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FOLDINGS OF ROOT SYSTEMS AND GABRIEL'S THEOREM

By

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1. Introduction.

Gabriel's theorem [5] (cf. below for precise statements) was generalized by Dlab-Ringel [3], [4] where Dynkin graphs of type B_n, C_n, F_4, G_2 also enter in the classification together with the graphs of type A_n, D_n, E_n in [5]. We give in this note another generalization of [5] using the fact that B_n, C_n, F_4, G_2 are obtained by the so-called folding-operation from A_n, D_n, E_6. Our formulation is rather similar to the original formulation in [5].

Let Γ be a finite graph. We denote its set of vertices by Γ_0 and its set of edges by Γ_1 (there may be several edges between two vertices and loops joining a vertex to itself). Let Λ be an orientation of Γ. For each $l \in \Gamma_1$ we denote its starting-point by $\alpha(l)$ and its end-point by $\beta(l)$.

For a fixed field k we define a category $\mathcal{L}(\Gamma, \Lambda)$ after Gabriel [5] as follows.

DEFINITION 1. Let (Γ, Λ) be a finite oriented graph. A pair (V, f) is an object of $\mathcal{L}(\Gamma, \Lambda)$ if $V = \{V_\alpha \mid \alpha \in \Gamma_0\}$ is a family of finite-dimensional vector spaces over k, and $f = \{f_l : V_{\alpha(l)} \to V_{\beta(l)} \mid l \in \Gamma_1\}$ is a family of k-linear mappings. $(V, f) \xrightarrow{\varphi} (W, g)$ is a morphism if $\varphi = \{\varphi_\alpha : V_\alpha \to W_\alpha \mid \alpha \in \Gamma_0\}$ is a family of k-linear mappings such that for each $l \in \Gamma_1$ the following diagram

$$\begin{array}{ccc} V_{\alpha(l)} & \xrightarrow{\ f_l\ } & V_{\beta(l)} \\ \Big\downarrow{\scriptstyle \varphi_{\alpha(l)}} & & \Big\downarrow{\scriptstyle \varphi_{\beta(l)}} \\ W_{\alpha(l)} & \xrightarrow[\ g_l\ ]{} & W_{\beta(l)} \end{array}$$

commutes.

The category $\mathcal{L}(\Gamma, \Lambda)$ is naturally an abelian category and in this category the theorem of Krull-Remak-Schmidt about the essential uniqueness of direct-

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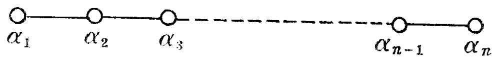
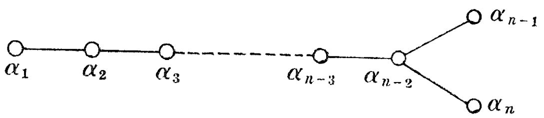
sum-decomposition of an object into indecomposable objects holds.

DEFINITION 2. For each object $(V, f)\in\mathcal{L}(\Gamma, \Lambda)$ we define an element **dim** V of the real vector space $\oplus_{\alpha\in\Gamma_0}\boldsymbol{R}\cdot\alpha$ by $\mathbf{dim}\ V=\sum_{\alpha\in\Gamma_0}(\dim V_\alpha)\alpha$.

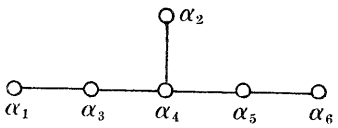
THEOREM 1 (Gabriel [5]). (i) *Let (Γ, Λ) be a finite connected oriented graph. Then there are only finitely many non-isomorphic indecomposable objects if and only if the graph Γ is one of the following graphs.*

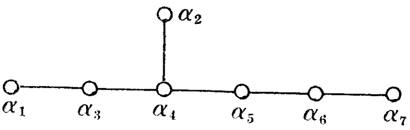
(A_n) 

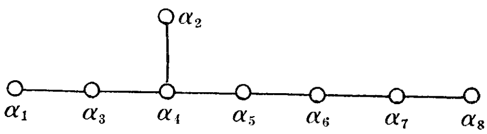
$(n\geqq 1)$

(D_n) 

