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A NOTE ON THE PROJECTIVE NORMALITY OF SPECIAL LINE BUNDLES ON ABELIAN VARIETIES

By

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

Introduction.

Let L be an ample line bundle on an abelian variety A of dimension g defined over an algebraically closed field k. It is well known that $L^{\otimes 2}$ is base point free and $L^{\otimes 3}$ is very ample and projectively normal. Moreover we know that

$$\Gamma(A, L^{\otimes a})\otimes\Gamma(A, L^{\otimes b}) \longrightarrow \Gamma(A, L^{\otimes a+b})$$

is surjective if $a\geqq 2$ and $b\geqq 3$ (Koizumi [3], Sekiguchi [8], [9]). But in the case of $a=b=2$, this map is not surjective in general. In this paper we determine the condition of projective normality of $L^{\otimes 2}$ for some ample line bundle L. Our result is as follows.

THEOREM. *If L is a symmetric ample line bundle of separable type, $l(A, L)$ is odd and assume that* char$(k)\neq 2$, *then $L^{\otimes 2}$ is projectively normal if and only if $Bs|L|\cap A[2]=\phi$.*

In §1 we prove the above theorem for abelian varieties defined over $\boldsymbol{C}$. In §2 we give the Mumford's theory of a theta group (Mumford [4], [5]). In §3 we prove the above theorem in general by the theory in §2.

Notations.

char(k): The characteristic of a field k

f^*: The pull back defined by a morphism f

$\underline{L}$: The invertible sheaf associated to a line bundle L

$\mathcal{O}_A$: The invertible sheaf of a variety A

(L^g): The self intersection number

$|L|$: The set of all effective Cartier divisors which define a line bundle L

$Bs|L|$: The set defined by $\bigcap_{D\in|L|} D$

$\Gamma(A, L)$: The set of global sections of a line bundle L

$l(A, L)$: The dimension of $\Gamma(A, L)$ as a vector space

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T_x: The translation morphism on an abelian variety A defined by $T_x(y)=x+y$ where x and y are elemets of A

$K(L)$: The subgroup of an abelian variety A defined by $\{x\in A;\ T_x^*L\cong L\}$ where L is a line bundle on A

$A[n]$: The set of all points of order n of an abelian variety A

$\boldsymbol{Z}$: The ring of integers

$\boldsymbol{R}$: The field of real numbers

$\boldsymbol{C}$: The field of complex numbers

§ 1. The C case.

First we recall a definition of projective normality.

DEFINITION. *Let M be an ample line bundle on an abelian variety A. We call that M is projectively normal if*

$$\Gamma(A, M)^{\otimes n} \longrightarrow \Gamma(A, M^{\otimes n})$$

is surjective for every $n\geqq 1$.

Next we define a theta function defined on $\boldsymbol{C}^g$.

DEFINITION. *Let m', m'' be elements of $\boldsymbol{R}^g$ and let τ be an element of a Siegel upper half space H_g. We define $\theta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)$ by*

$$\theta\begin{bmatrix} m' \\ m'' \end{bmatrix}(\tau, z)=\sum_{\zeta\in \boldsymbol{Z}} \boldsymbol{e}((1/2)^t(\zeta+m')\tau(\zeta+m')+{}^t(\zeta+m')(z+m''))$$

where $\boldsymbol{e}(x)$ means $e^{2\pi\sqrt{-1}x}$ and z is contained in $\boldsymbol{C}^g$.

Let $d_1, \cdots, d_g$ be positive integers with $d_1|\cdots|d_g$. We define an integral matrix e by

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{bmatrix}.$$

For an element τ of H_g we define an abelian variety A by $\boldsymbol{C}^g/\langle\tau, e\rangle$ where $\langle\tau, e\rangle$ is a lattice subgroup of $\boldsymbol{C}^g$ defined by $\tau\boldsymbol{Z}^g+e\boldsymbol{Z}^g$. Let $\mathcal{A}$ be a Riemann form on $\langle\tau, e\rangle$ defined by

$$\mathcal{A}(\tau x+ey, \tau x'+ey')={}^t xey'-{}^t x'ey$$

where x, x', y, y' are elements of $\boldsymbol{Z}^g$. It is well known that this $\mathcal{A}$ defines an algebraic equivalence class of line bundles on A. Now we take a line bundle L on A satisfying that L is symmetric and the global sections of L are generated by $\theta\begin{bmatrix}\eta\\0\end{bmatrix}(\tau, z)$ where η runs over a complete system of representative of $(1/d_1)\boldsymbol{Z}/\boldsymbol{Z} \oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$.

