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SATURATED SETS FOR GENERALIZED CARTAN MATRICES

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

By

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0. Introduction.

In the theory of Kac-Moody Lie algebras, it is important to know the set of imaginary roots and the set of weights for integrable modules. We will study these two kinds of sets using the idea of saturated sets (cf. [1]), and show the following theorems.

THEOREM 1 ([3], [4], [8]). *Let A be a generalized Cartan matrix, and $\mathfrak{g}$ the Kac-Moody Lie algebra of type A. Then the root system Δ, with simple roots $\Pi=\{\alpha_1, \cdots, \alpha_n\}$, of $\mathfrak{g}$ is uniquely characterized by the following properties*:

(1) *Δ is a saturated set,*

(2) $\Delta=-\Delta$,

(3) *$k\alpha_i\in\Delta\Leftrightarrow k=0, \pm1$ for all $\alpha_i\in\Pi$ and $k\in\boldsymbol{Z}$,*

(4) *$\beta<_{\Pi}0$ or $0<_{\Pi}\beta$ for each $\beta\in\Delta$,*

(5) *if $\beta\in\Delta$ and $ht(\beta)>1$, then there exists some $\alpha_i\in\Pi$ such that $\beta-\alpha_i\in\Delta$.*

A generalized Cartan matrix will be simply called a GCM.

THEOREM 2. *Let V be a standard $\mathfrak{g}$-module. Then the set Λ of weights for V is uniquely characterized by the following properties*:

(1) *Λ is a saturated set,*

(2) *there exists $\lambda\in\Lambda$ such that $\mu<_{\Pi}\lambda$ for all $\mu\in\Lambda$,*

(3) *if $ht(\lambda-\mu)>0$ for $\mu\in\Lambda$, then there exists some $\alpha_i\in\Pi$ such that $\mu+\alpha_i\in\Lambda$.*

Such a subset Λ is sometimes denoted $\Lambda(\lambda)$. Let $L=\bigoplus_{i=1}^{n}\boldsymbol{Z}\alpha_i$ and $L_-=\{\alpha\in L\,|\,\alpha<_{\Pi}0,\ \alpha\neq0\}$. A nonzero element $\alpha\in L$ is called connected if $\mathrm{Supp}(\alpha)$ is connected. Let R be the subset of L consisting of all the connected elements. Put $R_-=R\cap L_-$ and $C=R_-\cup\{0\}\cup(-R_-)$. Let Δ_-^{im} be the set of negative imaginary roots of $\mathfrak{g}$.

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THEOREM 3.

(1) Δ_-^{im} *is a unique maximal saturated subset of* R_-.

(2) Δ_-^{im} *is a unique maximal W-invariant subset of* R_-.

COROLLARY.

(1) Δ *is a unique maximal saturated subset* S *of* C *with the property* $S\cap \boldsymbol{Z}\alpha_i = \{0, \pm\alpha_i\}$ *for all* $1\leqq i\leqq n$.

(2) Δ *is a unique maximal W-invariant subset* S *of* C *with the property* $S\cap \boldsymbol{Z}\alpha_i=\{0, \pm\alpha_i\}$ *for all* $1\leqq i\leqq n$.

An element $\alpha\in\Delta_-^{im}$ is called a top element of Δ_-^{im} if there is no $\alpha_i\in\Pi$ such that $\alpha+\alpha_i\in\Delta_-^{im}$. Let T be the set of all the top elements of Δ_-^{im}.

THEOREM 4. $\Delta_-^{im}=\bigcup_{\alpha\in T}\Lambda(\alpha)$.

THEOREM 5. *Let A be a GCM. Suppose that A does not contain* $\begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$ $(a\geqq 5)$ *as a submatrix. Then the following two conditions are equivalent.*

(1) $\Delta_-^{im}=\Lambda(\lambda)$ *for some* λ.

(2) *A contains no affine GCM and contains a unique strictly hyperbolic GCM as submatrices.*

THEOREM 6. *Let A be a GCM. Suppose that A contains* $\begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$ $(a\geqq 5)$ *as a submatrix and choose a subset* $J=\{i, j\}$ *of* $\{1, \cdots, n\}$ *such that the corresponding submatrix* A_J *is* $\begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$. *Let* $\beta=-2\alpha_i-\alpha_j$. *If β is a unique highest element of* Δ_-^{im}, *then* $\Delta_-^{im}=\Lambda(\beta)$.

For general $\lambda\in P^+$, one can see the following.

THEOREM 7. *Let* $\lambda\in P^+\cap\sum_{i=1}^n \boldsymbol{C}\alpha_i$, *and suppose* $\lambda\neq 0$.

