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NON EXISTENCE OF GLOBAL SOLUTIONS OF PARABOLIC EQUATION IN CONICAL DOMAINS

By

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1. Introduction.

Let D be an unbounded domain in $\boldsymbol{R}^{N}$ and $T>0$. In this paper we study the initial-boundary value problem

$$\text{(P)}\qquad \begin{aligned} &u_{t}=\Delta u+|x|^{\sigma}u^{p} && \text{in } D\times(0, T),\\ &u(x, t)=0 && \text{on } \partial D\times(0, T),\\ &u(x, 0)=u_{0}(x) && \text{in } D, \end{aligned}$$

where $\sigma\geqq 0$, $p>1$, $u_{0}\geqq 0$, $\langle x\rangle^{\sigma/(p-1)}u_{0}$ ($\langle x\rangle=\sqrt{1+|x|^{2}}$) is continuous and bounded in $\bar{D}$ and $u_{0}=0$ on ∂D.

When $D=\boldsymbol{R}^{N}$ and $\sigma=0$, Fujita [1] and Weissler [2] proved that if $1<p\leqq 1+2/N$, there is no nontrivial nonnegative global solution of (P).

When D is a cone with vertex at the origin, that is $D=\{x\in\boldsymbol{R}^{N}\backslash 0;\, x/|x|\in\Omega\}$, where $\Omega\subset S^{N-1}$ is an open connected subset with smooth boundary, Levine and Meier, [3] and [4], proved that if $1<p<1+(2+\sigma)/(N+\gamma_{+})$ and $\sigma\geqq 0$, or $p=1+2/(N+\gamma_{+})$ and $\sigma=0$, there is no nontrivial nonnegative global solution of (P), where γ_{+} is the positive root of $\gamma(\gamma+N-2)=\omega_{1}$, and ω_{1} is the smallest Dirichlet eigenvalue for the Laplace-Beltrami operator on Ω.

In this paper we shall prove that there is no nontrivial global solution of (P) if $\sigma>0$ and $p=1+(2+\sigma)/(N+\gamma_{+})$ are valid. Moreover we can prove that when $D=\boldsymbol{R}^{N}$ there is no nontrivial global solution if $1+\sigma/(N-2)\leqq p\leqq 1+(2+\sigma)/N$ and $\sigma>0$.

DEFINITION 1.1. For $T>0$, $u=u(x, t)$ is called a solution of (P) in $(0, T)$, if

(A) u is continuous in $\bar{D}\times[0, T)$,

(B) u_{t}, $u_{x_{i}}$ and $u_{x_{i}x_{j}}$ ($i, j=1, \cdots, N$) are continuous in $D\times(0, T)$,

(C) $\|u(t)\|_{\sigma/(p-1)}$ is finite for each $t\in[0, T)$,

(D) u satisfies (P),

where $\|u(t)\|_{\sigma/(p-1)}:=\sup_{D}\langle x\rangle^{\sigma/(p-1)}|u(u, t)|$.

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Similarly, $\underline{u}$ is called a subsolution of (P) in $(0, T)$, if $\underline{u}$ satisfies (A), (B), (C) and

$$(\underline{\mathrm{P}}) \qquad \begin{array}{ll} \underline{u}_t \leqq \Delta \underline{u} + |x|^{\sigma} \underline{u}^p & \text{in } D \times (0, T), \\ \underline{u}(x, t) = 0 & \text{on } \partial D \times (0, T), \\ \underline{u}(x, 0) = u_0(x) & \text{in } D. \end{array}$$

DEFINITION 1.2. $\bar{T} := \sup\{T > 0 ; \|u(t)\|_{\sigma/(p-1)}$ is finite for $0 \leqq t < T\}$ is called the nontrivial existence time of u. If $\bar{T} = +\infty$, then u is called a global solution of (P).

REMARK. If $0 < T < \infty$ and $\|u(t)\|_{\sigma/(p-1)}$ is finite on $(0, T)$, the solution u can be extended beyond T (see Theorem 1.1).

We begin with stating the local existence theorem for (P).

