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ALMOST KÄHLER STRUCTURES ON THE RIEMANNIAN PRODUCT OF A 3-DIMENSIONAL HYPERBOLIC SPACE AND A REAL LINE

By

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1. Introduction.

An almost Hermitian manifold $M=(M,J,g)$ is called an almost Kähler manifold if the Kähler form is closed (or equivalently $\mathfrak{S}_{x,y,z} g \;\; \nabla_x J \; Y, Z \;\; = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{S}$ and $\mathfrak{X}(M)$ denotes the cyclic sum and the Lie algebra of all differentiable vector fields on M respectively). A Kähler manifold, which is defined by $\nabla J = 0$, is necessarily an almost Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([1]):

CONJECTURE. *A compact almost Kähler Einstein manifold is a Kähler manifold.*

The second author has proved that the above conjecture is true for the case where the scalar curvature is nonnegative ([4]). However, the above conjecture is still open in the case where the scalar curvature is negative. Recently, the authors proved that a $2n(\geq 4)$-dimensional hyperbolic space H^{2n} cannot admits (compatible) almost Kähler structure ([3]).

In the present paper, we consider about (compatible) almost Kähler structures on the Riemannian product $H^3 \times R$ of a 3-dimensional hyperbolic space H^3 and a real line R. We construct an example of strictly almost Kähler structure (J, g) on the Riemannian product $H^3 \times R$ and determine the automorphism group of the almost Kähler manifold $(H^3 \times R, J, g)$. To our knowledge, this is the first example of strictly almost Kähler symmetric space. Moreover, we prove that the Riemannian product $H^3 \times R$ provided with a (compatible) almost Kähler structure (J, g) cannot be a universal (almost Hermitian) covering of any compact almost Kähler manifold (Theorem 2 in

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section 3).

2. Preliminaries.

Let $\boldsymbol{H}^3$ be a 3-dimensional hyperbolic space of constant sectional curvature -1. Then, the Riemannian product $\boldsymbol{H}^3 \times \boldsymbol{R}$ can be regarded as a Riemanniam manifold $(\boldsymbol{R}^4_+, g)$ equipped with the Riemannian metric g defined by

$$g = \frac{1}{x_1^2}\sum_{i=1}^{3} dx_i \otimes dx_i + dx_4 \otimes dx_4,$$

where $\boldsymbol{R}^4_+ = \{(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4 \mid x_1 > 0\}$.

We put $X_i = x_1(\partial / \partial x_i), i =$ 1, 2, 3, and $X_4 = \partial / \partial x_4$. Then $\{X_1, X_2, X_3, X_4\}$ forms a global orthonormal frame field on $\boldsymbol{H}^3 \times \boldsymbol{R}$. Direct calculation implies

$$[X_1, X_i] = -[X_i, X_1] = X_i \tag{2.1}$$

for i = 2, 3, and are otherwise zero. We set

$$\nabla_{X_i} X_j = \sum_{k=1}^{4} \Gamma_{ijk} X_k,$$

for $1 \leq i, j \leq 4$, where ∇ denotes the Levi-Civita connection on $\boldsymbol{H}^3 \times \boldsymbol{R}$. Then, by (2.1), we have

$$\Gamma_{ii1} = -\Gamma_{i1i} = 1 \tag{2.2}$$

for i = 2, 3, and are otherwise zero.

Let (J, g) be an almost Hermitian structure on $\boldsymbol{H}^3 \times \boldsymbol{R}$. We put

$$JX_i = \sum_{j=1}^{4} J_{ij} X_j, \tag{2.3}$$

for $1 \leq i \leq 4$. Then we may easily observe that the 4×4 matrix (J_{ij}) is a skew-symmetric orthogonal matrix, i.e. the equalities

$$J_{ij} = -J_{ji}, \sum_{k=1}^{4} J_{ik} J_{jk} = \delta_{ij}$$

holds for $1 \leq i, j \leq 4$, , and, furthermore, that the matrix (J_{ij}) is of the form

$$\text{(I)} \quad \begin{pmatrix} 0 & J_{12} & J_{13} & J_{14} \\ -J_{12} & 0 & J_{14} & -J_{13} \\ -J_{13} & -J_{14} & 0 & J_{12} \\ -J_{14} & J_{13} & -J_{12} & 0 \end{pmatrix}$$

or

$$\text{(II)}\quad \begin{pmatrix} 0 & J_{12} & J_{13} & J_{14} \\ -J_{12} & 0 & -J_{14} & J_{13} \\ -J_{13} & J_{14} & 0 & -J_{12} \\ -J_{14} & -J_{13} & J_{12} & 0 \end{pmatrix}$$

with $J_{12}^2+J_{13}^2+J_{14}^2=1$.