(1) *If* $\lambda\in R_-$, *then* $\Lambda(\lambda)\subset\Delta_-^{im}$.

(2) *If A is of strictly hyperbolic type and* $\lambda(h_i)\equiv 0 \pmod{\det A}$ *for every* $1\leqq i\leqq n$, *then* $\Lambda(\lambda)\subset\Delta_-^{im}$.

(3) *If* $A=\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$, *then* $\Lambda(\lambda)\subset\Delta_-^{im}$.

Theorems 1 and 2 will be discussed in Section 1, and Theorems 3-7 will be established in Section 2. We will give several examples in Section 3. In Appendix I, strictly hyperbolic GCMs will be classified, and in Appendix II, GCMs satisfying the conditions of Theorem 6 will be classified. Finally we note that we do not assume the symmetrisability of GCMs.

1. Saturated sets, root systems and weight systems.

An $n\times n$ integral matrix $A=(a_{ij})$ is called a generalized Cartan matrix (GCM) if $a_{ii}=2$ $(1\leqq i\leqq n)$, $a_{ij}\leqq 0$ $(1\leqq i\neq j\leqq n)$, and $a_{ij}=0\Leftrightarrow a_{ji}=0$ $(1\leqq i, j\leqq n)$. A triplet $(\mathfrak{h}, \Pi, \Pi^{\vee})$ is called a realization of A (cf. [4]) if $\mathfrak{h}$ is a finite dimensional vector space over $\boldsymbol{C}$, $\Pi=\{\alpha_1, \cdots, \alpha_n\}$ is a set of n linearly independent elements of $\mathfrak{h}^*=\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{h}, \boldsymbol{C})$, $\Pi^{\vee}=\{h_1, \cdots, h_n\}$ is a set of n linearly independent elements of $\mathfrak{h}$, and $\alpha_i(h_j)=a_{ji}$ for all i, j.

Let $L=\bigoplus_{i=1}^n \boldsymbol{Z}\alpha_i$, the root lattice in $\mathfrak{h}^*$, and let $ht(\beta)=\sum_{i=1}^n c_i$, the height of β, for an element $\beta=\sum_{i=1}^n c_i\alpha_i\in L$. We define a partial order $<_{\Pi}$ on $\mathfrak{h}^*$ by saying that $\mu<_{\Pi}\nu$ if $\nu-\mu\in\sum_{i=1}^n \boldsymbol{Z}_{\geqq 0}\alpha_i$. An element $\mu\in\mathfrak{h}^*$ is called integral if $\mu(h_i)\in\boldsymbol{Z}$ for all $1\leqq i\leqq n$. We denote by P the set of all the integral elements of $\mathfrak{h}^*$. A non-empty subset S of P is called saturated (cf. [1]) if for all $\mu\in S$, $1\leqq i\leqq n$, and k between 0 and $\mu(h_i)$, the element $\mu-k\alpha_i$ also lies in S.

The Kac-Moody Lie algebra $\mathfrak{g}$ of type A is defined to be the Lie algebra over $\boldsymbol{C}$ generated by the so-called Cartan subalgebra $\mathfrak{h}$ and the so-called Chevalley generators $e_1, \cdots, e_n, f_1, \cdots, f_n$ with the following defining relations: $[h, h']=0$ $(h, h'\in\mathfrak{h})$, $[e_i, f_j]=\delta_{ij}h_i$ $(1\leqq i, j\leqq n)$, $[h, e_i]=\alpha_i(h)e_i$ $(1\leqq i\leqq n)$, $[h, f_i]=-\alpha_i(h)f_i$ $(1\leqq i\leqq n)$, $(\mathrm{ad}\, e_i)^{n(i,j)}e_j=0$ $(1\leqq i\neq j\leqq n)$, $(\mathrm{ad}\, f_i)^{n(i,j)}f_j=0$ $(1\leqq i\neq j\leqq n)$, where $n(i, j)=-a_{ij}+1$ (cf. [2], [3], [6]).

A $\mathfrak{g}$-module V is called integrable if

(1) V is the direct sum of the weight subspaces $V_{\mu}=\{v\in V\,|\,hv=\mu(h)v$ for all $h\in\mathfrak{h}\}$ with $\mu\in\mathfrak{h}^*$,

(2) the e_i and f_i are locally nilpotent on V.

The set of weights for V is denoted by $\Lambda(V)=\{\mu\in\mathfrak{h}^*\,|\,V_{\mu}\neq 0\}$. Then $\Lambda(V)\subset P$. The adjoint representation is integrable, and $\Delta=\Lambda(\mathfrak{g})$ is called the root system of $\mathfrak{g}$ (cf. [3], [7]).