LEMMA 1. *The basis of* $\Gamma(A, L^{\otimes 2^n})$ *is given by* $\theta\begin{bmatrix}\eta\\0\end{bmatrix}(2^n\tau, 2^n z)$ *where* η *runs over a complete system of representative of* $(1/2^n d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/2^n d_g)\boldsymbol{Z}/\boldsymbol{Z}$ $(n=1, 2, \cdots)$. *Moreover* $\Gamma(A, L^{\otimes 4})$ *is generated by* $\theta\begin{bmatrix}\xi\\ \sigma\end{bmatrix}(\tau, 2z)$ *where* ξ *runs over a complete system of representative of* $(1/2d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/2d_g)\boldsymbol{Z}/\boldsymbol{Z}$ *and* σ *runs over a complete system of representative of* $((1/2)\boldsymbol{Z}/\boldsymbol{Z})^g$.

PROOF. This is well known fact (cf. Igusa [2], p. 72, Theorem 4, and p. 84, Theorem 6).

LEMMA 2 (*Multiplication formula*). *If* $\eta', \eta'', \xi', \xi''$ *are contained in* $\boldsymbol{R}^g$ *and* τ *is contained in* H_g, *then*

$$\theta\begin{bmatrix}\eta'\\ \eta''\end{bmatrix}(\tau, z)\theta\begin{bmatrix}\xi'\\ \xi''\end{bmatrix}(\tau, z)=(1/2^g)\sum_{a''\in((1/2)\boldsymbol{Z}/\boldsymbol{Z})^g} e(-2\,{}^t\eta' a'')\cdot\theta\begin{bmatrix}\eta'+\xi'\\ (\eta''+\xi'')/2+a''\end{bmatrix}(\tau/2, (z_1+z_2)/2)\cdot\theta\begin{bmatrix}\eta'-\xi'\\ (\eta''-\xi'')/2+a''\end{bmatrix}(\tau/2, (z_1-z_2)/2)$$

where z_1 *and* z_2 *are contained in* $\boldsymbol{C}^g$.

PROOF. This is also well known fact (cf. Igusa [2], p. 139, Theorem 2).

LEMMA 3. *If* η, η' *are elements of* $(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$, d_g *is odd and* $\varepsilon, \varepsilon'$ *are contained in* $\boldsymbol{Z}^g$, *then*

$$\sum_{\sigma\in(\boldsymbol{Z}/2\boldsymbol{Z})^g}(-1)^{{}^t\sigma e\varepsilon'}\theta\begin{bmatrix}\eta+(\sigma/2)\\ 0\end{bmatrix}(2\tau, 2z)\theta\begin{bmatrix}\eta'+(\sigma+\varepsilon)/2\\ 0\end{bmatrix}(2\tau, 2z)$$
$$=\theta\begin{bmatrix}\eta+\eta'+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 2z)\theta\begin{bmatrix}\eta-\eta'+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0).$$

PROOF. By lemma 2, we obtain

$$\theta\begin{bmatrix}\eta+(\sigma/2)\\ 0\end{bmatrix}(2\tau, 2z)\theta\begin{bmatrix}\eta'+(\sigma+\varepsilon)/2\\ 0\end{bmatrix}(2\tau, 2z)$$
$$=(1/2^g)\sum_{a''\in((1/2)Z/Z)^g} e(-2^t(\eta+(\sigma/2))a'')$$
$$\cdot\theta\begin{bmatrix}\eta+\eta'+(\varepsilon/2)\\ a''\end{bmatrix}(\tau, 2z)\theta\begin{bmatrix}\eta-\eta'+(\varepsilon/2)\\ a''\end{bmatrix}(\tau, 0).$$