3. An example of strictly almost Kähler structure on $H^3 \times R$.

The aim of this section is to construct an example of a strictly almost Kähler structure on the Riemannian product $(H^3 \times R, g)$ and to show Theorem 2.

We assume that (J,g) is an almost Kähler structure on the Riemannian product $(H^3 \times R, g)$. Then, the almost Kähler condition $\mathfrak{S}_{i,j,k} g((\nabla_{X_i} J)X_j, X_k)=0$ and (2.2) yields the following system of first order partial differential equations:

$$\text{(3.1)}\quad \begin{cases} X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2J_{23} = 0, \\ X_1 J_{24} - X_2 J_{14} + X_4 J_{12} - J_{24} = 0, \\ X_1 J_{34} - X_3 J_{14} + X_4 J_{13} - J_{34} = 0, \\ X_2 J_{34} - X_3 J_{24} + X_4 J_{23} = 0. \end{cases}$$

We may regard the triple (J_{12}, J_{13}, J_{14}) as a unit vector in the 3-dimensional Euclidean space R^3. First of all, we may observe that the unit vector (J_{12}, J_{13}, J_{14}) has the following property.

PROPOSITION 1. *The vector* (J_{12}, J_{13}, J_{14}) *varies with the variable* x_4 *on an open subdt of* $H^3 \times R$.

PROOF. We assume that the vector (J_{12}, J_{13}, J_{14}) is independent on the variable x_4. Then, the system of partial differential equations (3.1) reduces to the following:

$$\text{(3.2)}\quad \begin{cases} X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2J_{23} = 0, \\ X_1 J_{24} - X_2 J_{14} - J_{24} = 0, \\ X_1 J_{34} - X_3 J_{14} - J_{34} = 0, \\ X_2 J_{34} - X_3 J_{24} = 0. \end{cases}$$

Now, we suppose that the matrix (J_{ij}) is of the form (I). Then, by (2.1), (2.2) and (3.2), we have

$$(3.3)\qquad \begin{cases} \Delta J_{12} - 2X_1J_{12} + 3J_{12} = 0, \\ \Delta J_{13} - 2X_1J_{13} + 3J_{13} = 0, \\ \Delta J_{14} - 2X_1J_{14} + 4J_{14} = 0. \end{cases}$$

From (3.3), we have

$$J_{12}\Delta J_{12} + J_{13}\Delta J_{13} + J_{14}\Delta J_{14} + 3 + J_{14}^2 = 0,$$

and hence

$$\sum_{i=1}^{4}\{(X_iJ_{12})^2 + (X_iJ_{13})^2 + (X_iJ_{14})^2\} = 3 + J_{14}^2,$$