THEOREM 1 ([3], [4], [8]). *Notation is as above. Then Δ is uniquely characterized by the following properties:*

(1) *Δ is a saturated set,*

(2) $\Delta=-\Delta$,

(3) *$k\alpha_i\in\Delta\Leftrightarrow k=0, \pm 1$ for all $1\leqq i\leqq n$ and $k\in\boldsymbol{Z}$,*

(4) *$\beta<_{\Pi}0$ or $0<_{\Pi}\beta$ for each $\beta\in\Delta$,*

(5) *if $\beta\in\Delta$ and $ht(\beta)>1$, then there exists some $\alpha_i\in\Pi$ such that $\beta-\alpha_i\in\Delta$.*

There exists a subset $S\subset P$ which satisfies the conditions (1)-(4) but does not satisfy (5) in Theorem 1. This is due to M. Kaneda.

An integrable $\mathfrak{g}$-module V is called a standard module (cf. [5]) if there exists a nonzero weight vector $v^+\in V_\lambda$ for some $\lambda\in\mathfrak{h}^*$ such that $U(\mathfrak{g})v^+=V$ and $e_iv^+=0$ for all $1\leqq i\leqq n$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. This weight λ is called the highest weight of V, and always lies in $P^+=\{\mu\in P\mid\mu(h_i)\geqq 0$ for all $1\leqq i\leqq n\}$. Conversely for each $\lambda\in P^+$ there exists a standard module whose highest weight is λ. The set of weights for such a $\mathfrak{g}$-module is denoted $\Lambda(\lambda)$.

THEOREM 2. *Notation is as above. Then $\Lambda(\lambda)$ is uniquely characterized by the following properties*:

(1) *$\Lambda(\lambda)$ is a saturated set,*

(2) *$\lambda\in\Lambda(\lambda)$ and $\mu<_{\Pi}\lambda$ for all $\mu\in\Lambda(\lambda)$,*

(3) *if $ht(\lambda-\mu)>0$ for $\mu\in\Lambda(\lambda)$, then there exists some $\alpha_i\in\Pi$ such that $\mu+\alpha_i\in\Lambda(\lambda)$.*

Notice that there is a saturated subset S of P which satisfies the condition (2) but does not satisfy (3) in Theorem 2 (see Example (3)).

PROOF OF THEOREM 2. It is enough to show the uniqueness. Let S and S' satisfy the conditions (1)-(3) in Theorem 2. Put $S_n=\{\mu\in S\mid ht(\lambda-\mu)\leqq n\}$ and $S'_n=\{\mu\in S'\mid ht(\lambda-\mu)\leqq n\}$ for $n=0, 1, 2, \cdots$. We will show $S_n=S'_n$ by induction on n. Of course $S_0=\{\lambda\}=S'_0$ by the condition (2). Suppose $S_{n-1}=S'_{n-1}$. Take an element $\mu\in S_n$. By the condition (3), there is some $\alpha_i\in\Pi$ such that $\nu=\mu+\alpha_i\in S$. Thus $\nu\in S_{n-1}=S'_{n-1}$. If $\nu(h_i)>0$, then $\nu-\alpha_i=\mu\in S'$ by the condition (1). Suppose $\nu(h_i)\leqq 0$. Then $\mu=\nu-\alpha_i\in S$ (resp. S') if and only if $\nu+(-\nu(h_i)+1)\alpha_i\in S_{n-1}$ (resp. S'_{n-1}). By our assumption: $S_{n-1}=S'_{n-1}$, we see $\mu\in S'$. Anyway $\mu\in S'_n$. Therefore $S_n\subset S'_n$. Similarly we can show $S'_n\subset S_n$. Hence $S_n=S'_n$. □

COROLLARY. *Let S satisfy the conditions* (1)-(3) *in Theorem* 2. *Put $S_n=\{\mu\in S\mid ht(\lambda-\mu)\leqq n\}$ for $n=0, 1, 2, \cdots$. Let $\mu\in S_n$ and $\alpha_i\in\Pi$.*

(1) *If $\mu(h_i)>0$, then $\mu-\alpha_i\in S_{n+1}$.*

(2) *If $\mu(h_i)\leqq 0$, then $\mu-\alpha_i\in S_{n+1}$ if and only if $\mu+(\mu(h_i)+1)\alpha_i\in S_n$.*

2. Negative imaginary roots.

Let r_i be the involutive linear transformation of $\mathfrak{h}^*$ defined by $r_i(x)=x-$