Hence

$$\sum_{\sigma\in(Z/2Z)^g}(-1)^{t\sigma e\varepsilon'}\theta\begin{bmatrix}\eta+(\sigma/2)\\ 0\end{bmatrix}(2\tau, 2z)\theta\begin{bmatrix}\eta'+(\sigma+\varepsilon)/2\\ 0\end{bmatrix}(2\tau, 2z)$$
$$=(1/2^g)\sum_{a''\in((1/2)Z/Z)^g} e(-2^t\eta a'')\sum_{\sigma\in(Z/2Z)^g}(-1)^{t\sigma(e\varepsilon'-2a'')}$$
$$\cdot\theta\begin{bmatrix}\eta+\eta'+(\varepsilon/2)\\ a''\end{bmatrix}(\tau, 2z)\theta\begin{bmatrix}\eta-\eta'+(\varepsilon/2)\\ a''\end{bmatrix}(\tau, 0)$$
$$=e({}^t\eta e\varepsilon')\theta\begin{bmatrix}\eta+\eta'+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 2z)\theta\begin{bmatrix}\eta-\eta'+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0)$$
$$=\theta\begin{bmatrix}\eta+\eta'+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 2z)\theta\begin{bmatrix}\eta-\eta'+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0).$$

Therefore we obtain this lemma.

LEMMA 4. *If M is an ample line boundle on an abelian variety A, then*

$$\Gamma(A, M^{\otimes a})\otimes\Gamma(A, M^{\otimes b})\longrightarrow\Gamma(A, M^{\otimes a+b})$$

is surjective for $a\geqq 2$ and $b\geqq 3$.

PROOF. See Koizumi [3] or Sekiguchi [8], [9].

LEMMA 5. *Under the notation of lemma 3, if there exists some $\eta_0\in(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$ with $\theta\begin{bmatrix}\eta_0+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0)\neq 0$, then $\theta\begin{bmatrix}\eta+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 2z)$ is in the image of $\Gamma(A, L^{\otimes 2})^{\otimes 2}\to\Gamma(A, L^{\otimes 4})$ for every $\eta\in(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$.*

PROOF. Let η_1 be an element of $(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$. In this case, we obtain that

$$\theta\begin{bmatrix}2\eta_1+\eta_0+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 2z)\theta\begin{bmatrix}\eta_0+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0)$$

is contained in the image of $\Gamma(A, L^{\otimes 2})^{\otimes 2}\to\Gamma(A, L^{\otimes 4})$ by lemma 3. Hence $\theta\begin{bmatrix}2\eta_1+\eta_0+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 2z)$ is contained in the image of $\Gamma(A, L^{\otimes 2})^{\otimes 2}\to\Gamma(A, L^{\otimes 4})$. As

d_g is odd, therefore the set $\{2\eta_1+\eta_0\,;\ \eta_1\in(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}\}$ is equal to

$$(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}.$$

Therefore we obtain this lemma.

LEMMA 6. *Under the assumption of lemma* 3, *the following conditions are equivalent:*

a) *For every* $\varepsilon, \varepsilon'\in\boldsymbol{Z}^g$, *there exists some* $\eta\in(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$ *with* $\theta\begin{bmatrix}\eta+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0)\neq 0$;

b) $Bs|L|\cap A[2]=\emptyset$.

PROOF. As

$$\theta\begin{bmatrix}m'+\xi'\\ m''+\xi''\end{bmatrix}(\tau, z)=\boldsymbol{e}((1/2)^t\xi'\tau\xi'+{}^t\xi'(z+\xi''+m''))\theta\begin{bmatrix}m'\\ m''\end{bmatrix}(\tau, z+\tau\xi'+\xi_{\prime\prime})$$

(cf. Igusa [2], p. 50, (θ.3)), therefore the condition $\theta\begin{bmatrix}\eta+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0)\neq 0$ is equivalent to $\theta\begin{bmatrix}\eta\\ 0\end{bmatrix}(\tau, (\tau\varepsilon+e\varepsilon')/2)\neq 0$. Hence this lemma is clear because $A=\boldsymbol{C}^g/\langle\tau, e\rangle$.

THEOREM. *If* $l(A, L)$ *is odd, then* $L^{\otimes 2}$ *is projectively normal if and only if* $Bs|L|\cap A[2]=\phi$.

