$ is injective for some non-positive integer m. Since $2|A|=\sum_{j=1}^{r} a_j|B_j|$, we conclude that $n=m=0$ and B_i is projective-injective.

(b) Next, assume that all B_i's are τ-periodic. Let $0\to\tau B_i\to A^{a'_i}\oplus C_i\to B_i\to 0$ be the Auslander-Reiten sequence for each i, where $a'_i=\dim N(A, B_i)_{F(A)}$. We get

$$a'_i|A|+|C_i|=|\tau B_i|+|B_i|\,,$$

hence

$$\begin{aligned}\Big(\sum_{i=1}^{r} a_i a'_i\Big)|A|+\sum_{i=1}^{r} a_i|C_i| &= \sum_{i=1}^{r} a_i|\tau B_i|+\sum_{i=1}^{r} a_i|B_i| \\ &=2|A|+2|A| \\ &=4|A|\,.\end{aligned}$$

Therefore we conclude that $\sum_{i=1}^{r} a_i a'_i \leqq 4$.

(c) Suppose $\sum_{i=1}^{r} a_i a'_i=4$. Then $C_i=0$ for all i. Hence we get a finite component $\{A, B_1, \cdots, B_r\}$ consisting of only τ-periodic modules, a contradiction (cf. [1, Theorem 6.5]).

(d) Suppose $r=1$, $a_1 a'_1=3$. By (b) we get $a_1|C_1|=|A|$, and clearly B_1 is τ-invariant. We get

$$\begin{aligned}2|B_1| &= a'_1|A|+|C_1| \\ &= a_1 a'_1|C_1|+|C_1| \\ &= 4|C_1|\,.\end{aligned}$$

Hence C_1 does not have a projective-injective summand, therefore by (b), (c) we get a contradiction.

(e) Suppose $r=2$, $a_1a'_1+a_2a'_2=3$. We may assume $a_1a'_1=2$, $a_2a'_2=1$. Clearly, each B_i is τ-invariant.

We prepare a lemma.

LEMMA 1. *Let X be an indecomposable module such that $\tau^2X\cong X$. Let $0\to\tau X\to Y\oplus Z\to X\to 0$ be the Auslander-Reiten sequence with Y indecomposable. Suppose $\tau^2Y\cong Y$, $|X|<|Y|$, $|\tau X|<|Y|$, $|X|<|\tau Y|$ and $|\tau X|<|\tau Y|$. Then either $Z=0$ or Z is indecomposable with $\tau^2Z\cong Z$.*

PROOF. We may assume $Z\neq 0$. Let $Z=\bigoplus_{i=1}^{s} Z_i^{d_i}$, where Z_i's are non-isomorphic indecomposable modules and $d_i = \dim_{F(Z_i)} N(\tau X, Z_i)$ for all i. Let $0 \to X \to \tau Y \oplus W \to \tau X \to 0$ be the Auslander-Reiten sequence. Since $|Z|<|X|$, $|Z|<|\tau X|$, $|W|<|X|$ and $|W|<|\tau X|$, both Z and W have neither projective nor injective summands. Hence $\tau Z \cong W$ and $\tau^{-1} Z \cong W$. Let $d_i' = \dim N(\tau X, Z_i)_{F(\tau X)}$ for each i. Using the Auslander-Reiten sequences ending in and starting from Z_i, we get

$$d_i' |\tau X| \leqq |Z_i| + |\tau Z_i| ,$$

$$d_i' |X| \leqq |Z_i| + |\tau^{-1} Z_i| ,$$

hence

$$\Big(\sum_{i=1}^{s} d_i d_i'\Big)(|\tau X| + |X|) \leqq 2 \sum_{i=1}^{s} d_i |Z_i| + \sum_{i=1}^{s} d_i |\tau Z_i| + \sum_{i=1}^{s} d^i |\tau^{-1} Z_i|$$

$$= 2|Z| + |W| + |W|$$

$$< 2(|\tau X| + |X|) .$$

Therefore we conclude that $\sum_{i=1}^{s} d_i d_i' = 1$. This finishes the proof.