$(n\geqq 4)$

(E_6) 

(E_7) 

(E_8) 

(ii) *Furthermore if the graph Γ coincides with one of the graphs (A_n), (D_n), (E_6), (E_7), (E_8), then* **dim** *gives a bijection from the set of all the classes of isomorphic indecomposable objects onto the set of all the positive roots of the root system of type (A_n), (D_n), (E_6), (E_7), (E_8) respectively.*

Since Gabriel established this theorem in [5] by rather individual treatment, Bernstein-Gelfand-Ponomarev [1] gave a simple unified proof using the theory of root systems and Weyl groups.

Now our generalization of this theorem is formulated as follows.

For a finite oriented graph (Γ, Λ) we denote by $Aut(\Gamma, \Lambda)$ the automorphism group of (Γ, Λ). Thus $Aut(\Gamma, \Lambda)=\{\sigma=(\sigma_0, \sigma_1)\in\mathfrak{S}^{\Gamma_0}\times\mathfrak{S}^{\Gamma_1}\,|\,\alpha(\sigma_1(l))=\sigma_0(\alpha(l)),$ $\beta(\sigma_1(l))=\sigma_0(\beta(l))$ for all $l\in\Gamma_1\}$, where $\mathfrak{S}^{\Gamma_i}$ means the symmetric group consisting of all permutations of the set Γ_i. Now for each $\sigma \in Aut(\Gamma, \Lambda)$ we define a functor $K^{\sigma}: \mathcal{L}(\Gamma, \Lambda) \rightarrow \mathcal{L}(\Gamma, \Lambda)$ as follows. For an object (V, f), $(W, g) = K^{\sigma} \cdot (V, f)$ is given by $W_{\alpha} = V\sigma_0^{-1}(\alpha)$ for all $\alpha \in \Gamma_0$ and $g_l = f_{\sigma_1^{-1}(l)}$ for all $l \in \Gamma_1$. For a morphism $(V, f) \xrightarrow{\varphi} (W, g)$, $K^{\sigma} \cdot (V, f) \xrightarrow{K^{\sigma} \cdot \varphi} K^{\sigma} \cdot (W, g)$ is given by $(K^{\sigma} \cdot \varphi)_{\alpha} = \varphi_{\sigma_0^{-1}(\alpha)}$ for all $\alpha \in \Gamma_0$.

Definition 3. Let G be a subgroup of $Aut(\Gamma, \Lambda)$. We define a category $\mathcal{L}^G(\Gamma, \Lambda)$ which is a full subcategory of $\mathcal{L}(\Gamma, \Lambda)$ as follows. For an object $(V, f) \in \mathcal{L}(\Gamma, \Lambda)$, (V, f) is an object of $\mathcal{L}^G(\Gamma, \Lambda)$ if for each $\sigma \in G$ $K^{\sigma} \cdot (V, f)$ is isomorphic to (V, f) in the category $\mathcal{L}(\Gamma, \Lambda)$.

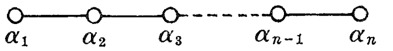
Our main theorem is the following.

Theorem 2. *Let (Γ, Λ) be a finite, connected, oriented graph and G be a subgroup of $Aut(\Gamma, \Lambda)$.*

(i) *In the category $\mathcal{L}^G(\Gamma, \Lambda)$, the theorem of Krull-Remak-Schmidt holds.*

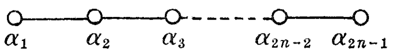
(ii) *There are only finitely many non-isomorphic indecomposable objects in $\mathcal{L}^G(\Gamma, \Lambda)$ if and only if the triple (Γ, Λ, G) is one of the following types.*

