TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 455—459

NOTE ON A PAPER BY MAHLER

By

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1. Introduction.

Let ω be a real quadratic irrational number with $0<\omega<1$, and put

$$F_{\omega}(z_1, z_2)=\sum_{h_1=1}^{\infty}\sum_{h_2=1}^{[h_1\omega]} z_1^{h_1}z_2^{h_2}. \tag{1}$$

The series $F_{\omega}(z_1, z_2)$ converges in the domain

$$\{|z_1|<1,\ |z_1||z_2|^{\omega}<1\}.$$

Mahler [3] proves that $F_{\omega}(\alpha_1, \alpha_2)$ is transcendental for algebraic α_1, α_2 with suitable properties. In the succeeding paper [4], he studies the algebraic independence of the values

$$\left.\frac{\partial^{k_1+k_2}F_{\omega}(z_1, z_2)}{\partial z_1^{k_1}\partial z_2^{k_2}}\right|_{(\alpha_1, \alpha_2)}, \qquad k_1\geqq 0,\ k_2\geqq 0. \tag{2}$$

To prove the algebraic independence of the values, he asserts the functions

$$\frac{\partial^{k_1+k_2}F_{\omega}(z_1, z_2)}{\partial z_1^{k_1}\partial z_2^{k_2}}, \qquad k_1\geqq 0,\ k_2\geqq 0, \tag{3}$$

are algebraically independent over the rational function field $C(z_1, z_2)$. But it is pointed out in Kubota [1] and Loxton and van der Poorten [2] that Mahler's criterion for algebraic independence (Satz 1 in [4]) is not correct. Although the correct criterion is given in [1] and [2], it seems that there is no proof of the algebraic independence of the functions (3). Here we will prove the following theorems.

THEOREM 1. *The functions* (3) *are algebraically independent over* $C(z_1, z_2)$.

Let ω be expanded in the continued fraction

$$\omega=\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}}, \tag{4}$$

Received June 10, 1992, Revised November 5, 1992.

and

(5) $$p_{-1}=0,\quad p_0=1,\quad p_1=a_1,\quad p_{\mu+1}=a_{\mu+1}p_\mu+p_{\mu-1},$$
$$q_{-1}=1,\quad q_0=0,\quad q_1=1,\quad q_{\mu+1}=a_{\mu+1}q_\mu+q_{\mu-1}.$$

From Theorem 1 and the main theorem in [4], we obtain the following theorem.

THEOREM 2. *Let α_1, α_2 be algebraic numbers satisfying*

(6) $$0<|\alpha_1|<1,\qquad 0<|\alpha_1||\alpha_2|^{\omega}<1,\qquad \alpha_1^{p_\mu}\alpha_2^{q_\mu}\neq 1\quad (\mu\geqq 0).$$

Then the values (2) *are algebraically independent.*

COROLLARY. *Let $f(z)=\sum_{h=1}^{\infty}[h\omega]z^h$ and α an algebraic number with $0<|\alpha|<1$. Then*

$$f^{(k)}(\alpha),\qquad k\geqq 0$$

are algebraically independent.

2. Proof of the theorems.

Define $\omega_1, \omega_2, \cdots$ by

$$\omega=\frac{1}{a_1+\omega_1},\qquad \omega_1=\frac{1}{a_2+\omega_2},\ \cdots$$

Because of the equality (see [3])

(7) $$F_\omega(z_1, z_2)=\sum_{\mu=0}^{\nu-1}(-1)^\mu\frac{z_1^{p_{\mu+1}+p_\mu}z_2^{q_{\mu+1}+q_\mu}}{(1-z_1^{p_{\mu+1}}z_2^{q_{\mu+1}})(1-z_1^{p_\mu}z_2^{q_\mu})}$$
$$+(-1)^\nu F_{\omega_\nu}(z_1^{p_\nu}z_2^{q_\nu},\ z_1^{p_{\nu-1}}z_2^{q_{\nu-1}}),$$

we may assume that the continued fraction of ω is purely periodic. Therefore there exists a natural number ν such that $\omega=\omega_\nu$. We may assume ν is even. Put

$$\Omega=\begin{pmatrix} p_\nu & q_\nu \\ p_{\nu-1} & q_{\nu-1}\end{pmatrix}\quad\text{and}\quad \Omega(z_1, z_2)=(z_1^{p_\nu}z_2^{q_\nu},\ z_1^{p_{\nu-1}}z_2^{q_{\nu-1}}).$$