(e′) Suppose $a_1 = 2$. Since $2|C_1| + |C_2| = |A|$, $|C_i| < |A|$ for all i. Suppose $|A| < |B_i|$ for some i, then we get $|A| < |B_i| < |C_i|$, a contradiction. Hence $|B_i| < |A|$, thus $|C_i| < |B_i| < |A|$ for all i. Suppose $C_i \neq 0$. By Lemma 1, C_i is indecomposable, and clearly τ-invariant. Let $0 \to C_i \to B_i \oplus D_i \to C_i \to 0$ be the Auslander-Reiten sequence. If $D_i \neq 0$, then again by Lemma 1, D_i is indecomposable and τ-invariant with $|D_i| < |C_i|$. Continuing these procedures, we get a finite component $\{A, B_1, B_2, C_1, C_2, D_1, D_2, \cdots\}$ consisting of only τ-invariant modules, a contradiction (cf. [1, Theorem 6.5]).

(e″) Suppose $a_1' = 2$. We get $|C_1| < |B_1|$, hence C_1 does not have a projective-injective summand. Therefore by (b), (c) and (e′) we get a contradiction.

(f) Suppose $r = 1$, $a_1 a_1' = 2$. Clearly, B_1 is τ-invariant.

(f′) If $a_1 = 2$, then we get $|A| = |B_1|$, a contradiction.

(f″) If $a_1' = 2$, then we get $|A^{a_1'}| = |B_1|$, a contradiction.

(g) Suppose $r = 3$, $a_i a_i' = 1$ for all i. Put $\sigma i = j$ if $\tau B_i \cong B_j$. Then σ is a permutation of the set $\{1, 2, 3\}$. Note that $\sum_{i=1}^{3} |B_i| = 2|A|$ and $\sum_{i=1}^{3} |C_i| = |A|$.

(g′) Suppose σ is cyclic. Suppose $|A| < |B_i|$ for some i. We get $|B_{\sigma i}| + |B_{\sigma^2 i}| < |A|$. On the other hand, using the Auslander-Reiten sequence ending in $B_{\sigma i}$, we get $|A| \leqq |B_{\sigma^2 i}| + |B_{\sigma i}|$, a contradiction. Hence $|B_i| < |A|$, thus $|C_i| < |B_{\sigma i}|$ for all i. Suppose $C_i = 0$ for some i. We get $|A| = |B_i \oplus B_{\sigma i}|$, a contradiction. Hence $C_i \neq 0$ for all i. Clearly, each C_i does not have a projective summand. Let X be an indecomposable summand of C_1. Using the Auslander-

Reiten sequences ending in X, τX and $\tau^2 X$, we get

$$\begin{aligned}2|A| &= |B_{\sigma 1}| + |B_{\sigma^2 1}| + |B_1| \\ &\leqq (|X| + |\tau X|) + (|\tau X| + |\tau^2 X|) + (|\tau^2 X| + |\tau^3 X|) \\ &\leqq 2(|C_1| + |C_{\sigma 1}| + |C_{\sigma^2 1}|) \\ &= 2|A| .\end{aligned}$$

Therefore each C_i is indecomposable and the Auslander-Reiten sequence ending in C_i is of the form $0 \to C_{\sigma i} \to B_{\sigma i} \to C_i \to 0$. Hence we get a finite component $\{A, B_1, B_2, B_3, C_1, C_2, C_3\}$ consisting of only τ-periodic modules, a contradiction.

(g″) Suppose σ is not cyclic. Suppose $|A| < |B_i|$ for some i. We get $|B_{\sigma i}| < |C_i| \leqq |A| < |B_i|$, thus $C_i \neq 0$ and C_i does not have an injective summand. Let X be an indecomposable summand of C_i. Using the Auslander-Reiten sequence starting from X, we get

$$\begin{aligned}|A| &< |B_i| \\ &\leqq |X| + |\tau^{-1} X| \\ &\leqq |C_i| + |C_{\sigma^{-1} i}| \\ &\leqq |A| ,\end{aligned}$$

a contradiction. Hence $|B_i| < |A|$ for all i. By Lemma 1, each C_i is either zero or indecomposable with $|C_i| < |B_i|$. Therefore, as in (e′), we get a finite component $\{A, B_1, B_2, B_3, C_1, C_2, C_3, \cdots\}$ consisting of only τ-periodic modules, a contradiction.

(h) Suppose $r=2$, $a_1 a_1' = a_2 a_2' = 1$ and $\tau B_1 \cong B_2$. Note that $\tau^2 B_i \cong B_i$ and $|C_i| = |A|$ for all i. We claim that each C_i is indecomposable.