THEOREM 1.1. *Let D be a cone in $\boldsymbol{R}^N$. Then for any nonnegative function u_0 in $C^0(\bar{D})$ satisfying $\|u_0\|_{\sigma/(p-1)} < \infty$ and $u_0 = 0$ on ∂D there is a solution $u(x, t)$ of* (P) *in $(0, t_0)$ such that $\|u(t)\|_{\sigma/(p-1)}$ is finite in $(0, t_0)$ where $t_0 > 0$ depends only on σ, p, N and $\|u_0\|_{\sigma/(p-1)}$.*

The main two theorem in this paper are the following.

THEOREM 1.2. *Let D be a cone in $\boldsymbol{R}^N$, $N \geqq 3$. If $u_0 \geqq 0$ and $u_0 \not\equiv 0$, $p = 1 + (2+\sigma)/(N+\gamma_+)$ and $0 < \sigma \leqq 2(N-2)/(\gamma_+ + 2)$, there is no global solution.*

THEOREM 1.3. *Let $D = \boldsymbol{R}^N$, $N \geqq 3$. If $u_0 \geqq 0$ and $u_0 \not\equiv 0$, $1 + \sigma/(N-2) \leqq p \leqq 1 + (2+\sigma)/N$ and $0 < \sigma \leqq N-2$, there is no global solution.*

To prove Theorem 1.2 and Theorem 1.3, we need the following estimate

$$v(x, t) < ((p-1)|x|^{\sigma} t)^{-1/(p-1)} \qquad \text{for } 0 < t < T,$$

where $T > 0$ is the maximum existence time of the solution of (P) and $v(x, t)$ is the solution of the heat equation with the same initial and boundary condition as (P).

The above inequality is true provided $0 < \sigma/(p-1) \leqq N-2$ (see Lemma 3.2).

2. Proof of Theorem 1.1.

Throughout this paper we take advantage of the following proposition proved by Protter and Weinberger [5] (Theorem 10, p.p. 183-184).

PROPOSITION 2.1. *Let D be an unbounded domain in $\boldsymbol{R}^N$. If u_t and $u_{x_i x_j}$ $(i, j=1, \cdots, N)$ are exist and continuous in $D\times(0, T)$ and $u=u(x, t)$ satisfies the following inequalities*

$$
(2.1)\qquad
\begin{aligned}
&u_t \leqq \Delta u + hu && \text{in } D\times(0, T),\\
&u(x, t)\leqq 0 && \text{on } \partial D\times(0, T),\\
&u(x, 0)\leqq 0 && \text{in } D,
\end{aligned}
$$

where $h=h(x, t)$ is bounded in $D\times[0, T)$. If there exist $c>0$ such that $\underline{\lim}_{r\to\infty} e^{-cr^2} \cdot \{\max_{|x|=r, 0\leq t\leq T} u(x, t)\} \leqq 0$, then $u(x, t)\leqq 0$ in $D\times(0, T)$.

REMARK. In the case $D=\boldsymbol{R}^N$, we can eliminate the boundary condition of u.

We introduce the Green's function $G(x, y; t)=G(r, \theta, \rho, \phi; t)$ $(r=|x|, \rho=|y|, \theta=x/|x|\in\Omega, \phi=y/|y|\in\Omega)$, for the linear heat equation in the cone.

Let $\{\psi_n(\theta)\}_{n=1}^{\infty}$ be a normalized orthogonal system for Δ_θ on Ω corresponding to the sequence $\{\omega_n\}$ of Dirichlet eigenvalues for this problem, especially we take $\psi_1>0$ in Ω and $\int_\Omega \psi_1(\theta)dS_\theta=c_1$.

Here

$$
\begin{aligned}
G(x, y, t)&=G(r, \theta, \rho, \phi; t)\\
&=\frac{1}{2t}(r\rho)^{-(N-2)/2}\exp\left(-\frac{\rho^2+r^2}{4t}\right)\sum_{n=1}^{\infty} I_{\nu_n}\left(\frac{r\rho}{2t}\right)\psi_n(\theta)\psi_n(\phi),
\end{aligned}
$$

where $\nu_n=\{((N-2)/2)^2+\omega_n\}^{1/2}$, and

$$
I_\nu(z)=\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{k!\,\Gamma(\nu+k+1)}\sim\begin{cases}(z/2)^{\nu}/\Gamma(\nu+1) & z\to 0^+\\ e^z/\sqrt{2\pi z} & z\to+\infty\end{cases}
$$

(see Watson [6]).