PROOF. By lemma 1, a basis of $\Gamma(A, L^{\otimes 4})$ consists of $\theta\begin{bmatrix}\eta\\ \sigma\end{bmatrix}(\tau, 2z)$ where η runs over a complete system of representative of $(12d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/2d_g)\boldsymbol{Z}/\boldsymbol{Z}$ and σ runs over a complete system of representative of $((1/2)\boldsymbol{Z}/\boldsymbol{Z})^g$. Hence $\Gamma(A, L^{\otimes 2})^{\otimes 2}\rightarrow\Gamma(A, L^{\otimes 4})$ is surjective if and only if for every $\varepsilon, \varepsilon'\in\boldsymbol{Z}^g$, there exists some $\eta_0\in(1/d_1)\boldsymbol{Z}/\boldsymbol{Z}\oplus\cdots\oplus(1/d_g)\boldsymbol{Z}/\boldsymbol{Z}$ such that $\theta\begin{bmatrix}\eta_0+(\varepsilon/2)\\ e\varepsilon'/2\end{bmatrix}(\tau, 0)\neq 0$ by lemma 5. Hence we obtain this theorem by lemma 4 and lemma 6.

§2. Review of a theta group.

In this section we recall the Mumford's theory for a theta group (cf. Mumford [4], [5]). Let A be an abelian variety of dimension g defined over an algebraically closed field k with char$(k)\neq 2$. We fix these notations.

DEFINITION. *Let* M *be an ample line bundle on* A. *We call that* M *is of separable type if* char$(k)\nmid l(A, M)$.

DEFINITION. *Let M be an ample line bundle on A and be of separable type. We define a theta group $G(M)$ by*

$$\{(x, \phi);\ x\in K(M) \text{ and } \phi: M \xrightarrow{\sim} T_x^*M\}.$$

This $G(M)$ is a group by the following way:

$$(x, \phi)\cdot(y, \rho)=(x+y,\ T_x^*\phi\cdot\rho).$$

It is well known that $K(M)$ is isomorphic to the following abelian group via Weil pairing:

$$K(M)=\boldsymbol{Z}/d_1\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}/d_g\boldsymbol{Z}\oplus(\widehat{\boldsymbol{Z}/d_1\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}/d_g\boldsymbol{Z}})$$

where $d_1|\cdots|d_g$ and $\hat{G}$ means $\mathrm{Hom}(G, k^*)$ for a group G. Here we denote by $k^*=k-\{0\}$. In this situation, we set $\delta_M=(d_1, \cdots, d_g)$ and put $H(\delta_M)=\boldsymbol{Z}/d_1\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}/d_g\boldsymbol{Z}$. We define a Heisenberg group $G(\delta_M)$.

DEFINITION. *In the above notations, we define a Heisenberg group $G(\delta_M)$ by*

$$G(\delta_M)=k^*\times H(\delta_M)\times H(\hat{\delta}_M).$$

This $G(\delta_M)$ is a group by the following way:

$$(t, (x, m))\cdot(t', (x', m'))=(tt'm'(x), (x+x', m+m'))$$

where $x, x'\in H)\delta_M)$, $m, m'\in H(\hat{\delta}_M)$ and $t, t'\in k^*$. The following theorem is foundamental.

THEOREM. *In the above notations, the following two horizontal sequences are exact and isomorphic:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^* & \longrightarrow & G(M) & \longrightarrow & K(M) & \longrightarrow & 0 \\ & & \| & \circlearrowright & \downarrow\wr & \circlearrowright & \downarrow\wr & & \\ 0 & \longrightarrow & k^* & \longrightarrow & G(\delta_M) & \longrightarrow & H(\delta_M)\times H(\hat{\delta}_M) & \longrightarrow & 0. \end{array}$$

PROOF. See Mumford [4], p. 294, Corollary of Th. 1.

DEFINITION. *Let $z=(x, \phi)$ be an element of $G(M)$. We define a map U_z as follows:*

$$U_z: \Gamma(A, M) \xrightarrow{\Gamma(\phi)} \Gamma(A, T_x^*M) \xrightarrow{T_{-x}^*} \Gamma(A, M).$$

It is clear that $\Gamma(A, M)$ is a $G(M)$-module by this way. Next we define a vector space $V(\delta_M)$ and its $G(\delta_M)$-module structure.

DEFINITION. *The vector space* $V(\delta_M)$ *is defined as follows:*

$$V(\delta_M)=\text{the set of all maps from } H(\delta_M) \text{ to } k.$$

Let $(t, (x, m))$ *be an element of* $G(\delta_M)$. *We define an automorphism* $U_{(t,(x,m))}$ *of* $V(\delta_M)$ *as follows:*

$$U_{(t,(x,m))}(f)(y)=tm(y)f(x+y)$$

where $f\in V(\delta_M)$ *and* $y\in H(\delta_M)$.

The following theorem is also foundamental.

