

A NOTE ON FREE DIFFERENTIAL GRADED ALGEBRA RESOLUTIONS

By

D. TAMBARA

Introduction

We work over a field k . A differential graded algebra (dga for short) in this paper is a graded k -algebra $U = \bigoplus_{n \geq 0} U_n$ with differential d of degree -1 . Given a k -algebra R , it is well-known that there exists a free dga resolution $\varepsilon: U \rightarrow R$ (Baues [2]). That is, U is a dga which is free as a graded algebra, ε is a dga map, and the sequence

$$\cdots \xrightarrow{d} U_n \xrightarrow{d} \cdots \xrightarrow{d} U_0 \xrightarrow{\varepsilon} R \rightarrow 0$$

is exact. Such a resolution is thought of as a prolongation of a presentation of R by generators and relations, and expected to contain lots of information about homology of R . Although free dga's frequently appear in homotopical algebra such as [2], not much seems to be known about the structure of free dga resolutions of algebras.

We study here a relationship between a free dga resolution of R and a free bimodule resolution of the R -bimodule R . Let U be a dga which is free on a graded space E , and $\varepsilon: U \rightarrow R$ an augmentation map. We construct a complex $R \otimes E \otimes R$ of free R -bimodules with augmentation $\sigma: R \otimes E \otimes R \rightarrow \Omega_R$, where Ω_R is the kernel of the multiplication map $R \otimes R \rightarrow R$. If ε is a resolution, then so is σ (Proposition 1.2). The converse is true when R is a connected graded algebra and U, ε are taken to be compatible with the grading of R (Theorem 3). Therefore, the verification of the exactness of $\varepsilon: U \rightarrow R$ reduces to that of $\sigma: R \otimes E \otimes R \rightarrow \Omega_R$, which is much easier.

Using this criterion, we give explicit free dga resolutions of Koszul algebras and their generalizations.

NOTATION. For a graded module $M = \bigoplus_{n \geq 0} M_n$, we write $M_+ = \bigoplus_{n > 0} M_n$. For a

Received August 31, 1994.

Revised May 31, 1995.

k -module V , $T(V)$ is the tensor algebra on V . When V is a graded k -module, we give $T(V)$ the induced grading.

1. The bimodule resolution associated with a dga resolution

A dga is a graded algebra $U = \bigoplus_{n \geq 0} U_n$ equipped with a linear map $d: U \rightarrow U$ such that $d^2 = 0$, $d(U_n) \subset U_{n-1}$ and $d(xy) = d(x)y + (-1)^p xd(y)$ for $x \in U_p, y \in U_q$. A dga (U, d) is said to be free if the graded algebra U is free, that is, $U = T(E)$ for some graded subspace E of U .

Any algebra can be viewed as a dga concentrating in degree 0. Let U be a dga and R an algebra. A dga map $\varepsilon: U \rightarrow R$ is called a resolution if

$$\dots \xrightarrow{d} U_n \xrightarrow{d} \dots \xrightarrow{d} U_0 \xrightarrow{\varepsilon} R \rightarrow 0$$

is exact.

It is well-known that given an algebra R , there exist a free dga U and a resolution $\varepsilon: U \rightarrow R$. For example, see [2, Lemma 7.21], where a more general statement is proved. Although our results are logically independent of this fact, we briefly review a construction of a free dga resolution.

First, take a surjective algebra map $U^{(0)} = T(E_0) \rightarrow R$ from a tensor algebra. Suppose we have constructed a dga $U^{(n)}$ which is free on a graded space $E_0 \oplus \dots \oplus E_n$, and a dga map $U^{(n)} \rightarrow R$ which induces isomorphism on homology in degree $< n$. Then take a linear map $\phi: E_{n+1} \rightarrow \text{Ker}(d: U_n^{(n)} \rightarrow U_{n-1}^{(n)})$ so that $\text{Im } \phi$ covers $H_n(U^{(n)})$. Put $U^{(n+1)} = T(E_0 \oplus \dots \oplus E_{n+1})$ and extend the differential of $U^{(n)}$ to the differential d of $U^{(n+1)}$ so that $d|_{E_{n+1}} = \phi$. Then $H_n(U^{(n+1)}) = 0$. Thus we obtain an increasing sequence of free dga's $U^{(n)}, n \geq 0$. Then $U = \bigcup_n U^{(n)}$ together with the map $U_0 = U^{(0)} \rightarrow R$ provides a free dga resolution of R .