Then we have

(8) $$F_\omega(z_1, z_2)=F_\omega(\Omega(z_1, z_2))+b(z_1, z_2),\qquad b(z_1, z_2)\in \boldsymbol{Q}(z_1, z_2).$$

Let $\rho_1=p_\nu+p_{\nu-1}\omega$, $\rho_2=q_{\nu-1}-p_{\nu-1}\omega$. Then ρ_1, ρ_2 are the eigen values of the matrix Ω and

$$\begin{pmatrix} o_1^{(\lambda)} \\ o_2^{(\lambda)}\end{pmatrix}=\begin{pmatrix} q_{\nu-1}-\rho_\lambda \\ -p_{\nu-1}\end{pmatrix}$$

is an eigenvector belonging to ρ_λ ($\lambda=1, 2$). Put

$$D_\lambda=o_1^{(\lambda)}z_1\frac{\partial}{\partial z_1}+o_2^{(\lambda)}z_2\frac{\partial}{\partial z_2}, \qquad \lambda=1, 2.$$

Then we have ([4], §9),

$$D_1^{k_1}D_2^{k_2}f(\Omega(z_1, z_2))$$
$$=\rho_1^{k_1}\rho_2^{k_2}D_1^{k_1}D_2^{k_2}f(z_1, z_2)|_{\Omega(z_1, z_2)}, \ k_1, k_2\geqq 0.$$

where $f(z_1, z_2)$ is any analytic function. By the equality (8), we have

(9) $$D_1^{k_1}D_2^{k_2}F_\omega(z_1, z_2)$$
$$=\rho_1^{k_1}\rho_2^{k_2}D_1^{k_1}D_2^{k_2}F_\omega(z_1, z_2)|_{\Omega(z_1, z_2)}+D_1^{k_1}D_2^{k_2}b(z_1, z_2).$$

We shall prove that the functions

(10) $$D_1^{k_1}D_2^{k_2}F_\omega(z_1, z_2), \qquad k_1, k_2\geqq 0,$$

are algebraically independent over $C(z_1, z_2)$, from which Theorem 1 and Theorem 2 follow, since $\det\begin{pmatrix} o_1^{(1)} & o_1^{(2)} \\ o_2^{(1)} & o_2^{(2)}\end{pmatrix}\neq 0$. The proof is by contradiction. We assume the functions (10) were algebraically dependent over $C(z_1, z_2)$. Let $K=\boldsymbol{Q}(\omega)$. Since the Taylor coefficients of the functions (10) are in K, the functions are algebraically dependent over $K(z_1, z_2)$. By Corollary 9 in [1], the functions (10) are K-linearly dependent mod $K(z_1, z_2)$. (Kubota [1] states the corollary over the field C, but it is easily checked that the above statement is also valid.) Therefore the functions

$$F^{(k_1, k_2)}(z_1, z_2)=\left(z_1\frac{\partial}{\partial z_1}\right)^{k_1}\left(z_2\frac{\partial}{\partial z_2}\right)^{k_2}F_\omega(z_1, z_2), \qquad k_1, k_2\geqq 0,$$

are also K-linearly dependent mod $K(z_1, z_2)$. Hence the functions

$$F^{(k_1, k_2)}(z, 1), \qquad k_1, k_2\geqq 0$$

are K-linearly dependent mod $K(z)$. We have

$$F^{(k_1, k_2)}(z, 1)=\sum_{h=1}^{\infty}h^{k_1}\{1+2^{k_2}+\cdots+[h\omega]^{k_2}\}z^h.$$