THEOREM. *If* $\alpha: G(M)\simeq G(\delta_M)$ *is an isomorphism given in the above theorem, then we obtain an isomorphism*

$$\Gamma(A, M)\xrightarrow{\sim} V(\delta_M)$$

as $G(M)\cong G(\delta_M)$*-modules.*

PROOF. See Mumford [4], p. 295, proposition 3, and p. 297, theorem 2.

DEFINITION. *Let* x *be an element of* $H(\delta_M)$. *We define* $\delta_x\in V(\delta_M)$ *by* $\delta_x(y)=1$ *if* $y=x$ *and* $\delta_x(y)=0$ *if* $y\neq x$.

It is clear that $U_{(t,(x,m))}(\delta_u)=tm(u-x)\delta_{u-x}$.

§ 3. The general case.

Let L be an ample line bundle on an abelian variety A of dimension g. Throughout of this section, we assume that L is of separable type and $l(A, L)$ is odd and L is symmetric. We fix an isomorphism $G(L^{\otimes 4})\cong G(4\delta_L)$ and identify two vector spaces $\Gamma(A, L^{\otimes 4})$ and $V(4\delta_L)$ by the isomorphism in § 2.

LEMMA 1. *Let* f *be an element of* $V(4\delta_L)\cong\Gamma(A, L^{\otimes 4})$ *defined by* $f=\sum_{u\in H(4\delta_L) \text{ and } 2u=0}\delta_u$. *Then* f *is in the image of* $2_A^*: \Gamma(A, L)\to\Gamma(A, L^{\otimes 4})$ *for some isomorphism* $2_A^*L\cong L^{\otimes 4}$ *where* $2_A(x)=2x$ $(x\in A)$.

PROOF. This lemma is trivial.

By the above lemma, we obtain $\theta\in\Gamma(A, L)$ with $2_A^*\theta=f$. We fix these notations through this section.

DEFINITION. *Let* x *be an element of* $H(4\delta_L)$ *and* σ *be an element of* $H(\hat{\delta}_L)$. *We define an element of* $\theta\begin{bmatrix} x \\ \sigma \end{bmatrix}$ *of* $V(4\delta_L)$ *by*

$$\theta\begin{bmatrix} x \\ \sigma \end{bmatrix}=U_z(2_A{}^*\theta)$$

where z *is an element of* $G(L^{\otimes 4})$ *corresponding to* $(1, (x, \sigma))$ *which is an element of* $G(4\delta_L)$.

LEMMA 2. *Let* x, u *be elements of* $H(4\delta_L)$ *and* σ, u^* *be elements of* $H(\hat{\delta}_L)$. *If* $2u=0$ *and* $2u^*=0$, *then* $\theta\begin{bmatrix} x+u \\ \sigma+u^* \end{bmatrix}=u^*(x)\theta\begin{bmatrix} x \\ \sigma \end{bmatrix}$.

PROOF. By the definition, we obtain that

$$\theta\begin{bmatrix} x \\ \sigma \end{bmatrix}=\sum_{2\zeta=0}\sigma(\zeta-x)\delta_{\zeta-x}.$$

Therefore

$$\begin{aligned}\theta\begin{bmatrix} x+u \\ \sigma+u^* \end{bmatrix}&=\sum_{2\zeta=0}(\sigma+u^*)(\zeta-x-u)\delta_{\zeta-x-u} \\ &=\sum_{2\zeta=0}(\sigma+u^*)(\zeta-x)\delta_{\zeta-x} \\ &=\sum_{2\zeta=0}u^*(\zeta-x)\sigma(\zeta-x)\delta_{\zeta-x}.\end{aligned}$$

As $2u^*=0$, hence $u^*(\zeta)=1$ for every $\zeta\in H(4\delta_L)$ with $2\zeta=0$. Therefore

$$\begin{aligned}\theta\begin{bmatrix} x+u \\ \sigma+u^* \end{bmatrix}&=u^*(-x)\sum_{2\zeta=0}\sigma(\zeta-x)\delta_{\zeta-x} \\ &=u^*(x)\theta\begin{bmatrix} x \\ \sigma \end{bmatrix}.\end{aligned}$$

So we obtain this lemma.

LEMMA 3. *The vector space* $\Gamma(A, L^{\otimes 4})$ *is sppaned by the elements* $\theta\begin{bmatrix} x \\ \sigma \end{bmatrix}$ *where* $x\in H(4\delta_L)$ *and* $\sigma\in(\widehat{\mathbf{Z}/4\mathbf{Z}})^g$ *which is regarded the subgroup of order of* $H(4\hat{\delta}_L)$.