(A_n) Γ



$(n \geqq 1)$ $G = \{1\}$

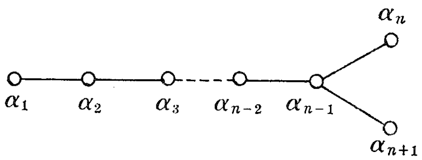
(B_n) Γ



$(n \geqq 2)$ $G = \{1, \tau\}$

$\tau(\alpha_i) = \alpha_{2n-i}$

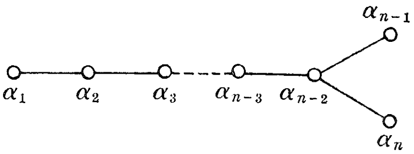
(C_n) Γ



$(n \geqq 3)$ $G = \{1, \tau\}$

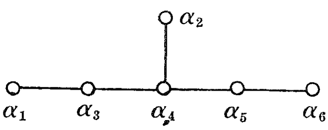
$$\tau(\alpha_i) \begin{cases} \alpha_i & (i \leqq n-1) \\ \alpha_{i+1} & (i = n) \\ \alpha_{i-1} & (i = n+1) \end{cases}$$

(D_n) Γ



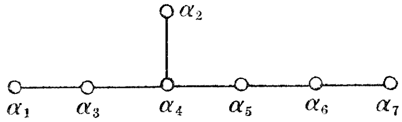
$(n \geqq 4)$ $G = \{1\}$

(E_6) Γ



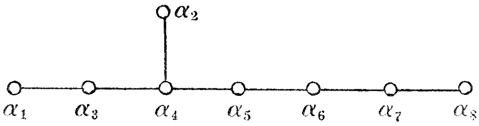
$G = \{1\}$

(E_7) Γ



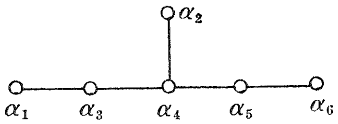
$G=\{1\}$

(E_8) Γ



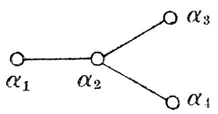
$G=\{1\}$

(F_4) Γ



$G=\{1, \tau\}$
$\tau(\alpha_1)=\alpha_6$, $\tau(\alpha_2)=\alpha_2$
$\tau(\alpha_3)=\alpha_5$, $\tau(\alpha_4)=\alpha_4$
$\tau(\alpha_5)=\alpha_3$, $\tau(\alpha_6)=\alpha_1$

(G_2) Γ



G acts transitively on the set $\{\alpha_1, \alpha_3, \alpha_4\}$ *and fixes* α_2.

Furthermore in the graphs above, the pair (Λ, G) *is assumed to have the property that* G *is a subgroup of* $Aut(\Gamma, \Lambda)$, *i.e.,* Λ *is* G*-invariant.*

(iii) *If the type of the triple* (Γ, Λ, G) *coincides with one of the* $(A_n)\sim(G_2)$ *above, then there is a natural one-to-one correspondence between the set of all the classes of isomorphic indecomposable objects and the set of all the positive roots of the root system of the type* $(A_n)\sim(G_2)$ *respectively.*

The author wishes to express his hearty gratitude to Professor N. Iwahori for his valuable advices.

2. Some categorical arguments.

Let C be an abelian category in which each object is isomorphic to a direct sum of finitely many indecomposable objects and the theorem of Krull-Remak-Schmidt holds. Let $\boldsymbol{H}$ be a finite set consisting of equivalent functors from C onto C. We assume that $\boldsymbol{H}$ forms a group with respect to the composition of functors.

DEFINITION 4. We define a full subcategory $C^{\boldsymbol{H}}$ of C in the following way. For an object M of C, M is an object of $C^{\boldsymbol{H}}$ if for all $F\in\boldsymbol{H}$ $F\cdot M$ is isomorphic to M in the category C.

PROPOSITION 1. (i) *In the category* $C^{\boldsymbol{H}}$ *the theorem of Krull-Remak-Schmidt* *holds.*

(ii) *For an indecomposable object $M\in\mathcal{C}$, let $\boldsymbol{H}=\bigcup_{i=1}^{m}F_i\cdot\boldsymbol{K}$ be the coset decomposition of $\boldsymbol{H}$ with respect to the subgroup $\boldsymbol{K}=\{F\in\boldsymbol{H}\mid F\cdot M\cong M\}$. Then $\tilde{M}=\bigoplus_{i=1}^{m}F_i\cdot M$ is an indecomposable object in the category $\mathcal{C}^{\boldsymbol{H}}$.*