since $J_{12}^2 + J_{13}^2 + J_{14}^2 = 1$, and we conclude that

$$(3.4)\qquad \sum_{i,j,k=1}^{4}(X_iJ_{jk})^2 = 4(3 + J_{14}^2).$$

Next, from the equality above, we have

$$\sum_{i,j,k=1}^{4}(X_iJ_{jk})X_lX_iJ_{jk} = 4J_{14}X_lJ_{14},$$

for each X_l. Thus, by a direct calculation, we obtain

$$\begin{aligned}
(3.5)\qquad & \sum_{l,i,j,k=1}^{4}(X_lX_iJ_{jk})^2 \\
&= 4\sum_l (X_lJ_{14})^2 + 4J_{14}\sum_l X_lX_lJ_{14} - \sum_{l,i,j,k}(X_iJ_{jk})X_lX_lX_iJ_{jk} \\
&= 4\sum_l (X_lJ_{14})^2 + 4J_{14}(\Delta J_{14} + 2X_1J_{14}) \\
&-\Big\{\sum_{l,i,j,k}(X_iJ_{jk})X_iX_lX_lJ_{jk} - 2\sum_{l,j,k}(X_1J_{jk})X_lX_lJ_{jk} \\
&+2\sum_{i,j,k}(X_iJ_{jk})X_1X_iJ_{jk} - \sum_{i\geqq 2}\sum_{j,k}(X_iJ_{jk})^2\Big\} \\
&= 4\sum_l (X_lJ_{14})^2 + 16J_{14}(X_1J_{14} - J_{14}) - \{32J_{14}X_1J_{14} \\
&-32(3 + J_{14}^2) - 4\sum_i (X_iJ_{14})^2 - 12\sum_i (X_1J_{1i})^2\} \\
&= 8\sum_i (X_iJ_{14})^2 + 12\sum_i (X_1J_{1i})^2 - 16J_{14}X_1J_{14} + 16J_{14}^2 + 96.
\end{aligned}$$

From (3.4) and (3.5), we find that $\sum_{i,j,k}(X_iJ_{jk})^2$ and $\sum_{i,j,k,l}(X_lX_iJ_{jk})^2$ are both bounded. By applying the similar argument in [3] along x_1-curve, we can deduce a contradiction. More precisely, let γ_1 be any integral curve of X_1. Then, we obtain

$$\lim_{x_1\to\infty} X_1 J_{ij} = 0 \quad (1 \le i, j \le 4). \tag{3.6}$$

along the geodesic γ_1 (see [3]). We denote by $\overline{\varphi}_a (a=2,3)$ isometries of $\boldsymbol{H}^3$ such that $(\overline{\varphi}_a)_* X_1$ is orthogonal to X_1 and, $(\overline{\varphi}_2)_* X_1$ and $(\overline{\varphi}_3)_* X_1$ are orthogonal to each other along γ_1. Let $\varphi_a(x_1,x_2,x_3,x_4) = (\overline{\varphi}_a(x_1,x_2,x_3),x_4)(a=2,3)$ be the naturally induced isometries of $\boldsymbol{H}^3 \times \boldsymbol{R}$, and we define almost complex structures $J_{(a)}(a=2,3)$on $\boldsymbol{H}^3 \times \boldsymbol{R}$ by$J_{(a)} = (\varphi_a)_*^{-1} \circ J \circ (\varphi_a)_*$. Because J is independent on x_4, so are $J_{(a)}$. We may easily check that $(J_{(a)}, g)$ are almost Kähler structures on $\boldsymbol{H}^3 \times \boldsymbol{R}$. Thus, by similar argument as above, we obtain

$$\lim_{x_1\to\infty} X_1 J_{(a)ij} = 0 \quad (1 \le i, j \le 4, a=2,3). \tag{3.7}$$

along the geodesic γ_1. Moreover, by semi-Kähler condition $\Sigma_{a=1}^4 \nabla_a J_{aj} = 0$ $(j=1,2,3,4)$, we have

$$\begin{aligned}
&\sum_{i=1}^{4}\{(\nabla_i J_{12})^2 + (\nabla_i J_{13})^2 + (\nabla_i J_{14})^2\}\\
&= \sum_i \{(X_i J_{12})^2 + (X_i J_{13})^2 + (X_i J_{14})^2\} + 1 + J_{14}^2\\
&+2(J_{23}X_2J_{13} + J_{24}X_2J_{14} + J_{32}X_3J_{12} + J_{34}X_3J_{14})\\
&= 4 + 2J_{14}^2 - 2(J_{13}X_2J_{23} + J_{14}X_2J_{24} + J_{12}X_3J_{32} + J_{14}X_3J_{34})\\
&= 2 - 2(J_{13}\nabla_2 J_{23} + J_{14}\nabla_2 J_{24} + J_{12}\nabla_3 J_{32} + J_{14}\nabla_3 J_{34})\\
&= 2 + 2(J_{13}\nabla_1 J_{13} + J_{14}(\nabla_1 J_{14} + \nabla_3 J_{34}) + J_{12}\nabla_1 J_{12} + J_{14}(\nabla_1 J_{14} + \nabla_2 J_{24}))\\
&= 2,
\end{aligned}$$

and hence, we have

$$\sum_{i,j,k=1}^{4} (\nabla_i J_{jk})^2 = 8, \tag{3.8}$$

where $\nabla_i J_{jk} = g((\nabla_{X_i} J)X_j, X_k)$. From (3.6), (3.7) and (3.8), we can derive a contradiction (see [3]).