LEMMA 2.2. *Let D be a cone in $\boldsymbol{R}^N$. Assume that v_0 is a bounded continuous function in D and vanishes on ∂D. Then there exist a unique solution of the heat equation*

$$
(2.2)\qquad
\begin{aligned}
&v_t=\Delta v && \text{in } D\times(0, \infty),\\
&v(x, t)=0 && \text{on } \partial D\times(0, \infty),\\
&v(x, 0)=v_0(x) && \text{in } D,
\end{aligned}
$$

in $C(\bar{D}\times[0, \infty))\cap C^2(D\times(0, \infty))$, which has the form

$$
(2.3)\qquad v(x, t)=\int_D G(x, y; t)v_0(y)dy\,.
$$

Especially, if $v_0(x)\geqq 0$ in D, then $v(x, t)\geqq 0$ in $D\times(0, \infty)$.

LEMMA 2.3. *Let* $v=v(x, t)$ *be a solution of* (2.2) *and* $\alpha:=\max\{0, -(\sigma/2(p-1))\cdot(N-2-\sigma/(p-1))\}$. *If* $\|v\|_{\sigma/(p-1)}<\infty$, *then for* $0<t<\infty$

$$\|v(t)\|_{\sigma/(p-1)}\leqq\|v_0\|_{\sigma/(p-1)}\exp(\alpha t).$$

REMARK. Moreover if we take $0<t_0\leqq(\log 2)/\alpha$, then for $0<t<t_0$

$$\|v(t)\|_{\sigma/(p-1)}\leqq 2\|v_0\|_{\sigma/(p-1)}.$$

PROOF OF LEMMA 2.3. Let $w(x, t):=v(x, t)-\|v_0\|_{\sigma/(p-1)}\langle x\rangle^{-\sigma/(p-1)}\exp(\alpha t)$, then we have

$$\begin{aligned}\Delta w-w_t&=\left\{\alpha|x|^4+\left\{2\alpha+\frac{\sigma}{p-1}\left(N-2-\frac{\sigma}{p-1}\right)\right\}|x|^2+\frac{N\alpha}{p-1}+\alpha\right\}\\&\qquad\times\|v_0\|_{\sigma/(p-1)}\langle x\rangle^{-\sigma/(p-1)-4}\exp(\alpha t)\\&\geqq 0.\end{aligned}$$

Combining this with Proposition 2.1, we get $w(x, t)\leqq 0$ for $D\times[0, \infty)$. This shows Lemma 2.3.

PROOF OF THEOREM 1.1. Define $\alpha:=\max\{0, -(\sigma/2(p-1))(N-2-\sigma/(p-1))\}$ and $t_0:=\min\{(\log 2)/\alpha, 4^p(\|u_0\|_{\sigma/(p-1)})^{1-p}\}$.

First, we consider the following initial-boundary value problem

$$(\mathrm{P}_1)\qquad\begin{cases}\partial_t V_1=\Delta V_1+|x|^\sigma V_0^p & \text{in } D\times(0, t_0),\\ V_1(x, t)=0 & \text{on } \partial D\times(0, t_0),\\ V_1(x, 0)=u_0(x) & \text{in } D,\end{cases}$$

where V_0 is a solution of (2.2) with the initial condition $v_0=u_0$.

The solution of (P_1) is

$$V_1(x, t)=V_0(x, t)+\int_0^t\int_D G(x, y; t-\tau)|y|^\sigma V_0^p(y, \tau)dyd\tau$$

for $(x, t)\in D\times(0, t_0)$.

Since $V(x, t; \tau):=\int_D G(x, y; t-\tau)|y|^\sigma V_0^p(y, \tau)dy$ is a solution of (2.2) with the initial condition $v_0=|x|^\sigma V_0^p(x, \tau)$ for arbitrarily fixed $\tau\in(0, t)$, so we have $\|V(t; \tau)\|_{\sigma/(p-1)}\leqq 2(2\|u_0\|_{\sigma/(p-1)})^p$ for $0<\tau<t_0$ by using the above remark.