(iii) *Any indecomposable object of $\mathcal{C}^{\boldsymbol{H}}$ is isomorphic to $\tilde{M}$ which is obtained as in* (ii) *for some indedomposable object M of $\mathcal{C}$.*

(iv) *There are only finitely many non-isomorphic indecomposable objects in $\mathcal{C}^{\boldsymbol{H}}$ if and only if there are only finitely many non-isomorphic indecomposable objects in $\mathcal{C}$.*

PROOF. We first note that every $\tilde{M}$ of $\mathcal{C}^{\boldsymbol{H}}$ is a direct sum of finitely many indecomposable objects of $\mathcal{C}^{\boldsymbol{H}}$. In fact this is easily seen by induction on the 'length' k of $\tilde{M}$ expressed as a direct sum of k indecomposable objects of $\mathcal{C}$.

(ii) It is clear that $\tilde{M}$ is an object of $\mathcal{C}^{\boldsymbol{H}}$ by construction. Let us prove that $\tilde{M}$ is indecomposable in $\mathcal{C}^{\boldsymbol{H}}$. There exist indecomposable objects $\tilde{M}_1, \cdots, \tilde{M}_k$ of $\mathcal{C}^{\boldsymbol{H}}$ such that $\tilde{M}$ is isomorphic to $\tilde{M}_1\oplus\cdots\oplus\tilde{M}_k$. By the theorem of Krull-Remak-Schmidt, M is isomorphic to an indecomposable component of some $\tilde{M}_i$ in $\mathcal{C}$. Since $\tilde{M}_i\cong F\cdot\tilde{M}_i$ for every $F\in\boldsymbol{H}$ and the theorem of Krull-Remak-Schmidt holds, $\tilde{M}$ is isomorphic to a direct sum component of $\tilde{M}_i$ in $\mathcal{C}$. Thus $\tilde{M}$ coincides with $\tilde{M}_i$.

(iii) Let N be an indecomposable object of $\mathcal{C}^{\boldsymbol{H}}$. If M is an indecomposable component of N in $\mathcal{C}$, $F\cdot M$ is also isomorphic to an indecomposable component of N in $\mathcal{C}$ for all $F\in\boldsymbol{H}$. So there exists $N'\in\mathcal{C}$ such that N is isomorphic to $\tilde{M}\oplus N'$. Because N and $\tilde{M}$ are objects of $\mathcal{C}^{\boldsymbol{H}}$, N' is an object of $\mathcal{C}^{\boldsymbol{H}}$, too. On the other hand N is indecomposable in $\mathcal{C}^{\boldsymbol{H}}$. Thus N is isomorphic to $\tilde{M}$.

(i) In the category $\mathcal{C}$ the theorem of Krull-Remak-Schmidt holds. So by (ii) and (iii) the same theorem also holds in $\mathcal{C}^{\boldsymbol{H}}$.

(iv) Let Φ_1 (resp. Φ_2) be the set of all the classes of isomorphic indecomposable objects in the category $\mathcal{C}$ (resp. $\mathcal{C}^{\boldsymbol{H}}$). By (ii) and (iii) there is a natural mapping from Φ_1 onto Φ_2. And the inverse image of one element of Φ_2 is a finite set and its cardinality is less than the order of $\boldsymbol{H}$. So Φ_1 is a finite set if and only if Φ_2 is a finite set.

3. Proof of the main theorem.

Let (Γ, Λ) be a finite oriented graph and G be a subgroup of $Aut(\Gamma, \Lambda)$. We first remark the following obvious lemma.

LEMMA 1. (i) *$K^{\sigma}\circ K^{\tau}=K^{\sigma\tau}$ for all $\sigma, \tau\in G$.*

(ii) *For each $\sigma \in G$, K^{σ} is an equivalence of the category.*

(iii) *The set $\boldsymbol{H}=\{K^{\sigma} \mid \sigma \in G\}$ forms a group with respect to the composition of functors.*

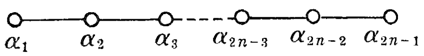
By the lemma above we can apply the arguments in §2 to our situation. If we set $\mathcal{C}=\mathcal{L}(\Gamma, \Lambda)$ and $\boldsymbol{H}=\{K^{\sigma} \mid \sigma \in G\}$, then the category $\mathcal{C}^{\boldsymbol{H}}$ equals to $\mathcal{L}^{G}(\Gamma, \Lambda)$.