PROOF. We regard that $H(\delta_L)$ and $(\mathbf{Z}/4\mathbf{Z})^g$ are the subgroups of $H(4\delta_L)$ in the canonical way. For every $\xi\in H(\delta_L)$, $\tau\in(\mathbf{Z}/4\mathbf{Z})^g$ and $\sigma\in(\widehat{\mathbf{Z}/4\mathbf{Z}})^g$, we obtain

$$\begin{aligned}\theta\begin{bmatrix} \xi+\tau \\ \sigma \end{bmatrix}&=\sum_{2\zeta=0}\sigma(\zeta-\xi-\tau)\delta_{\zeta-\xi-\tau} \\ &=\sum_{2\zeta=0}\sigma(\zeta-\tau)\delta_{\zeta-\xi-\tau}\end{aligned}$$

because as $l(A, L)$ is odd, $\sigma(\xi)=1$. Therefore

$$\sum_{\sigma\in(\widehat{Z/4Z})^g}\sigma(\tau)\theta\begin{bmatrix}\xi+\tau\\ \sigma\end{bmatrix}=\sum_{\sigma\in(\widehat{Z/4Z})^g}\sum_{2\zeta=0}\sigma(\tau)\sigma(\zeta-\tau)\delta_{\zeta-\xi-\tau}$$
$$=\sum_{2\zeta=0}\Big(\sum_{\sigma\in(\widehat{Z/4Z})^g}\sigma(\zeta)\Big)\delta_{\zeta-\xi-\tau}$$
$$=4^g\delta_{\zeta-\xi-\tau}.$$

Therefore we obtain this lemma.

Let x be a closed point of A and M be an ample line bundle on A of separable type. We define $M(x)$ cy

$$M(x)=\underline{M}_x\underset{\mathcal{O}_{A,x}}{\otimes}k(x)$$

where $\underline{M}_x$ and $\mathcal{O}_{A,x}$ are the stalk of $\underline{M}$ and $\mathcal{O}_A$ at x respectively, and $k(x)$ is a residue field of $\mathcal{O}_{A,x}$. It is clear that $M(x)\cong k$. We choose an isomorphism $\lambda_0: M(0)\xrightarrow{\sim}k$. We fix an isomorphism $G(M)\cong G(\delta_M)$. For every $w\in K(M)$, we take $(w, \phi_w)\in G(M)$ which is corresponding to an element of $G(\delta_M)$ with a type $(1, (x, m))$ by the above isomorphism.

DEFINITION. *We defined $\lambda_w: M(w)\to k$ by*

$$\lambda_w: M(w)=(T_x{}^*M)(0)\xleftarrow[\phi_w(0)]{} M(0)\xrightarrow[\lambda_0]{} k$$

where $w\in K(M)$ and $\phi_w(0)$ is given by ϕ_w.

DEFINITION. *Under the above notations, we define q_w^M by*

$$q_w^M:\Gamma(A, M)\xrightarrow[\text{canonical map}]{} M(w)\xrightarrow[\lambda_w]{} k.$$

REMARK. *For any $z=(x, \phi)\in G(M)$ and any $s\in\Gamma(A, M)$, the conditions $q_w^M(U_z(s))=0$ and $q_{w+x}^M(s)=0$ are eqivalent.*

REMARK. *The condition that $q_w^M(s)=$ for every $s\in\Gamma(A, M)$ implies that w is contained in $Bs|M|$.*

REMARK. *If M is a symmetric ample line bundle on A, then the conditions $q_w^{M^{\otimes 4}}(2_A{}^*s)=0$ and $q_{2w}^M(s)=0$ are equivalent for any $s\in\Gamma(A, M)$.*

DEFINITION. *We define $q_{L^{\otimes 4}}(x)$ by*

$$q_{L^{\otimes 4}}(x)=q_0^{L^{\otimes 4}}(\delta_x)$$

where $x\in K(L^{\otimes 4})$ and $\delta_x\in V(4\delta_L)\cong\Gamma(A, L^{\otimes 4})$. Moreover we define $q\begin{bmatrix}x\\ \sigma\end{bmatrix}$ by

$$q\begin{bmatrix}x\\ \sigma\end{bmatrix}=q_0^{L^{\otimes 4}}\left(\theta\begin{bmatrix}x\\ \sigma\end{bmatrix}\right)$$

where $x \in H(4\delta_L)$ *and* $\sigma \in (\boldsymbol{Z}/4\boldsymbol{Z})^g$.