The solution V_1 of (P_1) satisfies

$$\|V_1(t)\|_{\sigma/(p-1)} \leqq \|V_0(t)\|_{\sigma/(p-1)} + \int_0^t \|V(t\,;\tau)\|_{\sigma/(p-1)} d\tau$$

$$\leqq 2\|u_0\|_{\sigma/(p-1)} + \int_0^t 2(2\|u_0\|_{\sigma/(p-1)})^p d\tau$$

$$\leqq 4\|u_0\|_{\sigma/(p-1)} \qquad \text{for } 0<t<t_0 .$$

Next we consider the following problem

$$(\mathrm{P}_{i+1}) \qquad \begin{cases} \partial_t V_{i+1}(x,t) = \Delta V_{i+1}(x,t) + |x|^\sigma V_i^p(x,t) & \text{in } D\times(0,t_0), \\ V_{i+1}(x,t)=0 & \text{on } \partial D\times(0,t_0), \\ V_{i+1}(x,0)=u_0(x) & \text{in } D, \end{cases}$$

where $i=1, 2, \cdots$.

Then

$$\|V_i(t)\|_{\sigma/(p-1)} \leqq 4\|u_0\|_{\sigma/(p-1)} \qquad \text{for } 0<t<t_0 .$$

As can be seen from the argument to obtain the estimate of V_1, the above inequality is true for V_{i+1}.

Moreover we can obtain

$$\|V_{i+1}(t)-V_i(t)\|_{\sigma/(p-1)} \leqq \int_0^t 2p(4\|u_0\|_{\sigma/(p-1)})^{p-1}\mathcal{M}_i(\tau)d\tau ,$$

where $\mathcal{M}_i(\tau) := \sup_{0<s<\tau} \|V_i(s)-V_{i-1}(s)\|_{\sigma/(p-1)}$ ($i=1, 2, \cdots$), because

$$\langle x\rangle^{\sigma/(p-1)} \| x|^\sigma V_i^p(x,\tau) - |x|^\sigma V_{i-1}^p(x,\tau)|$$

$$\leqq 2p(4\|u_0\|_{\sigma/(p-1)})^{p-1}\langle x\rangle^{\sigma/(p-1)}|V_i(x,\tau)-V_{i-1}(x,\tau)|$$

$$\leqq 2p(4\|u_0\|_{\sigma/(p-1)})^{p-1}\mathcal{M}_i(\tau) .$$

So,

$$\mathcal{M}_{i+1}(t) \leqq \int_0^t 2p(4\|u_0\|_{\sigma/(p-1)})^{p-1}\mathcal{M}_i(\tau)d\tau \qquad i=1, 2, \cdots .$$

Note that $\mathcal{M}_1(t) \leqq 2(2\|u_0\|_{\sigma/(p-1)})^p t$.

Thus,

$$\mathcal{M}_i(t) \leqq \frac{2^{2-p}\|u(t)\|_{\sigma/(p-1)}(2p(2\|u_0\|_{\sigma/(p-1)})^{p-1}t)^i}{i\,!\,p} \qquad i=1, 2, \cdots .$$

We conclude that there is a solution u of (P) such that $\{V_i\}$ converges to u uniformly in $D\times(0,t_0)$. It is clear that u is the unique solution of (P).

3. Proof of Theorem 1.2.

LEMMA 3.1. *Let $T>0$ be the existence time of u, and $\underline{u}$ a subsolution in $(0, T_1)$ for some $T_1>0$. Then we have*

$$\underline{u}(x, t) \leqq u(x, t) \quad in\ D \times [0, T_2),$$

where $T_2 = \min\{T_1, T\}$.

PROOF. Let

$$U(x, t) := \underline{u}(x, t) - u(x, t),$$

then $U_t \leqq \Delta U + |x|^\sigma(\underline{u}^p - u^p)$.

By using the mean value theorem, there exist $0 < \zeta < 1$ such that

$$|x|^\sigma(\underline{u}^p - u^p) = h(x, t)(\underline{u} - u)$$

where,

$$h(x, t) = p|x|^\sigma\{(1-\zeta)\underline{u} + \zeta u\}^{p-1}$$
$$\leqq p \max\{\|\underline{u}(t)\|_{\sigma/(p-1)}, \|u(t)\|_{\sigma/(p-1)}\}^{p-1}$$

Combining this and Proposition 2.1, we get

$$U(x, t) \leqq 0 \quad \text{in } D \times [0, T_2).$$

LEMMA 3.2. *We assume* $0 < \sigma/(p-1) \leqq N-2$. *Let* $\bar{T}$ *be the maximal existence time of* u, *and* v *be the solution of the linear heat equation with the same initial-boundary condition as* u.