$x(h_i)\alpha_i$ for all $x\in\mathfrak{h}^*$. Then Δ is r_i-stable for each $i=1, \cdots, n$. Put $W=\langle r_i | i=1, \cdots, n\rangle\subset GL(\mathfrak{h}^*)$, called the Weyl group. Let $\Delta^{re}=\{w\alpha_i | w\in W, i=1, \cdots, n\}$, the set of real roots, and $\Delta^{im}=\Delta-\Delta^{re}$, the set of imaginary roots. Set $\Delta_-^{im}=\{\alpha\in\Delta^{im} | \alpha\neq 0, \alpha<_\Pi 0\}$, the set of negative imaginary roots. Put $L_-=\{\mu\in L | \mu<_\Pi 0, \mu\neq 0\}$.

For a GCM $A=(a_{ij})_{1\leq i,j\leq n}$ and a non-empty subset J of $\{1, \cdots, n\}$ with some ordering, the matrix $A_J=(a_{ij})_{i,j\in J}$ is also a GCM. Such a GCM is called a submatrix of A, and we say that A contains A_J as a submatrix. A GCM $A=(a_{ij})_{1\leq i,j\leq n}$ is called indecomposable if for any distinct i, j, there exist some indices $i(1)=i, i(2), i(3), \cdots, i(r)=j$ such that $a_{i(k),i(k+1)}\neq 0$ for $k=1, \cdots, r-1$.

For a nonzero element $\alpha=\sum_{i=1}^n c_i\alpha_i\in L$, put $\mathrm{Supp}(\alpha)=\{i | c_i\neq 0\}$, the support of α. Simply we write $A_\alpha=A_{\mathrm{Supp}(\alpha)}$. Then α is called connected if $\mathrm{Supp}(\alpha)$ is connected in terms of Dynkin diagrams, equivalently, if A_α is indecomposable. We regard 0 as a non-connected element. Let R be the subset of L consisting of all the connected elements. Put $R_-=R\cap L_-$ and $C=R_-\cup\{0\}\cup(-R_-)$.

THEOREM 3. *Notation is as above.*

(1) Δ_-^{im} *is a unique maximal saturated subset of* R_-.

(2) Δ_-^{im} *is a unique maximal* W*-invariant subset of* R_-.

COROLLARY.

(1) Δ *is a unique maximal saturated subset* S *of* C *with the property* $S\cap\boldsymbol{Z}\alpha_i=\{0, \pm\alpha_i\}$ *for all* $1\leqq i\leqq n$.

(2) Δ *is a unique maximal* W*-invariant subset* S *of* C *with the property* $S\cap\boldsymbol{Z}\alpha_i=\{0, \pm\alpha_i\}$ *fer all* $1\leqq i\leqq n$.

PROOF OF COROLLARY. It is enough to show (2). Put $S_-=S\cap R_-$ and $S_+=S\cap(-R_-)$. Since $-S$ also has the same property, we see $S_+=-S_-$ by the maximality. Let $\alpha\in S_-$, and set $S'=\{w(\alpha) | w\in W\}$. If $S'\subset S_-$, then S' is a W-invariant subset of R_-, so $S'\subset\Delta_-^{im}$ by Theorem 3(2). Therefore $\alpha\in\Delta_-^{im}$. If $S'\not\subset S_-$, then there is $\beta\in S'$ such that $\beta\in S_-$ and $r_i(\beta)\in S_+$ for some $1\leqq i\leqq n$. This implies that $\beta=\alpha_i$ and $\alpha\in\Delta^{re}$ by the trichotomy. Hence $\alpha\in\Delta$ and $S_-\subset\Delta$. Then S_+ is also a subset of Δ by Theorem 1(2). Therefore $S\subset\Delta$, and $S=\Delta$ by the maximality of S. $\square$

PROOF OF THEOREM 3. Since a saturated set is always W-invariant, we need only to prove (2). Let S be a W-invariant subset in R_-. For any $\lambda\in S$, we can choose $w_0\in W$ such that $w_0(\lambda)\in P^+$. Then $w_0(\lambda)\in K=K(A)=P^+\cap R_-$, and so $\lambda\in\bigcup_{w\in W}w(K)=\Delta_{-}^{im}$ (cf. [3], [4; Th. 5.4]), which completes the proof. □

An element $\alpha\in\Delta_{-}^{im}$ is called a top element if there is no $\alpha_i\in\Pi$ such that $\alpha+\alpha_i\in\Delta_{-}^{im}$. Let T be the set of all the top elements of Δ_{-}^{im}.