So Theorem 2 (i), (ii) is a consequence of Proposition 1 (i), (iv) and Theorem 1 (i). At the end of this section we prove Theorem 2 (iii).

By the Proposition 1 (ii), (iii) we can construct all the indecomposable objects of $\mathcal{L}^{G}(\Gamma, \Lambda)$ from the indecomposable objects of $\mathcal{L}(\Gamma, \Lambda)$. And the indecomposable objects of $\mathcal{L}(\Gamma, \Lambda)$ are described in the Theorem 1 (ii). So Theorem 2 (iii) is a consequence of the following proposition about the so-called foldings of the root systems.

PROPOSITION 2. *Let Δ be a reduced irreducible root system and Π be a fundamental root system of Δ (cf. N. Bourbaki [2]). For each root system of the following types we give a subgroup G of $Aut(\Pi)$ as follows. (Note that $G=Aut(\Pi)$ except the case* (iv) *and the case* (ii) *with $n=3$.)*

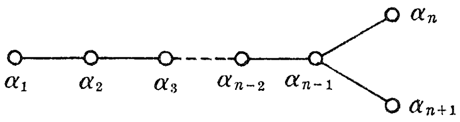
(*i*) $\Delta=A_{2n-1}$ $(n \geqq 2)$



$G=\{1, \tau\}$

$\tau(\alpha_i)=\alpha_{2n-i}$ $(1 \leqq i \leqq 2n-1)$

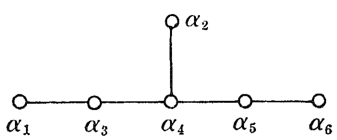
(*ii*) $\Delta=D_{n+1}$ $(n \geqq 3)$



$G=\{1, \tau\}$

$$\tau(\alpha_i)\begin{cases} \alpha_i & (i \leqq n-1) \\ \alpha_{i+1} & (i=n) \\ \alpha_{i-1} & (i=n+1) \end{cases}$$

(*iii*) $\Delta=E_6$



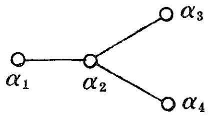
$G=\{1, \tau\}$

$\tau(\alpha_1)=\alpha_6$, $\tau(\alpha_2)=\alpha_2$

$\tau(\alpha_3)=\alpha_5$, $\tau(\alpha_4)=\alpha_4$

$\tau(\alpha_5)=\alpha_3$, $\tau(\alpha_6)=\alpha_1$

(iv) $\Delta=D_4$



G operates transitively on the set $\{\alpha_1, \alpha_3, \alpha_4\}$ *and fixes* α_2. *Thus* $G\cong \boldsymbol{Z}/3\boldsymbol{Z}$ *or* $G\cong\mathfrak{S}_3$.

In each case of (i)$\sim$(iv) *above, we define* $\tilde{\alpha}$ *for each* $\alpha\in\Delta$ *as follows. Let* $G=\bigcup_{i=1}^{k}\sigma_i\cdot G^{\alpha}$ *be the coset decomposition of G relative to the subgroup* $G^{\alpha}=\{\sigma\in G\,|\,\sigma(\alpha)=\alpha\}$. *We define* $\tilde{\alpha}$ *by* $\tilde{\alpha}=\sum_{i=1}^{k}\sigma_i(\alpha)$.

Then $\tilde{\Delta}=\{\tilde{\alpha}\,|\,\alpha\in\Delta\}$ *is a root system of type* B_n, C_n, F_4, G_2 *respectively, and* $\tilde{\Pi}=\{\tilde{\alpha}\,|\,\alpha\in\pi\}$ *is a fuudamental root system of* $\tilde{\Delta}$ *respectively. Moreover for* α, $\beta\in\Delta$, $\tilde{\alpha}=\tilde{\beta}$ *holds if and only if there exists an element* σ *of G such that* $\sigma(\alpha)=\beta$.