In the case where (J_{ij}) is of the form (II), we also arrive at a contradiction. Now, we write down an example of strictly almost Kähler structure (J,g) on $\boldsymbol{H}^3 \times \boldsymbol{R}$.

EXAMPLE. We define an almost complex structure J by

$$(J_{ij}) = \begin{pmatrix} 0 & \cos x_4 & \sin x_4 & 0 \\ -\cos x_4 & 0 & 0 & -\sin x_4 \\ -\sin x_4 & 0 & 0 & \cos x_4 \\ 0 & \sin x_4 & -\cos x_4 & 0 \end{pmatrix}$$

with respect to the orthonormal frame field $\{X_i\}_{i=1,2,3,4}$ defined in the preceding section. Then, it is easy to verify that (J,g) is a strictly almost Kähler structure on $\boldsymbol{H}^3 \times \boldsymbol{R}$.

In the rest of this section, we shall prove the following Theorem 2. First of all, we recall an integral formula on a 4-dimensional compact almost Kähler manifold. Let $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$be a 4-dimensional compact almost Kähler manifold. Then, we see that the square of the first Chern class $c_1(\overline{M})$ is given by the following formula (cf. [2], [5]):

$$c_1(\overline{M})^2 = \frac{1}{16\pi^2}\int_{\overline{M}}\{(\overline{\tau}^*)^2 - 2\|\overline{\rho}^{*sym}\|^2 + 2\|\overline{\rho}^{*skew}\|^2 - \frac{1}{4}(\overline{\tau}^* + \overline{\tau})\|\overline{\nabla}\overline{J}\|^2 + (\overline{\rho}, \overline{D})\}d\overline{M}, \tag{3.9}$$

where $\overline{\rho}^*, \overline{\tau}^*, \overline{\tau}, \overline{\nabla}, d\overline{M}$ denote the Ricci $*$-tensor, $*$-scalar curvature, scalar curvature, Levi-Civita connection on $\overline{M}$ and the volume element of $\overline{M}$ respectively, and $\overline{\rho}^{*sym}$ (resp. $\overline{\rho}^{*skew}$) the symmetric (resp. skew-symmetric) part of $\overline{\rho}^*$, and $(\overline{\rho}, \overline{D}) = \sum_{a,b,i,j=1}^{4} \overline{\rho}_{ab}(\overline{\nabla}_a\overline{J}_{ij})\overline{\nabla}_b\overline{J}_{ij}$. Here we put $\overline{\nabla}_a\overline{J}_{ij} = \overline{g}((\overline{\nabla}_{\overline{X}_a}\overline{J})\overline{X}_i, \overline{X}_j)$ and $\overline{\rho}_{ab} = \overline{\rho}(\overline{X}_a, \overline{X}_b)$ for a local orthonormal frame field $\{\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4\}$ on $\overline{M}$.

THEOREM 2. *Let (g, J) be a (compatible) almost Kähler structure on the Riemannian product $\boldsymbol{H}^3 \times \boldsymbol{R}$. Then, the almost Kähler manifold $(\boldsymbol{H}^3 \times \boldsymbol{R}, J, g)$ cannot be a universal (almost Hermitian) covering of any compact almost Kähler manifold.*