THEOREM 4. $\Delta_{-}^{im}=\bigcup_{\alpha\in T}\Lambda(\alpha)$.

PROOF OF THEOREM 4. Let $\alpha\in T$, and take $\beta\in\Lambda(\alpha)$. Suppose that Supp(β) is not connected. Then there is an element $\gamma\in\Lambda(\alpha)$ such that $\gamma-\alpha_i\in\Lambda(\alpha)$ and $\gamma(h_i)=0$ for some $i\notin\mathrm{Supp}(\gamma)$, which implies that $\gamma+\alpha_i\in\Lambda(\alpha)$ and $\gamma+\alpha_i\notin L_{-}$. This is a contradiction. Hence $\Lambda(\alpha)$ is a saturated subset of R_{-}. Therefore $\bigcup_{\alpha\in T}\Lambda(\alpha)\subset\Delta_{-}^{im}$. Conversely let $\beta\in\Delta_{-}^{im}$. We will show that β belongs to $\Lambda(\alpha)$ for some $\alpha\in T$ by induction on $n=|ht(\beta)|$. If there is no $\alpha_i\in\Pi$ such that $\beta+\alpha_i\in\Delta_{-}^{im}$, then $\beta\in T$ and $\beta\in\Lambda(\beta)$. Suppose that $\beta'=\beta+\alpha_i\in\Delta_{-}^{im}$ $(\alpha_i\in\Pi)$. Then, by induction, $\beta'\in\Lambda(\alpha)$ for some $\alpha\in T$. If $\beta'(h_i)>0$, then $\beta=\beta'-\alpha_i\in\Lambda(\alpha)$. If $\beta'(h_i)\leqq 0$, then $\gamma=\beta-\beta(h_i)\alpha_i=\beta'+(-\beta'(h_i)+1)\alpha_i\in\Delta_{-}^{im}$ and $|ht(\gamma)|<n$. Then $\gamma\in\Lambda(\alpha')$ for some $\alpha'\in T$. Hence $\beta\in\Lambda(\alpha')$. Therefore $\Delta_{-}^{im}\subset\bigcup_{\alpha\in T}\Lambda(\alpha)$. □

Cartan matrices arising from the classification of finite dimensional complex semisimple Lie algebras are called GCMs of finite type. An indecomposable GCM $A=(a_{ij})_{1\leq i,j\leq n}$ is called of affine type if there exist some positive integers $b_1,\cdots,b_n$ such that $(b_1,\cdots,b_n)\cdot(a_{ij})=(0,\cdots,0)$. An indecomposable GCM A is called of strictly hyperbolic type if A is not of finite type and not of affine type, and any proper submatrix of A is of finite type.

THEOREM 5. *Let A be a GCM, and Δ_{-}^{im} the set of negative imaginary roots of the associated Kac-Moody Lie algebra. Suppose that A does not contain $\begin{pmatrix}2 & -a\\ -1 & 2\end{pmatrix}$ $(a\geqq 5)$ as a submatrix. Tnen the following two conditions are equivalent.*

(1) *$\Delta_{-}^{im}=\Lambda(\lambda)$ for some λ.*

(2) *A contains no affine GCM and contains a unique strictly hyperbolic GCM as submatrices.*

We will proceed in several steps.

LEMMA ([4; Prop. 11. 2b)]). *Let $\lambda\in P^{+}$, and $J=\{i\,|\,\lambda(h_i)=0\}$. If J is empty or A_J is of finite type, then $\Lambda(\lambda)=W\cdot\{\mu\in P^{+}\,|\,\mu<_{\Pi}\lambda\}$.*

PROOF OF THEOREM 5.

STEP 0. If A is of strictly hyperbolic type, then $\Delta_{-}^{im}=\Lambda(\lambda)$ for some λ.

This follows from Theorem 3, Lemma, and Appendix I.

STEP 1. In Theorem 5, (1) implies (2).

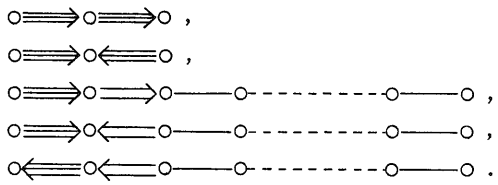
If A contains an affine submatrix, then Δ_-^{im} does not satisfy the condition (3) in Theorem 2. If A contains two strictly hyperbolic GCMs as submatrices, then Δ_-^{im} does not satisfy the condition (2) in Theorem 2.

STEP 2. Suppose that A satisfies the condition (2) in Theorem 3. Let B be a unique, up to permutations of indices, strictly hyperbolic submatrix of A, and β the highest element of $K(B)$ (cf. Theorem 3, Step 0). If $\alpha\in K(A)$, then $\mathrm{Supp}(\beta)\subset\mathrm{Supp}(\alpha)$.