Put

(11) $$f_{ij}(z)=\sum_{h=1}^{\infty}h^i[h\omega]^j z^h, \qquad i\geqq 0, \ j\geqq 1.$$

Then $\{F^{(k_1, k_2)}(z, 1)\}_{0\leq k_1, k_2\leq M}$ and $\{f_{ij}(z)\}_{0\leq i\leq M, 1\leq j\leq M+1}$ generate the same vector space over K, and so $\{f_{ij}(z)\}_{i\geq 0, j\geq 1}$ are K-linearly dependent mod $K(z)$. Since the coefficients of $f_{ij}(z)$ are all in $\boldsymbol{Q}$, $\{f_{ij}(z)\}_{i\geq 0, j\geq 1}$ are $\boldsymbol{Q}$-linearly dependent mod $\boldsymbol{Q}(z)$. Then there are integers e_{ij}, not all zero, such that

(12) $$g(z)=\sum_{i=0}\sum_{j=1}e_{ij}f_{ij}(z)=p(z)/q(z)\in\boldsymbol{Q}(z),$$

where $p(z)$ and $q(z)$ are relatively prime polynomials with integer coefficients. Let $\xi_1, \cdots, \xi_n$ be the distinct roots of $q(z)$ and $g(z)=\sum_{h=0}^{\infty}c_h z^h$. Then we have $c_h\in\boldsymbol{Z}$ and

$$c_h=P_1(h)\xi_1{}^h+\cdots+P_n(h)\xi_n{}^h, \qquad h\gg 0.$$

We choose a subset S of $\{\xi_1, \cdots, \xi_n\}$ such that for every i $(1\leqq i\leqq n)$, there exists an unique $\xi\in S$ with ξ_i/ξ is a root of unity. We may assume $S=\{\xi_1, \cdots, \xi_m\}$. By the choice of S, ξ_i/ξ_j is not a root of unity for any distinct i, j. For a suitable natunrnal number N, we have

$$c_{hN}=Q_1(h)\xi_1{}^{hN}+\cdots+Q_{m'}(h)\xi_{m'}{}^{hN}, \qquad h\geqq 1,$$

where $Q_i(h)$ are nonzero polynomials of h and $m'\leqq m$. By (11) and (12), c_h are rational integers and

$$|c_h|\leqq c_1 h^{c_2}, \qquad h\geqq 1,$$

where c_1 and c_2 are positive constants. When $m'\geqq 1$, by the lemma in [5], we have

$$|\xi_i{}^{\sigma}|_p\leqq 1, \qquad i=1, \cdots, m',$$

where p is ∞ or a prime number and σ is any automorphism of $\bar{\boldsymbol{Q}}$. Therefore we conclude that ξ_i are roots of unity. Hence we have

(13) $$c_{hN}=a_s h^s+a_{s-1}h^{s-1}+\cdots+a_0, \qquad h\geqq 1,$$

for a suitable natural number N. If $m'=0$, then $c_{hN}=0$ for any $h\geqq 1$. In any case, we have the equality (13). On the other hand, by (12), we have

(14) $$c_{hN}=\sum_{i=0}\sum_{j=1}e_{ij}(hN)^i(hN\omega-\{hN\omega\})^j$$
$$=P_t(\{hN\omega\})h^t+P_{t-1}(\{hN\omega\})h^{t-1}+\cdots+P_0(\{hN\omega\}),$$

where $\{x\}$ denotes the fractional part of x, P_i are polynomials, $P_t\neq 0$ and at least one of $P_t, \cdots, P_0$ is not constant. Let t_0 be the largest integer such that p_{t_0} is not constant. Comparing (13) and (14), we see that $s=t$, $a_i=P_i$ for $i=t_0+1, \cdots, t$ and

$$a_{t_0}=\lim_{h\to\infty}P_{t_0}(\{hN\omega\}).$$

This is a contradiction, since $\{\{hN\omega\}\}_{h=1}^{\infty}$ is dense in the interval $[0, 1)$.

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