Then

$$v(x, t) < ((p-1)|x|^\sigma t)^{-1/(p-1)} \quad in\ D \times (0, \bar{T}).$$

PROOF. If $\bar{T} \leqq T_0 := (\|u_0\|_{\sigma/(p-1)}^{-(p-1)})/(p-1)$, then by Lemma 2.3

$$v(x, t) \leqq \|u_0\|_{\sigma/(p-1)} \langle x \rangle^{-\sigma/(p-1)}$$
$$\leqq ((p-1)\bar{T})^{-1/(p-1)} \langle x \rangle^{-\sigma/(p-1)}$$
$$< ((p-1)|x|^\sigma t)^{-1/(p-1)} \quad \text{for } 0 \leqq t < \bar{T}.$$

Now we assume $\bar{T} > T_0$ and let

$$\underline{u}(x, t) = \{(v(x, t))^{-(p-1)} - (p-1)|x|^\sigma t\}^{-1/(p-1)}.$$

We shall prove $\|\underline{u}(t)\|_{\sigma/(p-1)}$ is finite for $0 < t < \bar{T}$.

When $0 < t < T_0$, from Lemma 2.3, we see that

$$\|\underline{u}(t)\|_{\sigma/(p-1)} \leqq \{\|u_0\|_{\sigma/(p-1)}^{-(p-1)} - (p-1)t\}^{-1/(p-1)} < \infty.$$

We assume that there exists $\tau < \bar{T}$ such that $\|\underline{u}\|_{\sigma/(p-1)} \to \infty$ as $t \uparrow \tau$, and let t_1 be the smallest one of all such τ.

On the other hand, let t_2 be an arbitrary constant with $0 < t_2 < t_1$. Then $\underline{u}$ satisfies for $t \leqq t_2$,

$$\begin{aligned}&\Delta \underline{u}+|x|^{\sigma}\underline{u}^{p}-\underline{u}_{t}\\&=\sigma(N-2-\sigma/(p-1))|x|^{\sigma-2}t\{v^{-(p-1)}-(p-1)|x|^{\sigma}t\}^{-p/(p-1)}\\&\quad+p(p-1)v^{-(p+1)}|x|^{\sigma}t\sum_{i=1}^{N}\Big(v_{i}+\frac{\sigma}{p-1}\frac{x_{i}}{|x|^{2}}v\Big)^{2}\{v^{-(p-1)}-(p-1)|x|^{\sigma}t\}^{(-2p+1)/(p-1)}\\&\geqq 0\,.\end{aligned}$$

So $\underline{u}$ is a subsolution for $0<t<t_2<t_1$, it follows from Lemma 3.1 that $\underline{u}(x, t)\leqq u(x, t)$ in $D\times[0, t_2)$. Hence we see from the definition of $\|u(t)\|_{\sigma/(p-1)}$,

$$\|\underline{u}(t)\|_{\sigma/(p-1)}\leqq\|u(t)\|_{\sigma/(p-1)}\qquad\text{for } 0<t\leqq t_2\,.$$

On the other hand,

$$\|\underline{u}(t_2)\|_{\sigma/(p-1)}\longrightarrow\infty\qquad\text{as } t_2\uparrow t_1\,.$$

We have reached the contradiction.