Let A' be a minimal indecomposable non-finite submatrix of A_α. Then $A'=B$ (modulo permutations of indices).

STEP 3. Under the same situation as in Step 2, β is a unique highest element of $K=K(A)$.

The result is clear if B is none of the GCMs of type (iii)-tree (b), (c) in Appendix I. In the remaining cases, the posibility of A is given by the following Dynkin diagrams:



In any case, it is confirmed by direct computation that β is the highest element of K.

STEP 4. Under the same situation as in step 2, $\Delta_-^{im}=\Lambda(\lambda)$.

By Step 3, we see $K\subset P^+(\beta)=\{\mu\in P^+ \mid \mu<_\Pi\beta\}$. On the other hand, take $\mu\in P^+(\beta)$. Then $\mu\in L$ and $\mu\lneqq_\Pi 0$. Suppose that μ is not connected. Then there is an indecomposable finite submatrix, say A_J, of A satisfying that there is an index $j\in J$ such that $\mu(h_j)<0$, a contradiction. Therefore $\mu\in K$. Hence $K=P^+(\beta)$. Then Lemma leads to $\Delta_-^{im}=\Lambda(\lambda)$. □

Similarly we can show the following.

THEOREM 6. *Let A be a GCM. Suppose that A contains $\begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$ $(a\geqq 5)$ as a submatrix and choose a subset $J=\{i, j\}$ of $\{1, \cdots, n\}$ such that the corresponding submatrix A_J is $\begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$. Let $\beta=-2\alpha_i-\alpha_j$. If β is a unique highest element of Δ_-^{im}, then $\Delta_-^{im}=\Lambda(\beta)$.*

Such an indecomposable GCM in Theorem 6 will be classified in Appendix II. To attract a one's attention, we will write Step 0 again.

COROLLARY. *If A is of strictly hyperbolic type, then*

(1) *Δ_{-}^{im} has a unique highest element β (listed in Appendix I),*

(2) *$\Delta_{-}^{im}=\Lambda(\beta)$.*

As a condition of the relation $\Lambda(\lambda)\subset\Delta_{-}^{im}$, we get the following.

THEOREM 7. *Let $\lambda\in P^{+}\cap\sum_{i=1}^{n}C\alpha_i$ and suppose $\lambda\neq 0$.*

(1) *If $\lambda\in R_{-}$, then $\Lambda(\lambda)\subset\Delta_{-}^{im}$.*

(2) *If A is of strictly hyperbolic type and $\lambda(h_i)\equiv 0$ (mod $\det A$) for every $1\leqq i\leqq n$, then $\Lambda(\lambda)\subset\Delta_{-}^{im}$.*

(3) *If $A=\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$, then $\Lambda(\lambda)\subset\Delta_{-}^{im}$.*

PROOF OF THEOREM 7 (1): One sees, as in the proof of Theorem 4, that $\Lambda(\lambda)\subset R_{-}$. Thus, by the maximality of Δ_{-}^{im} (cf. Theorem 3), $\Lambda(\lambda)\subset\Delta_{-}^{im}$. (2): We notice that $\det A<0$ and every entry of the cofactor matrix $\tilde{A}$ of A is a positive integer (cf. Appendix I). Write $\lambda=\sum_{i=1}^{n}k_i\alpha_i$ ($k_i\in\boldsymbol{C}$), and put $m_i=\lambda(h_i)$ ($1\leqq i\leqq n$). Then $(m_1, \cdots, m_n)=(k_1, \cdots, k_n)\cdot{}^tA$, and $(k_1, \cdots, k_n)=(m_1', \cdots, m_n')\cdot{}^t\tilde{A}$, where $m_i'=m_i/(\det A)$. Therefore $k_i\in\boldsymbol{Z}_{<0}$ for all $1\leqq i\leqq n$, and $\lambda\in R_{-}$. (3) is obvious from (2) since $\det A$ is -1. $\square$

In each case, it is possible to write down an explicit condition (cf. Example (6)-(12)).

3. Examples.

(1) Let $A=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then

$$T=\{-k(\alpha_1+\alpha_2)\,|\,k=1, 2, 3, \cdots\},$$

$$\Lambda(\alpha)=\{\alpha\} \quad \text{for each } \alpha\in T,$$

$$\Delta_{-}^{im}=\bigcup_{\alpha\in T}\Lambda(\alpha)=T.$$

(2) Let $A=\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. Then

$$T=\{-\alpha_1-\alpha_2\},$$

$$\Delta_{-}^{im}=\Lambda(-\alpha_1-\alpha_2).$$

(3)-(i) Let A be an affine GCM and $S_1=\Delta_-^{im}$.