PROOF. Let (g,J) be a (compatible) almost Kähler structure on the Riemannian product $\boldsymbol{H}^3 \times \boldsymbol{R}$. We assume that there exists a compact almost Kähler manifold $(\overline{M}, \overline{J}, \overline{g})$ whose universal (almost Hermitian) covering is the almost Kähler manifold $(\boldsymbol{H}^3 \times \boldsymbol{R}, J, g)$. We denote by p the covering map from $\boldsymbol{H}^3 \times \boldsymbol{R}$ onto $\overline{M}$. For any point $\overline{p} \in \overline{M}$, we may choose a local orthonormal frame field $\{\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4\}$ near $\overline{p}$ in such a way that $p_*(X_i) = \overline{X}_i, (i = 1,2,3,4)$. We set $\overline{J}\overline{X}_i = \sum_{j=1}^{4} \overline{J}_{ij}\overline{X}_j (i = 1,2,3,4)$. for the proof, without loss of essentially, it is sufficient to consider the case where (J_{ij}) (and hence $(\overline{J}_{ij})$) is of the form (**I**). We may easily observe that $\sum_{i,j=1}^{4}(\overline{X}_4\overline{J}_{ij})^2$ gives rise to a differentiable function on $\overline{M}$. Since $\overline{M}$ is a locally product Riemannian manifold of 3-dimensional hyperbolic space and a real line, it follows that the Euler class $\chi(\overline{M})$ of $\overline{M}$ vanishes. Further, since $\overline{M}$ is conformally flat, it follows that the first Pontrjagin class $p_1(\overline{M})$ of $\overline{M}$ also vanishes. Thus, by the Wu's theorem ([6]), we have $c_1(\overline{M})^2 = 0$. On one hand, by direct calculation, from (3.9), we see that

$$c_1(\overline{M})^2 = \frac{1}{8\pi^2}\int_{\overline{M}}\left(\| \overline{\nabla} \overline{J} \|^2 - \sum_{i,j=1}^{4}\sum_{a=1}^{3}(\overline{\nabla}_a \overline{J}_{ij})^2\right)d\overline{M}$$

$$= \frac{1}{8\pi^2}\int_{\overline{M}} \sum_{i,j=1}^{4}(\overline{\nabla}_a \overline{J}_{ij})^2\, d\overline{M}.$$

Thus, it must follows that $\nabla_4 \overline{J}_{ij} = 0 (1 \le i, j \le 4)$ everywhere on $\overline{M}$, and hence, $\nabla_4 J_{ij} = X_4 J_{ij} = 0 (1 \le i, j \le 4)$ everywhere on M. But this contradicts Proposition 1, which completes the proof of theorem.

4. Automorphisms of the example of almost Kähler manifold $(\boldsymbol{H}^3 \times \boldsymbol{R}, J, g)$.

A differentiable transformation φ on an almost hermitian manifold (M, J, g) is called an automorphism if φ is a isometry and pseudo-holomorphic transformation, that is, φ satisfies

$$\varphi^* g = g \text{ and } \varphi_* \circ J = J \circ \varphi_*,$$

where φ_* denotes the differential map of φ and φ^* its dual map. We denote by $\mathrm{Aut}_M(J, g)$ the set of all automorphisms on (M, J, g). It is obvious that the set $\mathrm{Aut}_M(J, g)$ is a group under the composition of maps, and we call it the automorphism group on (M, J, g). In this section, we shall determine the automorphism group $\mathrm{Aut}_{\boldsymbol{H}^3\times \boldsymbol{R}}(J, g)$ of the example of strictly almost Kähler manifold $(\boldsymbol{H}^3 \times \boldsymbol{R}, J, g)$ constructed in the preceding section.

Let $\varphi \in \mathrm{Aut}_{\boldsymbol{H}^3\times \boldsymbol{R}}(J, g)$. We set $\varphi_*(X_i) = \sum_{j=1}^{4} \varphi_{ij} X_j$ for $i = 1,2,3,4$. Then, we see that 4×4 matrix (φ_{ij}) is of the following form

$$(4.1) \qquad \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } (\varphi_{ij})_{1\le i,j\le 3} \in SO(3),$$

$$(4.2)\qquad \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ with } -(\varphi_{ij})_{1\le i,j\le 3} \in SO(3),$$

since φ is an orientation-preserving isometry. We notice that $\varphi_{ij}(1\le i,j\le 3)$ are independent on the variable x_4. Since φ satisfies $\varphi_*^{-1}\circ J\circ\varphi_* = J$, we have, in particular,

$$(\varphi_*^{-1}\circ J\circ\varphi_*)X_1 = JX_1.$$

We now suppose that the matrix (φ_{ij}) is of the form (4.1). Then, by a direct calculation, we have