PROOF OF THEOREM 1.2. We assume that there exists a global solution of (P), then from Lemma 3.2 we have

$$(p-1)^{-1/(p-1)}>|x|^{\sigma/(p-1)}t^{1/(p-1)}v(x, t)\qquad\text{in } D\times(0, \infty)\,.$$

Integrating the above both sides over Ω with respect to $\phi_1(\theta)dS_\theta$, we can estimate by use of (2.3),

$$\begin{aligned}c_2&>\int_{\Omega}r^{\sigma/(p-1)}t^{1/(p-1)}v(r, \theta\,; t)\phi_1(\theta)dS_\theta\\&=\int_{\Omega}r^{\sigma/(p-1)}t^{1/(p-1)}\int_0^\infty\int_{\Omega}G(r, \theta, \rho, \phi\,; t)u_0(\rho, \phi)\rho^{N-1}dS_\phi d\rho\phi_1(\theta)dS_\theta\\&=c_1r^{\sigma/(p-1)}t^{1/(p-1)}\int_0^\infty\int_{\Omega}\frac{1}{2t}(r\rho)^{-(N-2)/2}I_{\nu_1}\Big(\frac{r\rho}{2t}\Big)\exp\Big(-\frac{r^2+\rho^2}{4t}\Big)u_0(\rho, \phi)\phi_1(\phi)\\&\qquad\times\int_{\Omega}\phi_1^2(\theta)\rho^{N-1}dS_\theta dS_\phi d\rho\\&\geqq c_3r^{\sigma/(p-1)-(N-2)/2}t^{1/(p-1)-1}\exp\Big(-\frac{r^2}{4t}\Big)\\&\qquad\times\int_0^\infty\int_{\Omega}\rho^{-(N-2)/2+N-1}\Big(\frac{r\rho}{2t}\Big)^{\nu_1}\exp\Big(-\frac{\rho^2}{4t}\Big)u_0(\rho, \phi)\phi_1(\phi)dS_\phi d\rho\,,\end{aligned}$$

where c_1, c_2 and c_3 are constants and $r=|x|$, $\theta=x/|x|$.

Let $r=t^{1/2}$. Then $\nu_1=\gamma_++(N-2)/2$, there exists $c_4>0$ independent of t such that

$$\begin{aligned}c_4&>t^{(\sigma/(p-1)-(N-2)/2-\nu_1)/2+1/(p-1)-1}\int_0^\infty\int_{\Omega}\rho^{\gamma_++N-1}\exp\Big(-\frac{\rho^2}{4t}\Big)u_0(\rho, \phi)\phi_1(\phi)dS_\phi d\rho,\\&=\int_0^\infty\int_{\Omega}\rho^{\gamma_++N-1}\exp\Big(-\frac{\rho^2}{4t}\Big)u_0(\rho, \phi)\phi_1(\theta)dS_\phi d\rho\,.\end{aligned}$$

Since u is a global solution, we can replace $u_0(\rho, \phi)$ by $u(r, \theta ; t_0)$ for any $t_0>0$. Thus, for $t>t_0$

$$c_4>\int_0^\infty\int_\Omega r^{\gamma_+ +N-1}\exp\Bigl(-\frac{r^2}{4(t-t_0)}\Bigr)u(r, \theta ; t_0)\phi_1(\theta)dS_\theta dr\,.$$

Here let $t\uparrow\infty$ and replace t_0 by t. Then there exists $c_5>0$ such that

$$\begin{aligned}c_4 &\geqq \int_0^\infty\int_\Omega r^{\gamma_+ +N-1}\int_0^t\int_0^\infty\int_\Omega G(r, \theta, \rho, \phi ; t-s)\\ &\quad\times\rho^{\sigma+N-1}u^p(r, \theta, s)d\rho dS_\phi ds\phi_1(\theta)dS_\theta dr\\ &\geqq c_5\int_0^t\int_0^\infty\int_\Omega u^p(\rho, \phi, s)\phi_1(\phi)(t-s)^{-(\gamma_+ +N/2)}\rho^{\gamma_+ +\sigma+N-1}\\ &\quad\times\exp\Bigl(-\frac{\rho^2}{4(t-s)}\Bigr)\int_0^\infty r^{2\gamma_+ +N-1}\exp\Bigl(-\frac{r^2}{4(t-s)}\Bigr)dr dS_\phi d\rho ds\,.\end{aligned}$$

From Hölder's inequality, it follows that there exists $c_6>0$ such that

$$c_6\geqq\int_0^t\int_0^\infty\Bigl(\int_\Omega u(\rho, \phi, s)\phi_1(\phi)dS_\phi\Bigr)^p\rho^{\gamma_+ +\sigma+N-1}\exp\Bigl(-\frac{\rho^2}{4(t-s)}\Bigr)d\rho ds\,.$$