PROOF. If we put $\Pi=\{\alpha_i\,|\,1\leqq i\leqq k\}$ where $k=2n-1$, $n+1$, 6, 4 for the cases (i)$\sim$(iv) respectively, then $\tilde{\alpha}_i=\sum_{j\in I_i}\alpha_j$ with $I_i=\{1\leqq j\leqq k\,|\,{}^{\exists}\sigma\in G$ s.t. $\sigma(\alpha_i)=\alpha_j\}$. So the elements of $\tilde{\Pi}$ are linearly independent. And for any $\tilde{\alpha}=\sum_{i=1}^{k}m_i\alpha_i\in\tilde{\Delta}$, $m_i=m_j$ if there exists some $\sigma\in G$ such that $\sigma(\alpha_i)=\alpha_j$, because $\sigma(\tilde{\alpha})=\tilde{\alpha}$ for any $\sigma\in G$. So each $\tilde{\alpha}\in\tilde{\Delta}$ can be written as $\tilde{\alpha}=\sum_{\beta\in\tilde{\Pi}}n_\beta\beta$ with integral coefficients n_β which are all non-negative or all non-positive.

Thus it is enough to show that $\tilde{\Delta}$ is a root system of type B_n, C_n, F_4, G_2 respectively and that if $\tilde{\alpha}=\tilde{\beta}$ for α, $\beta\in\Delta$, then there exists some $\sigma\in G$ such that $\sigma(\alpha)=\beta$. This can be seen by straightforward verifications. For example we give the verifications for the cases (i), (iii), using the notations of N. Bourbaki [2].

(i) $\Delta=\{e_i-e_j\,|\,1\leqq i, j\leqq 2n, i\neq j\}$ and $\Pi=\{\alpha_i=e_i-e_{i+1}\,|\,1\leqq i\leqq 2n-1\}$. τ is given by $\tau(e_i)=-e_{2n+1-i}$, so for each $\alpha=e_i-e_j$ $\tau(\alpha)=\alpha$ if and only if $i+j=2n+1$. Thus

$$\tilde{\alpha}=\begin{cases}\alpha=e_i-e_j & (i+j=2n+1)\\ \alpha+\tau(\alpha)=(e_i-e_{2n+1-i})-(e_j-e_{2n+1-j}) & (i+j\neq 2n+1)\,.\end{cases}$$

So $\tilde{\alpha}=\tilde{\beta}$ implies that there exists an element σ of G such that $\sigma(\alpha)=\beta$. If we set $f_i=e_i-e_{2n+1-i}$ $(1\leqq i\leqq n)$, then $\tilde{\Delta}=\{\pm f_i\,|\,1\leqq i\leqq n\}\cup\{\pm f_i\pm f_j\,|\,i\neq j\}$. So $\tilde{\Delta}$ is a root system of type B_n.

(iii) $\Delta=\{\pm e_i\pm e_j\,|\,1\leqq i<j\leqq 5\}\cup\{\pm(e_8-e_7-e_6+\sum_{i=1}^{5}(-1)^{\nu(i)}e_i)/2\,|\,\sum_{i=1}^{5}\nu(i)$: even$\}$ and $\Pi=\{\alpha_i\,|\,1\leqq i\leqq 6\}$ with

$$\alpha_1=(e_1+e_8)/2-(e_2+e_3+e_4+e_5+e_6+e_7)/2$$

$$\alpha_1=e_1+e_2$$

$$\alpha_i=e_{i-1}-e_{i-2}\quad(3\leqq i\leqq 6)\,.$$

τ is given by

$$\tau(e_i)=-e_{5-i}+x \qquad (1\leqq i\leqq 4)$$

$$\tau(e_5)=(y-e_5)/2$$

$$\tau(y)=(y+3e_5)/2$$

where $x=(e_1+e_2+e_3+e_4)/2$

$y=e_3-e_6-e_7$.