$$\begin{aligned}
&(\varphi_*^{-1}\circ J\circ\varphi_*)X_1 \\
&= \{(\varphi_{11}\varphi_{22}-\varphi_{12}\varphi_{21})J_{12}+(\varphi_{11}\varphi_{23}-\varphi_{13}\varphi_{21})J_{13}\}X_2 \\
&+\{(\varphi_{11}\varphi_{32}-\varphi_{12}\varphi_{31})J_{12}+(\varphi_{11}\varphi_{33}-\varphi_{13}\varphi_{31})J_{13}\}X_3 \\
&+(\varphi_{13}J_{12}-\varphi_{12}J_{13})X_4 \\
&= (\varphi_{33}\cos(x_4+c_4)-\varphi_{32}\sin(x_4+c_4))X_2 \\
&+(-\varphi_{23}\cos(x_4+c_4)+\varphi_{22}\sin(x_4+c_4))X_3 \\
&+(\varphi_{13}\cos(x_4+c_4)-\varphi_{12}\sin(x_4+c_4))X_4,
\end{aligned}$$

for some constant $c_4\in\boldsymbol{R}$, and hence

$$(4.3)\qquad \begin{pmatrix} \varphi_{33} & -\varphi_{32} & \varphi_{31} \\ -\varphi_{23} & \varphi_{22} & -\varphi_{21} \\ \varphi_{13} & -\varphi_{12} & \varphi_{11} \end{pmatrix}\begin{pmatrix} \cos(x_4+c_4) \\ \sin(x_4+c_4) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos x_4 \\ \sin x_4 \\ 0 \end{pmatrix}.$$

Similarly, we see that the condition $(\varphi_*^{-1}\circ J\circ\varphi_*)X_i = JX_i, (i=2,3,4)$ implies the same equality (4.3). From (4.3), it follows that

$$(\varphi_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos c_4 & \sin c_4 & 0 \\ 0 & -\sin c_4 & \cos c_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the automorphism $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$ satisfies the following system system of first order partial differential equations:

$$\frac{\partial\varphi_1}{\partial x_1} = \frac{1}{x_1}\varphi_1, \qquad \frac{\partial\varphi_1}{\partial x_2} = \frac{\partial\varphi_1}{\partial x_3} = \frac{\partial\varphi_1}{\partial x_4} = 0,$$

$$\frac{\partial\varphi_2}{\partial x_2}=\frac{1}{x_1}\varphi_1\cos c_4,\quad \frac{\partial\varphi_2}{\partial x_3}=-\frac{1}{x_1}\varphi_1\sin c_4,\quad \frac{\partial\varphi_2}{\partial x_1}=\frac{\partial\varphi_2}{\partial x_4}=0,$$

$$\frac{\partial\varphi_3}{\partial x_2}=\frac{1}{x_1}\varphi_1\sin c_4,\quad \frac{\partial\varphi_3}{\partial x_3}=\frac{1}{x_1}\varphi_1\cos c_4,\quad \frac{\partial\varphi_3}{\partial x_1}=\frac{\partial\varphi_3}{\partial x_4}=0,$$

$$\frac{\partial\varphi_4}{\partial x_4}=1,\qquad \frac{\partial\varphi_4}{\partial x_1}=\frac{\partial\varphi_4}{\partial x_2}=\frac{\partial\varphi_4}{\partial x_3}=0.$$

Solving this system of differential equations, we find that the automorphism φ can be express as the form

$$\varphi(x_1,x_2,x_3,x_4)=(e^{c_1}x_1,e^{c_1}((\cos c_4)x_2-(\sin c_4)x_3)+c_2,$$
$$e^{c_1}((\sin c_4)x_2+(\cos c_4)x_3)+c_3,x_4+c_4)$$

for $c_i\in\boldsymbol{R}, i=1,2,3,4$.