Moreover since we see $u(x, t)\geqq v(x, t)$ from Lemma 3.1, we can estimate

$$\begin{aligned}&\int_\Omega u(\rho, \phi, s)\phi_1(\phi)dS_\phi\\ &\quad\geqq\int_\Omega v(\rho, \phi, s)\phi_1(\phi)dS_\phi\\ &\quad\geqq c_3 s^{-(\gamma_+ +N/2)}\rho^{\gamma_+}\exp\Bigl(-\frac{\rho^2}{4s}\Bigr)\\ &\qquad\times\int_0^\infty\int_\Omega r^{\gamma_+ +N-1}\exp\Bigl(-\frac{r^2}{4s}\Bigr)u_0(r, \theta)\phi_1(\theta)dS_\theta dr\\ &\quad\geqq c_7 s^{-(\gamma_+ +N/2)}\rho^{\gamma_+}\exp\Bigl(-\frac{\rho^2}{4s}\Bigr)\end{aligned}$$

for $0<t'\leqq s$.

Thus we obtain

$$c_6\geqq(c_7)^p\int_{t'}^t s^{-p(\gamma_+ +N/2)}\int_0^\infty\rho^{(p+1)\gamma_+ +\sigma+N-1}\exp\Bigl(-\frac{\rho^2}{4s}\Bigl(p+\frac{s}{t-s}\Bigr)\Bigr)d\rho ds\,.$$

Since $p>1$, we can see that for $\delta\in(0, 1)$ such that $p-1+1/\delta>0$. Noting that $p+s/(t-s)=p+t/(t-s)-1\leqq p+1/\delta-1$, for $s\in[t', (1-\delta)t]$, we have $c_8>0$, such that

$$c_8 \geqq \int_{t'}^{(1-\delta)t} s^{((p+1)\gamma_+ + \sigma + N)/2 - p(\gamma_+ + N/2)} ds$$

$$= \int_{t'}^{(1-\delta)t} \frac{1}{s} ds \longrightarrow \infty$$

as $t \to \infty$.

This is a contradiction. Thus we have proved Theorem 1.2.

4. Proof of Theorem 1.3.

In this section we consider next problem

$$(4.1) \qquad \begin{aligned} &u_t = \Delta u + |x|^\sigma u^p && \text{in } \boldsymbol{R}^N \times (0, T), \\ &u(x, 0) = u_0(x) && \text{in } \boldsymbol{R}^N, \end{aligned}$$

where $N \geqq 3$ and $\|u(t)\|_{\sigma/(p-1)}$ is finite and not zero.

REMARK 4.1. If $D = \boldsymbol{R}^N$, the statements of Lemma 3.1 and Lemma 3.2 are also true without the boundary condition of (P).

PROOF OF THEOREM 1.3. We assume that there exists a global solution $u = u(x, t)$ of (4.1) such that $\|u(t)\|_{\sigma/(p-1)}$ is finite for any $t > 0$. Moreover, let $v = v(x, t)$ be a solution of the heat equation with an initial datum $u_0(x)$. Since $|x-y|^2 \leqq 2(|x|^2 + |y|^2)$ we have

$$(4.2) \qquad \begin{aligned} v(x, t) &= \int_{\boldsymbol{R}^N} \left(\frac{1}{2\sqrt{\pi t}}\right)^N \exp\left(-\frac{|x-y|^2}{4t}\right) u_0(y) dy \\ &\geqq \int_{\boldsymbol{R}^N} \left(\frac{1}{2\sqrt{\pi t}}\right)^N \exp\left(-\frac{|x|^2 + |y|^2}{2t}\right) u_0(y) dy. \end{aligned}$$

By use of Lemma 3.2 we have

$$(p-1)^{-1/(p-1)} |x|^{-\sigma/(p-1)} t^{-1/(p-1)} \geqq \left(\frac{1}{2\sqrt{\pi t}}\right)^N \exp\left(-\frac{|x|^2}{2t}\right) \int_{\boldsymbol{R}^N} \exp\left(-\frac{|y|^2}{2t}\right) u_0(y) dy.$$

Therefore, for $|x| = t^{1/2}$ we have

$$2^N (p-1)^{-1/(p-1)} \pi^{N/2} \exp\left(\frac{1}{2}\right) t^{(N-(2+\sigma)/(p-1))/2} \geqq \int_{\boldsymbol{R}^N} \exp\left(-\frac{|y|^2}{2t}\right) u_0(y) dy.$$

If $p < 1 + (2+\sigma)/N$, the left side of the above inequality goes to 0 as $t \to \infty$. This is a contradiction.