LEMMA 2. *Let X be an indecomposable module such that $\tau^2 X \cong X$. Let $0 \to \tau X \to Y \oplus Z \to X \to 0$ be the Auslander-Reiten sequence with Y indecomposable. Suppose $\tau^2 Y \cong Y$, $|\tau Y| = |Y|$ and $|X| + |\tau X| = 2|Y|$. Then Z is indecomposable with $\tau^2 Z \cong Z$.*

PROOF. We may assume $Z \not\cong Y$. First, assume $|\tau X| < |Y| < |X|$. Let $0 \to X \to \tau Y \oplus W \to \tau X \to 0$ be the Auslander-Reiten sequence. Since $|Z| = |W| < |X|$, Z does not have an injective summand and W does not have a projective summand. Hence $W \cong \tau^{-1} Z$. Let $Z = \bigoplus_{i=1}^{s} Z_i^{d_i}$, where Z_i's are non-isomorphic indecomposable modules and $d_i = \dim_{F(Z_i)} N(\tau X, Z_i)$ for all i. Let $d_i' = \dim N(\tau X, Z_i)_{F(\tau X)}$ for each i. Using the Auslander-Reiten sequence starting from Z_i, we get

$$d_i'|X| \leqq |Z_i| + |\tau^{-1}Z_i| ,$$

hence

$$\left(\sum_{i=1}^{s} d_i d_i'\right)|X| \leqq \sum_{i=1}^{s} d_i |Z_i| + \sum_{i=1}^{s} d_i |\tau^{-1}Z_i|$$

$$= |Z| + |W|$$

$$< 2|X| .$$

Therefore $\sum_{i=1}^{s} d_i d_i' = 1$, thus Z is indecomposable. Suppose Z is projective. Let $0 \to Z \to X \oplus E \to W \to 0$ be the Auslander-Reiten sequence. Since $|E| = |\tau X| < |Z|$, E does not have a projective summand. Let F be an indecomposable summand of E. Using the Auslander-Reiten sequence ending in F, we get

$$|Z| \leqq |F| + |\tau F|$$

$$\leqq |E| + |\tau F|$$

$$= |\tau X| + |\tau F| .$$

On the other hand, since $\tau X \oplus \tau F$ is a summand of rad Z, we get $|\tau X| + |\tau F| < |Z|$, a contradiction. Therefore $\tau Z \cong W$, thus $\tau^2 Z \cong Z$. Exchainging W for Z, the above arguments imply the case in which $|X| < |Y| < |\tau X|$. This finishes the proof.

By Lemma 2, each C_i is indecomposable. Clearly, $\tau C_1 \cong C_2$ and $\tau C_2 \cong C_1$. Let $0 \to \tau C_i \to \tau B_i \oplus D_i \to C_i \to 0$ be the Auslander-Reiten sequence for each i. Clearly, $|D_i| = |B_i|$ for all i. We claim that each D_i is indecomposable with $\tau^2 D_i \cong D_i$.

LEMMA 3. *Let X be an indecomposable module such that $\tau^2 X \cong X$ and $|\tau X| = |X|$. Let $0 \to \tau X \to Y \oplus Z \to X \to 0$ be the Auslander-Reiten sequence with Y indecomposable. Suppose $\tau^2 Y \cong Y$, $|Y| + |\tau Y| = 2|X|$. Let $Z = \oplus_{i=1}^{s} Z_i^{d_i}$, where Z_i's are non-isomorphic indecomposable modules and $d_i = \dim_{F(Z_i)} N(\tau X, Z_i)$ for all i. Let $d_i' = \dim N(\tau X, Z_i)_{F(\tau X)}$ for each i. Then $\sum_{i=1}^{s} d_i d_i' \leqq 2$:*

(1) *If $\sum_{i=1}^{s} d_i d_i' = 1$, then Z is indecomposable with $\tau^2 Z \cong Z$.*

(2) *If $\sum_{i=1}^{s} d_i d_i' = 2$, then each Z_i is neither projective nor injective and the Auslander-Reiten sequences ending in and starting from Z_i are of the form*

$$0 \longrightarrow \tau Z_i \longrightarrow \tau X^{d_i'} \longrightarrow Z_i \longrightarrow 0 ,$$

$$0 \longrightarrow Z_i \longrightarrow X^{d_i'} \longrightarrow \tau^{-1} Z_i \longrightarrow 0$$

respectively.