Next, we suppose that the matrix (φ_{ij}) is of the form (4.2). Then, in the same way, we have

$$\begin{pmatrix}-\varphi_{33} & \varphi_{32} & -\varphi_{31}\\ \varphi_{23} & -\varphi_{22} & \varphi_{21}\\ -\varphi_{13} & \varphi_{12} & -\varphi_{11}\end{pmatrix}\begin{pmatrix}\cos(-x_4+c_4)\\ \sin(-x_4+c_4)\\ 0\end{pmatrix}=\begin{pmatrix}\cos x_4\\ \sin x_4\\ 0\end{pmatrix},$$

which implies

$$(\varphi_{ij})=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos c_4 & \sin c_4 & 0\\ 0 & \sin c_4 & -\cos c_4 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$

and hence, we have

$$\varphi(x_1,x_2,x_3,x_4)=(e^{c_1}x_1,e^{c_1}((\cos c_4)x_2+(\sin c_4)x_3)+c_2,$$
$$e^{c_1}((\sin c_4)x_2-(\cos c_4)x_3)+c_3,-x_4+c_4)$$

for $c_i\in\boldsymbol{R}, i=1,2,3,4$.

We can summarize the above arguments as follows.

PROPOSITION 3. *The automorphism group* $\mathrm{Aut}_{\boldsymbol{H}^3\times\boldsymbol{R}}(J,g)$ *of the almost Kähler manifold* $(\boldsymbol{H}^3\times\boldsymbol{R},J,g)$ *is isomorphic to a solvable subgroup of affine transformation group* $GL(4,\boldsymbol{R})\times\boldsymbol{R}^4$ *(embedded in* $GL(5,\boldsymbol{R})$*), which consists of the elements*

$$
(4.4)\qquad \begin{pmatrix} e^{c_1} & 0 & 0 & 0 & 0 \\ 0 & e^{c_1}\cos c_4 & -e^{c_1}\sin c_4 & 0 & c_2 \\ 0 & e^{c_1}\sin c_4 & e^{c_1}\cos c_4 & 0 & c_3 \\ 0 & 0 & 0 & 1 & c_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

and

$$
(4.5)\qquad \begin{pmatrix} e^{c_1} & 0 & 0 & 0 & 0 \\ 0 & e^{c_1}\cos c_4 & e^{c_1}\sin c_4 & 0 & c_2 \\ 0 & e^{c_1}\sin c_4 & -e^{c_1}\cos c_4 & 0 & c_3 \\ 0 & 0 & 0 & -1 & c_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

where $c_i \in \boldsymbol{R}, i = 1,2,3,4.$.

From Proposition 3, we may easily see that the group $\mathrm{Aut}_{H^3\times R}(J,g)$ acts transitively on $\boldsymbol{H}^3 \times \boldsymbol{R}$ and that the identity component $\mathrm{Aut}_{H^3\times R}(J,g)_0$ of $\mathrm{Aut}_{H^3\times R}(J,g)$ is a subgroup consists of the elements of the form (4.4) and acts simply transitively on $\boldsymbol{H}^3 \times \boldsymbol{R}$. Taking account of Theorem 2, we see that there does not exist a discrete uniform subgroup of $\mathrm{Aut}_{H^3\times R}(J,g)_0$ (i.e. discrete subgroup Γ of $\mathrm{Aut}_{H^3\times R}(J,g)_0$ such that the orbit space $\Gamma \setminus \boldsymbol{H}^3 \times \boldsymbol{R}$ is compact).

REMARK 1. We may easily find that the system of differential equations $\nabla_{X_i} J = 0 (i = 1,2,3,4)$ has no solution, and thus the Riemannian product $(\boldsymbol{H}^3 \times \boldsymbol{R}, g)$ can not admit a (compatible) Kähler structure.

REMARK 2. Let ψ be an isometry on $(\boldsymbol{H}^3 \times \boldsymbol{R}, g)$ and (J,g) be an almost Kähler structure constructed in the preceding section. Then, almost Hermitian structure $(\psi(J), g)$ is also an almost Kähler structure on $(\boldsymbol{H}^3 \times \boldsymbol{R}, g)$, where $\psi(J)$ is defined by $\psi_*^{-1} \circ J \circ \psi_*$. The automorphism group $\mathrm{Aut}_{H^3\times R}(\psi(J), g)$ is determined by the automorphism group $\mathrm{Aut}_{H^3\times R}(J,g)$. Indeed, the map $\mathrm{Aut}_{H^3\times R}(J,g) \ni \varphi \mapsto \psi^{-1} \circ \varphi \circ \psi \in \mathrm{Aut}_{H^3\times R}(\psi(J), g)$ is an isomorphism.

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