Next, we assume $p = 1 + (2+\sigma)/N$ and let $c_9 = 2^N (p-1)^{-1/(p-1)} \pi^{N/2} \exp(1/2)$. Then

$$c_9 \geqq \int_{\boldsymbol{R}^N} \exp\left(-\frac{|y|^2}{2t}\right) u_0(y) dy.$$

As we discussed in the proof of Theorem 1.2 we can replace $u_0(y)$ by $u(y, t_0)$ for arbitrarily $t_0>0$. Thus,

$$c_9 \geqq \int_{R^N} \exp\left(-\frac{|y|^2}{2(t-t_0)}\right) u(y, t_0)dy \qquad \text{for } t>t_0 .$$

The right side of the above inequality goes to

$$\int_{R^N} u(y, t_0)dy \qquad \text{as } t\to\infty .$$

Replace t_0 by t. Then we have

$$\begin{aligned} c_9 &\geqq \int_{R^N} u(y, t)dy \\ &\geqq \int_{R^N}\int_0^t\int_{R^N}\left(\frac{1}{2\sqrt{\pi(t-s)}}\right)^N \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)|y|^p u^p(y, s)dydsdx \\ &\geqq \int_0^t\int_{R^N}|y|^\sigma u^p(y, s)\int_{R^N}\left(\frac{1}{2\sqrt{\pi(t-s)}}\right)^N \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)dxdyds \\ &= \int_0^t\int_{R^N}|y|^\sigma u^p(y, s)dyds \\ &= \int_0^t\int_0^\infty\int_{S^{N-1}} u^p(\rho, \phi, s)\rho^{\sigma+N-1}dS_\phi d\rho ds . \end{aligned}$$

By Hölder's inequality we have c_{10} such that,

$$c_{10} > \int_0^t\int_0^\infty\left(\int_{S^{N-1}} u(\rho, \phi, s)dS_\phi\right)^p \rho^{\sigma+N-1}d\rho ds .$$

Moreover, by using Lemma 3.2 and (4.2) we get

$$\begin{aligned} &\int_{S^{N-1}} u(\rho, \phi, s)dS_\phi \\ &\qquad \geqq 2^{-N}\pi^{-N/2}\int_{S^{N-1}} s^{-N/2}\exp\left(-\frac{\rho^2}{2s}\right)\int_{R^N}\exp\left(-\frac{|x|^2}{2s}\right)u_0(x)dxdS_\phi . \end{aligned}$$

Since $u_0\not\equiv 0$, for any $t'>0$ there exists $c_{11}=c_{11}(u_0\,;t')>0$ such that

$$c_{11} < 2^{-N}\pi^{-N/2}\int_{S^{N-1}}\int_{R^N}\exp\left(-\frac{|x|^2}{2s}\right)u_0(x)dxdS_\phi \qquad \text{for } s\geqq t' .$$

Hence we get

$$\begin{aligned} c_{10} &> \int_{t'}^t\int_0^\infty\left(c_{11}s^{-N/2}\exp\left(-\frac{\rho^2}{2s}\right)\right)^p \rho^{\sigma+N-1}d\rho ds \\ &= (c_{11})^p\int_{t'}^t\int_0^\infty \rho^{\sigma+N-1}\exp\left(-\frac{p\rho^2}{2s}\right)d\rho\, s^{-(Np)/2}ds \\ &= (c_{11})^p\frac{1}{2}\left(\frac{2}{p}\right)^{(\sigma+N)/2}\Gamma\left(\frac{\sigma+N}{2}\right)\int_{t'}^t s^{(\sigma+N)/2-(Np)/2}ds . \end{aligned}$$

Since $p=1+N/(\sigma+2)$, the right side of the above inequality goes to ∞ as $t\rightarrow\infty$. This is a contradiction. Thus we have proved Theorem 1.3.

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