PROOF. First, assume $|\tau Y| < |X| < |Y|$. Let $0 \to X \to \tau Y \oplus W \to \tau X \to 0$ be the Auslander-Reiten sequence. Since $|Z| < |X| = |\tau X|$, each Z_i is neither projective nor injective. Using the Auslander-Reiten sequence starting from Z_i, we get

$$d'_i|X| \leqq |Z_i| + |\tau^{-1}Z_i| ,$$

hence

$$\left(\sum_{i=1}^{s} d_i d'_i\right)|X| \leqq \sum_{i=1}^{s} d_i|Z_i| + \sum_{i=1}^{s} d_i|\tau^{-1}Z_i|$$
$$\leqq |Z| + |W|$$
$$= 2|X| .$$

Therefore $\sum_{i=1}^{s} d_i d'_i \leqq 2$. Suppose $\sum_{i=1}^{s} d_i d'_i = 2$. Then $\tau^{-1}Z \cong W$, thus W does not have a projective summand and the Auslander-Reiten sequence starting from Z_i is of the form

$$0 \longrightarrow Z_i \longrightarrow X^{d'_i} \longrightarrow \tau^{-1}Z_i \longrightarrow 0$$

for all i. Using the Auslander-Reiten sequences ending in Z_i's, we conclude also that if $\sum_{i=1}^{s} d_i d'_i = 2$, then $\tau Z \cong W$, thus W does not have an injective summand and the Auslander-Reiten sequence ending in Z_i is of the form

$$0 \longrightarrow \tau Z_i \longrightarrow \tau X^{d'_i} \longrightarrow Z_i \longrightarrow 0$$

for all i. Assume $\sum_{i=1}^{s} d_i d'_i = 1$. Clearly, Z is indecomposable. Suppose $\tau^2 Z \neq Z$. Then τZ is projective and $\tau^{-1}Z$ is injective, thus we get

$$2|X| = |X| + |\tau X|$$
$$< |\tau Z| + |\tau^{-1}Z|$$
$$\leqq |W|$$
$$< 2|X| ,$$

a contradiction. Hence $\tau^2 Z \cong Z$. Suppose $\tau Z \not\cong W$ and let $W \cong \tau Z \oplus W'$. Then W' is projective-injective, thus we get

$$|Z| + |\tau Z| = |\tau Y| + |\tau Z|$$
$$< |\tau X| .$$

On the other hand, using the Auslander-Reiten sequence ending in Z, we get $|\tau X| \leqq |Z| + |\tau Z|$, a contradiction. Hence $\tau Z \cong W$. Exchainging W for Z, the above arguments imply the case in which $|Y| < |X| < |\tau Y|$. This finishes the proof.

Let $D_1 = \bigoplus_{j=1}^{s} E_j^{d_j}$, where E_j's are non-isomorphic indecomposable modules and $d_j = \dim_{F(E_j)} N(C_2, E_j)$ for all j. Let $d'_j = \dim N(C_2, E_j)_{F(C_2)}$ for each j. Suppose $\sum_{j=1}^{s} d_j d'_j \neq 1$. Then by Lemma 3(2), we get a finite component $\{A, B_1, B_2, C_1,$

$C_2, E_1, \cdots, E_s, \tau E_1, \cdots, \tau E_s\}$ consisting of only τ-periodic modules, a contradiction. Therefore, by Lemma 3(1), D_1 is indecomposable with $\tau^2 D_1 \cong D_1$. Note that $D_2 \cong \tau D_1$, since, by Lemma 3, D_2 does not have an injective summand. Thus D_2 is also indecomposable with $\tau^2 D_2 \cong D_2$. Therefore, by induction, we get a bounded length component $\{A, B_1, B_2, C_1, C_2, D_1, D_2, \cdots\}$ consisting of only τ-periodic modules, a contradiction.

This finishes the proof of Theorem 1.

2. Proof of Theorem 2.

Let X, Y be inecomposable modules such that $N(X, Y) \neq 0$ and $N(Y, X) \neq 0$. We claim that either X or Y is τ-invariant. Note that $N(\tau X, \tau Y) \neq 0$ and $N(\tau Y, \tau X) \neq 0$ if neither X nor Y is projective, and that $N(\tau^{-1}X, \tau^{-1}Y) \neq 0$ and $N(\tau^{-1}Y, \tau^{-1}X) \neq 0$ if neither X nor Y is injective. Therefore, it is sufficient to consider the following three cases:

(1) Either X or Y is projective.