Now the isomorphism $G(L^{\otimes 4}) \cong G(4\delta_L)$ induces an isomorphism $G(L^{\otimes 2}) \cong G(2\delta_L)$; these isomorphisms define the symmetric theta structure for $(L^{\otimes 2}, L^{\otimes 4})$ (cf. Mumford [4], p. 317). We identify the two vector spaces $\Gamma(A, L^{\otimes 2})$ and $V(2\delta_L)$ by means of the isomorphism $G(L^{\otimes 2}) \cong G(2\delta_L)$.

LEMMA 4 (*Multiplication formula*). *If* δ_x *and* $\delta_{x'}$ *are elements of* $\Gamma(A, L^{\otimes 2})$, *then*

$$\delta_x \cdot \delta_{x'} = \sum_{\zeta \in H(4\delta_L) \text{ and } 2\zeta=0} q_{L^{\otimes 4}}(\underline{x} - \underline{x}' + \zeta)\delta_{\underline{x}+\underline{x}'+\zeta}$$

where $\cdot$ *is a canonical map* $\Gamma(A, L^{\otimes 2})^{\otimes 2} \to \Gamma(A, L^{\otimes 4})$ *and* $\underline{x}, \underline{x}' \in H(4\delta_L)$ *satisfying* $2\underline{x} = x$, $2\underline{x}' = x'$. *Here we regard* $H(2\delta_L)$ *as a subgroup of* $H(4\delta_L)$ *in the canonical way.*

PROOF. See Mumford [4], p. 330.

Let x, x' be elements of $H(\delta_L)$, and ξ, ξ' be elements of $(\boldsymbol{Z}/2\boldsymbol{Z})^g$. We take $\underline{x}, \underline{x}' \in H(\delta_L)$ and $\underline{\xi}, \underline{\xi}' \in (\boldsymbol{Z}/4\boldsymbol{Z})^g$ satisfying $2\underline{x} = x$, $2\underline{x}' = x'$, $2\underline{\xi} = \xi$ and $2\underline{\xi}' = \xi'$.

LEMMA 5. *Under the above notations,*

$$\delta_{x+\xi} \cdot \delta_{x'+\xi'} = (1/4^g) \sum_{\sigma \in (\boldsymbol{Z}/4\boldsymbol{Z})^g} \sigma(\xi) q\begin{bmatrix} -\underline{x} + \underline{x}' - \underline{\xi} + \underline{\xi}' \\ \sigma \end{bmatrix} \theta \begin{bmatrix} -\underline{x} - \underline{x}' - \underline{\xi} - \underline{\xi}' \\ \sigma \end{bmatrix}$$

PROOF. For $\delta_{x+\xi}$ and $\delta_{x'+\xi'} \in \Gamma(A, L^{\otimes 2})$, we obtain that

$$\begin{aligned}
\delta_{x+\xi} \cdot \delta_{x'+\xi'} &= \sum_{\zeta \in H(4\delta_L) \text{ and } 2\zeta=0} q_{L^{\otimes 4}}(\underline{x} - \underline{x}' + \underline{\xi} - \underline{\xi}' + \zeta)\delta_{\underline{x}+\underline{x}'+\underline{\xi}+\underline{\xi}'+\zeta} \\
&= \sum_{2\zeta=0} q_{L^{\otimes 4}}(\underline{x} - \underline{x}' + \underline{\xi} - \underline{\xi}' + \zeta)(1/4^g) \sum_{\sigma \in (\boldsymbol{Z}/4\boldsymbol{Z})^g} \sigma(-\underline{\xi} - \underline{\xi}' - \zeta) \\
&\quad \cdot \theta \begin{bmatrix} -\underline{x} - \underline{x}' - \underline{\xi} - \underline{\xi}' - \zeta \\ \sigma \end{bmatrix} \\
&= (1/4^g) \sum_{\sigma} \sum_{2\zeta=0} \sigma(-\underline{\xi} - \underline{\xi}' - \zeta) q_{L^{\otimes 4}}(\underline{x} - \underline{x}' + \underline{\xi} - \underline{\xi}' + \zeta) \\
&\quad \cdot \theta \begin{bmatrix} -\underline{x} - \underline{x}' - \underline{\xi} - \underline{\xi}' - \zeta \\ \sigma \end{bmatrix}.
\end{aligned}$$