-(ii) Let A be a non-finite GCM and $S_2=\{0\}\cup\Delta_-^{im}$.

Then S_1 and S_2 satisfy the conditions (1), (2) in Theorem 2, but does not satisfy the condition (3) in Theorem 2.

(4) Let $A=\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$. Then

$$T=\{-2\alpha_1-\alpha_2\},$$

$$\Delta_-^{im}=\Lambda(-2\alpha_1-\alpha_2).$$

(5) Let $A=\begin{pmatrix} 2 & -5 & 0 \\ -1 & 2 & -1 \\ 0 & -5 & 2 \end{pmatrix}$. Then

$$T=\{-2\alpha_1-\alpha_2,\ -\alpha_1-\alpha_2-\alpha_3,\ -\alpha_2-2\alpha_3\},$$

$$\Delta_-^{im}=\Lambda(-2\alpha_1-\alpha_2)\cup\Lambda(-\alpha_1-\alpha_2-\alpha_3)\cup\Lambda(-\alpha_2-2\alpha_3).$$

In (6)-(12), we suppose that λ belongs to $P^+\cap\sum_{i=1}^n C\alpha_i$ and $\lambda\neq 0$.

(6) Let $A=\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$. Then

$$\lambda(h_3)\equiv 0 \pmod 2 \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im}.$$

(7) Let $A=\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Then

$$\lambda(h_1+h_3)\equiv 0 \pmod 2 \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im}.$$

(8) Let $A=\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Then

$$\lambda(h_1-h_3)\equiv 0 \pmod 3 \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im}.$$

(9) Let $A=\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$. Then

$$\lambda(h_1+h_3)\equiv 2\lambda(h_2) \pmod 4 \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im}.$$

(10) Let $A=\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$. Then

$$\lambda(h_1-h_3)\equiv 2\lambda(h_2) \pmod 4 \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im}.$$

(11) Let $A=\begin{pmatrix} 2 & -1 & 0 & -1 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$. Then

$$\lambda(h_1-h_3)\equiv 2\lambda(h_4) \pmod 4 \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im}.$$

(12) Let $A=\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -2 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}$. Then

$$\begin{cases} a+2b\equiv 0 \pmod 5 \\ 2a-b\equiv 0 \pmod 5 \end{cases} \Leftrightarrow \Lambda(\lambda)\subset\Delta_-^{im},$$

where $a=\lambda(h_2-h_3)$, $b=\lambda(h_1-h_4)$.

Appendix I

(List of strictly hyperbolic GCMs)

(size), GCM A, det A, $\tilde{A}$, a unique highest element of Δ_-^{im}

(ii)-(α) $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ $(a,\ b\geqq 2,\ ab\geqq 5)$, $4-ab$, $\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$, $-\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(β) $\begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$ $(a\geqq 5)$, $4-a$, $\begin{pmatrix} 2 & a \\ 1 & 2 \end{pmatrix}$, $-\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(iii)-loop