(2) Either X or Y is injective.

(3) Both X and Y are stable. (Recall that an indecomposable module X is said to be stable if for any integer n, $\tau^n X$ is neither projective nor injective).

CASE 1. We may assume X is projective. Then Y is a summand of rad X, thus $|Y| < |X|$. Hence Y is not projective. Using the Auslander-Reiten sequence ending in Y, we get $|X| \leqq |\tau Y| + |Y|$. Suppose Y is not τ-invariant. Then $\tau Y \oplus Y$ is a summand of rad X, thus $|\tau Y| + |Y| < |X|$, a contradiction. Therefore Y is τ-invariant.

CASE 2. By the dual arguments, we conclude that either X or Y is τ-invariant.

CASE 3. Suppose neither X nor Y is τ-invariant. For any integer n, using the Auslander-Reiten sequence ending in $\tau^n X$, we get $|\tau^{n+1}Y| + |\tau^n Y| \leqq |\tau^{n+1}X| + |\tau^n X|$, hence, by symmetry, $|\tau^{n+1}Y| + |\tau^n Y| = |\tau^{n+1}X| + |\tau^n X|$. Therefore, for any integer n the Auslander-Reiten sequences ending in $\tau^n X, \tau^n Y$ are of the form

$$0 \longrightarrow \tau^{n+1}X \longrightarrow \tau^{n+1}Y \oplus \tau^n Y \longrightarrow \tau^n X \longrightarrow 0,$$

$$0 \longrightarrow \tau^{n+1}Y \longrightarrow \tau^{n+1}X \oplus \tau^n X \longrightarrow \tau^n Y \longrightarrow 0$$

respectively. We may assume X is of minimal length in the component $\{\tau^n X, \tau^m Y \mid n, m \in \mathbf{Z}\}$. Let $f : \tau Y \to X$ be an irreducible map. Extending f to the minimal right almost split map ending in X, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & \tau Y & \xrightarrow{f} & X & \longrightarrow & 0 \\
 & & \uparrow \wr & & \uparrow \beta' & & \uparrow \alpha' & & \\
0 & \longrightarrow & \operatorname{Ker} f' & \longrightarrow & \tau X & \xrightarrow{f'} & Y & \longrightarrow & 0,
\end{array}$$

where α', β' and f' are irreducible maps. Next, extending f' to the minimal right almost split map ending in Y, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f' & \longrightarrow & \tau X & \xrightarrow{f'} & Y & \longrightarrow & 0 \\
 & & \uparrow \wr & & \uparrow \beta'' & & \uparrow \alpha'' & & \\
0 & \longrightarrow & \operatorname{Ker} g & \longrightarrow & \tau Y & \xrightarrow{g} & X & \longrightarrow & 0,
\end{array}$$

where α'', β'' and g are irreducible maps. Hence, putting $\alpha=\alpha'\alpha''$ and $\beta=\beta'\beta''$, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & \tau Y & \xrightarrow{f} & X & \longrightarrow & 0 \\
 & & \uparrow \wr & & \uparrow \beta & & \uparrow \alpha & & \\
0 & \longrightarrow & \operatorname{Ker} g & \longrightarrow & \tau Y & \xrightarrow{g} & X & \longrightarrow & 0,
\end{array}$$

where $\alpha \in \operatorname{rad} \operatorname{End}(X)$, $\beta \in \operatorname{rad} \operatorname{End}(\tau Y)$ and g is an irreducible map. Clearly, the above arguments hold for any irreducible maps from τY to X. Therefore, by induction, we conclude that for any positive integer n, there is an irreducible map $f_n : \tau Y \to X$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & \tau Y & \xrightarrow{f} & X & \longrightarrow & 0 \\
 & & \uparrow \wr & & \uparrow \beta_n & & \uparrow \alpha_n & & \\
0 & \longrightarrow & \operatorname{Ker} f_n & \longrightarrow & \tau Y & \xrightarrow{f_n} & X & \longrightarrow & 0,
\end{array}$$

where $\alpha_n \in (\operatorname{rad} \operatorname{End}(X))^n$ and $\beta_n \in (\operatorname{rad} \operatorname{End}(\tau Y))^n$, this contradicts the fact that $\operatorname{rad} \operatorname{End}(X)$ and $\operatorname{rad} \operatorname{End}(\tau Y)$ are nilpotent.

This finishes the proof of Theorem 2.

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