On the other hand, $\theta\begin{bmatrix} x+u \\ \sigma+u^* \end{bmatrix} = u^*(x)\ \theta\begin{bmatrix} x \\ \sigma \end{bmatrix}$ for $2u=0$ and $2u^*=0$. Moreover in above situation, $\sigma(\underline{x}) = \sigma(\underline{x}') = 1$. Hence

$$\delta_{x+\xi}\cdot\delta_{x'+\xi'}=(1/4^g)\sum_\sigma\sigma(-2\xi)\theta\begin{bmatrix}-\underline{x}-\underline{x}'-\underline{\xi}-\underline{\xi}'\\ \sigma\end{bmatrix}\sum_\zeta\sigma(\underline{x}-\underline{x}'+\underline{\xi}-\underline{\xi}'+\zeta)$$

$$\cdot q_{L^{\otimes 4}}(\underline{x}-\underline{x}'+\underline{\xi}-\underline{\xi}'+\zeta)$$

$$=(1/4^g)\sum_\sigma\sigma(\xi)\theta\begin{bmatrix}-\underline{x}-\underline{x}'-\underline{\xi}-\underline{\xi}'\\ \sigma\end{bmatrix}$$

$$\cdot q_0^{L^{\otimes 4}}(\sum_\zeta\sigma(\underline{x}-\underline{x}'+\underline{\xi}-\underline{\xi}'+\zeta)\delta_{\underline{x}-\underline{x}'+\underline{\xi}-\underline{\xi}'+\zeta})$$

$$=(1/4^g)\sum_\sigma\sigma(\xi)\theta\begin{bmatrix}-\underline{x}-\underline{x}'-\underline{\xi}-\underline{\xi}'\\ \sigma\end{bmatrix}q\begin{bmatrix}-\underline{x}+\underline{x}'-\underline{\xi}-\underline{\xi}'\\ \sigma\end{bmatrix}$$

Therefore we obtain this lemma.

THEOREM. *Under the above notations,* $L^{\otimes 2}$ *is projectively normal if and only if* $Bs|L|\cap A[2]=\phi$.

PROOF. Replacing the lemmas for the theorem in § 1 by the above lemmas, the proof of the theorem in § 1 still works in general case.

COROLLARY. *If* M *is an ample line bundle and is of separable type on an abelian variety* A, *then* $Bs|M|=\phi$ *and* $l(A, M)=\text{odd}$ *imply that* $M^{\otimes 2}$ *is projectively normal.*

To conclude this section, we give an easy criterion for the base point freeness of a line bundle M on an abelian variety A. We assume that M is of separable type. Let $\alpha: G(M)\to G(\delta_M)$ be an isomorphism. As α induces $\bar{\alpha}: K(M)\to H(\delta_M)\oplus H(\hat{\delta}_M)$, we put $H(M)$ by $\bar{\alpha}^{-1}(H(\delta_M))$. Let B be an abelian variety defined by $A/H(M)$ and $\pi: A\to B$ the canonical morphism. In this situation, the line bundle M is given by $M\cong\pi^*N$ where N is a principal line bundle on B. In this notations, we obtain the following proposition.

PROPOSITION. $Bs|M|=\pi^{-1}(\bigcap_{x\in\pi(K(M))}T_x{}^*\theta)$ *where* $\theta\in|N|$.

PROOF. As there exists a canonical isomorphism

$$\Gamma(A, M)\cong\bigoplus_{x\in\pi(K(M))}\Gamma(B, T_x{}^*N)$$

therefore this proposition is clear.

The following proposition is also clear.

PROPOSITION. *Let* M *be as in above. If* $Bs|M|$ *is finite and* $(M^g)>(g\,!)^2$, *then* $Bs|M|=\phi$.

PROOF. If $Bs|M|\neq\phi$, then there is a point $q\in Bs|M|$. By the definition of $K(M)$, $q+K(M)$ is also contained in $Bs|M|$. As

$$\text{the cardinality of } Bs|M|\leqq(M^g),$$

hence

$$\text{the order of } K(M)=((M^g)/g\,!)^2\leqq(M^g).$$

Therefore we obtain $(M^g)\leqq(g\,!)^2$.

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