So it is easily seen that $\tilde{\alpha}=\tilde{\beta}$ implies the existence of an element σ of G with $\sigma(\alpha)=\beta$. If we set

$$f_1=x+(e_5+y)/2$$

$$f_2=-x+(e_5+y)/2$$

$$f_3=e_3-e_2$$

$$f_4=e_4-e_1,$$

then $\tilde{\Delta}=\{\pm f_i \mid 1\leqq i\leqq 4\}\cup\{\pm f_i\pm f_j \mid 1\leqq i<j\leqq 4\}\cup\{(\pm f_1\pm f_2\pm f_3\pm f_4)/2\}$. So $\tilde{\Delta}$ is a root system of type F_4.

4. Some remarks.

REMARK 1. In the Theorem 2 the assumption that Γ is connected is not essential.

Indeed if Γ is not connected let $\Gamma_0=\bigcup_{i=1}^{k}\Gamma_0^{(i)}$ be the decomposition into connected components. We can assume that G acts transitively on the set $\{\Gamma_0^{(i)} \mid 1\leqq i\leqq k\}$. Now let $G^{(i)}$ be the subgroup of $Aut(\Gamma^{(i)}, \Lambda^{(i)})$ induced by the subgroup $\{\sigma\in G \mid \sigma_0(\Gamma_0^{(i)})=\Gamma_0^{(i)}\}$. Then by restriction we obtain a natural bijection from the set of all the classes of isomorphic indecomposable objects of $\mathcal{L}^G(\Gamma, \Lambda)$ onto the set of all the classes of isomorphic indecomposable objects of $\mathcal{L}^{G^{(i)}}(\Gamma^{(i)}, \Lambda^{(i)})$.

REMARK 2. Let Γ be one of the Dynkin graphs A_n, D_n, E_6, E_7, E_8. For the category $\mathcal{C}=\mathcal{L}(\Gamma, \Lambda)$ and for any finite group $\boldsymbol{H}$ consisting of equivalent functors from $\mathcal{C}$ onto $\mathcal{C}$, the arguments in §2 also hold. However, if K is an equivalent functor from $\mathcal{C}$ onto $\mathcal{C}$, there exists some $\sigma\in Aut(\Gamma, \Lambda)$ such that $K\cdot M\cong K^{\sigma}\cdot M$ for any $M\in\mathcal{C}$. So essentially we can limit the arguments in §2 only for the case $\boldsymbol{H}=\{K^{\sigma} \mid \sigma\in G\}$ where G is a subgroup of $Aut(\Gamma, \Lambda)$.

We can show the statement above as follows. If M is a simple object, then $K\cdot M$ is also a simple object of $\mathcal{C}$. So K induces a permutation σ_0 of the set Γ_0.

For each edge $l\in\Gamma_1$ we define an object (V, f) by $V_{\alpha(l)}=V_{\beta(l)}=k$, $V_\gamma=0$ $(\gamma\neq\alpha(l), \beta(l))$, $f_l=id$ and $f_{l'}=0$ $(l'\neq l)$. Considering the Jordan-Hölder sequences of the objects (V, f) and $K\cdot(V, f)$, K induces some $\sigma\in Aut(\Gamma, \Lambda)$. It is enough to show that for each indecomposable object M, $(K^{\sigma^{-1}}\circ K)\cdot M$ is isomorphic to M. By the way $\mathbf{dim}((K^{\sigma^{-1}}\circ K)\cdot M)=\mathbf{dim}\, M$ (If N is simple, $(K^{\sigma^{-1}}\circ K)\cdot N\cong N$. So if N appears n-times in the Jordan-Hölder sequence of M, it appears n-times in the Jordan-Hödler sequence of $(K^{\sigma^{-1}}\circ K)\cdot M$, too). Thus by the Theorem 1 (ii), $(K^{\sigma^{-1}}\circ K)\cdot M$ is isomorphic to M.(This remark is due to Yohei Tanaka.)

Note added in proof.

After the preparation of this paper, the author realized that the notion of "folding" has been already given by R. Steinberg: in [6] a theorem similar to our Proposition 2 is proved in a unified manner.

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