In this section we give a construction of a free R -bimodule resolution of the R -bimodule R from a free dga resolution of R . This is based on an idea of Shukla in [5].

Let R be an algebra, $U = (T(E), d)$ a free dga and $\varepsilon: U \rightarrow R$ a dga map. Define an R -bimodule map $\rho: R \otimes U \otimes R \rightarrow R \otimes E \otimes R$ by

$$\rho(1 \otimes x_1 \cdots x_n \otimes 1) = \sum_{i=1}^n \varepsilon(x_1 \cdots x_{i-1}) \otimes x_i \otimes \varepsilon(x_{i+1} \cdots x_n)$$

for $x_1, \dots, x_n \in E$ and

$$\rho(1 \otimes 1 \otimes 1) = 0.$$

Then $\rho(R \otimes U_n \otimes R) \subset R \otimes E_n \otimes R$, because $\varepsilon(U_+) = 0$.

Define an R -bimodule map $\partial: R \otimes E \otimes R \rightarrow R \otimes E \otimes R$ as the composite

$$R \otimes E \otimes R \hookrightarrow R \otimes U \otimes R \xrightarrow{1 \otimes d \otimes 1} R \otimes U \otimes R \xrightarrow{\rho} R \otimes E \otimes R.$$

Then $\partial(R \otimes E_n \otimes R) \subset R \otimes E_{n-1} \otimes R$.

Define an R -bimodule map $\sigma : R \otimes E \otimes R \rightarrow R \otimes R$ by

$$\sigma(1 \otimes x \otimes 1) = \varepsilon(x) \otimes 1 - 1 \otimes \varepsilon(x).$$

σ vanishes on $R \otimes E_+ \otimes R$.

PROPOSITION 1.1. $\partial^2 = 0, \sigma\partial = 0$.

Thus we obtain a complex of R -bimodules

$$\dots \xrightarrow{\partial} R \otimes E_n \otimes R \xrightarrow{\partial} \dots \xrightarrow{\partial} R \otimes E_0 \otimes R \xrightarrow{\sigma} R \otimes R \xrightarrow{\text{mult}} R \rightarrow 0.$$

PROPOSITION 1.2. *If $\varepsilon : U \rightarrow R$ is a resolution, then this complex is exact.*

2. Proof of Propositions 1.1 and 1.2

Viewing U as just an algebra, we form the standard free resolution of the U -bimodule U ([3]):

$$\dots \xrightarrow{\delta} U^{\otimes(n+2)} \xrightarrow{\delta} \dots \xrightarrow{\delta} U \otimes U \xrightarrow{\text{mult}} U \rightarrow 0$$

where

$$\delta(u_0 \otimes \dots \otimes u_{n+1}) = \sum_{i=0}^n (-1)^i u_0 \otimes \dots \otimes u_i u_{i+1} \otimes \dots \otimes u_{n+1}.$$

Each term of the resolution is a complex as a tensor product of the complex U , and each δ is a chain map, because the multiplication $U \otimes U \rightarrow U$ is so.