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, -3, \begin{pmatrix} 3 & 3 & 3 \\ 5 & 3 & 4 \\ 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, -6, \begin{pmatrix} 3 & 3 & 3 \\ 7 & 3 & 5 \\ 5 & 3 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}, -6, \begin{pmatrix} 2 & 3 & 4 \\ 6 & 3 & 6 \\ 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}, -6, \begin{pmatrix} 2 & 6 & 4 \\ 3 & 3 & 3 \\ 4 & 6 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}, -7, \begin{pmatrix} 2 & 4 & 3 \\ 5 & 3 & 4 \\ 6 & 5 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -9, \begin{pmatrix} 1 & 3 & 5 \\ 7 & 3 & 8 \\ 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -9, \begin{pmatrix} 1 & 7 & 4 \\ 3 & 3 & 3 \\ 5 & 8 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}, -10, \begin{pmatrix} 2 & 4 & 6 \\ 6 & 2 & 8 \\ 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -11, \begin{pmatrix} 1 & 5 & 3 \\ 5 & 3 & 4 \\ 8 & 7 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -11, \begin{pmatrix} 1 & 5 & 8 \\ 5 & 3 & 7 \\ 3 & 4 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -12, \begin{pmatrix} 1 & 3 & 5 \\ 9 & 3 & 9 \\ 5 & 3 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -12, \begin{pmatrix} 1 & 9 & 5 \\ 3 & 3 & 3 \\ 5 & 9 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}, -13, \begin{pmatrix} 2 & 6 & 5 \\ 5 & 2 & 6 \\ 6 & 5 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -13, \begin{pmatrix} 1 & 4 & 7 \\ 7 & 2 & 10 \\ 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -14, \begin{pmatrix} 1 & 10 & 6 \\ 3 & 2 & 4 \\ 5 & 8 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -2 & -3 & 2 \end{pmatrix}, -14, \begin{pmatrix} 1 & 5 & 3 \\ 6 & 2 & 4 \\ 10 & 8 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -16, \begin{pmatrix} 1 & 5 & 3 \\ 7 & 3 & 5 \\ 11 & 7 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -3 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -17, \begin{pmatrix} 1 & 4 & 7 \\ 9 & 2 & 12 \\ 5 & 3 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 & -2 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -17, \begin{pmatrix} 1 & 12 & 7 \\ 3 & 2 & 4 \\ 5 & 9 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -19, \begin{pmatrix} 1 & 8 & 5 \\ 5 & 2 & 6 \\ 8 & 7 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 & -2 \\ -1 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -19, \begin{pmatrix} 1 & 8 & 13 \\ 5 & 2 & 8 \\ 3 & 5 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}, -22, \begin{pmatrix} 1 & 5 & 9 \\ 9 & 1 & 15 \\ 5 & 3 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -27, \begin{pmatrix} 1 & 8 & 5 \\ 7 & 2 & 8 \\ 11 & 7 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -3 \\ -3 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, -38, \begin{pmatrix} 1 & 11 & 7 \\ 7 & 1 & 11 \\ 11 & 7 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

tree (a)

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}, -2, \begin{pmatrix} 1 & 4 & 2 \\ 2 & 4 & 2 \\ 3 & 6 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}, -4, \begin{pmatrix} 1 & 6 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

tree (b)

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}, -2, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}, -2, \begin{pmatrix} 1 & 4 & 6 \\ 2 & 4 & 6 \\ 1 & 2 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}, -4, \begin{pmatrix} 1 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}, -4, \begin{pmatrix} 1 & 2 & 3 \\ 6 & 4 & 6 \\ 3 & 2 & 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

tree (c)

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}, -2, \begin{pmatrix} 1 & 2 & 1 \\ 4 & 4 & 2 \\ 6 & 6 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(iv)
$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, -4, \begin{pmatrix} 4 & 4 & 4 & 4 \\ 7 & 4 & 5 & 6 \\ 6 & 4 & 2 & 4 \\ 5 & 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

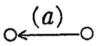
$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix}, -7, \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 2 & 5 & 8 \\ 8 & 5 & 2 & 6 \\ 5 & 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -2 & 2 \end{pmatrix}, -8, \begin{pmatrix} 2 & 4 & 6 & 4 \\ 5 & 2 & 7 & 6 \\ 6 & 4 & 2 & 4 \\ 7 & 6 & 5 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(v)
$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}, -5, \begin{pmatrix} 5 & 5 & 5 & 5 & 5 \\ 9 & 5 & 6 & 7 & 8 \\ 8 & 5 & 2 & 4 & 6 \\ 7 & 5 & 3 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 \end{pmatrix}, -\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

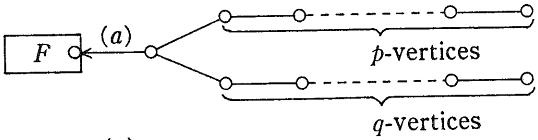
Appendix II

(Dynkin diagrams with the properties in Theorem 6)

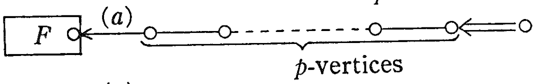


$= \begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix} \quad (a \geq 5).$

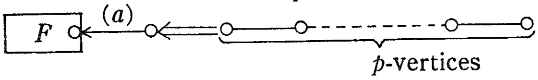
$\boxed{\;F\;}$ = a Dynkin diagram of finite type.

(1) 

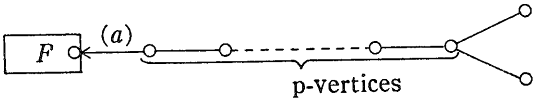
$(p, q \geq 0)$

(2) 

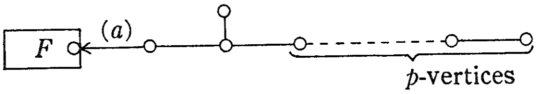
$(p \geq 1)$

(3) 

$(p \geq 2)$

(4) 

$(p \geq 2)$

(5) 

$(p \geq 2)$

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