Now regard R as a U -bimodule through the map $\varepsilon : U \rightarrow R$. Applying the functor $R \otimes_U (\) \otimes_U R$ to the standard resolution, we obtain a complex

$$\dots \xrightarrow{\gamma} R \otimes U^{\otimes n} \otimes R \xrightarrow{\gamma} \dots \xrightarrow{\gamma} R \otimes R \xrightarrow{\text{mult}} R \rightarrow 0,$$

whose terms are complexes and differentials γ are chain maps. So we have a double complex B having terms $B_{pq} = (R \otimes U^{\otimes p} \otimes R)_q$ for $p, q \geq 0$. This was considered by Shukla [5]. The propositions are proved by relating $H_p^I H_q^{II}(B)$ and $H_q^{II} H_p^I(B)$ with $H_{p+q}(\text{tot } B)$. Here H^I, H^{II} mean the homology with respect to the first, second index respectively, and $\text{tot } B$ is the total complex of B .

We first treat $H^I H^{II}(B)$.

(i) We have a diagram

$$\begin{array}{ccccccc}
 R \otimes U \otimes U \otimes R & \xrightarrow{\gamma} & R \otimes U \otimes R & \xrightarrow{\rho} & R \otimes E \otimes R & \longrightarrow & 0 \\
 & & \gamma \downarrow & \nearrow \sigma & & & \\
 & & R \otimes R & & & &
 \end{array}$$

where the first row is exact and the triangle is commutative.

PROOF. Forget the differential graded structure of U for a moment. As U is the tensor algebra on E , a minimal free resolution of the U -bimodule U is given by

$$0 \rightarrow U \otimes E \otimes U \xrightarrow{\tau} U \otimes U \xrightarrow{\text{mult}} U \rightarrow 0$$

where $\tau(1 \otimes x \otimes 1) = x \otimes 1 - 1 \otimes x$ for $x \in E$ ([2, p. 181, Ex2]). Define a U -bimodule map $\theta: U \otimes U \otimes U \rightarrow U \otimes E \otimes U$ by

$$\begin{aligned}
 \theta(1 \otimes x_1 \cdots x_n \otimes 1) &= \sum_{i=1}^n x_1 \cdots x_{i-1} \otimes x_i \otimes x_{i+1} \cdots x_n \\
 \theta(1 \otimes 1 \otimes 1) &= 0
 \end{aligned}$$

for $x_1, \dots, x_n \in E$. θ is the identity on $U \otimes E \otimes U$ and the diagram

$$\begin{array}{ccc}
 U \otimes U \otimes U & \xrightarrow{\delta} & U \otimes U \\
 \theta \downarrow & & \parallel \\
 U \otimes E \otimes U & \xrightarrow{\gamma} & U \otimes U
 \end{array}$$

is commutative. By the exactness of the standard and the minimal resolutions, it follows that the sequence

$$U \otimes U \otimes U \otimes U \xrightarrow{\delta} U \otimes U \otimes U \xrightarrow{\theta} U \otimes E \otimes U \rightarrow 0$$

is exact. Now apply $R \otimes_U (\) \otimes_U R$ to the above diagram and the sequence. As $\rho = R \otimes_U \theta \otimes_U R$, $\sigma = R \otimes_U \tau \otimes_U R$, the assertion follows.

PROOF OF PROPOSITION 1.1. Since γ is a chain map, it follows from the exact sequence of (i) that $R \otimes E \otimes R$ becomes a complex with differential ∂' and ρ becomes a chain map. By the definition of ∂ and the fact that ρ is the identity on $R \otimes E \otimes R$, we know $\partial' = \partial$. Thus $(R \otimes E \otimes R, \partial)$ is a complex and ρ, σ are chain maps.

(ii) The homology of the complex

$$\dots \xrightarrow{\partial} R \otimes E_n \otimes R \xrightarrow{\partial} \dots \xrightarrow{\partial} R \otimes E_0 \otimes R \xrightarrow{\sigma} R \otimes R$$

at $R \otimes E_n \otimes R$ is isomorphic to $H_{n+1}(\text{tot } B)$.

PROOF. As noted in (i), the U -bimodule U has projective dimension 1. So $H_p^1(B) = \text{Tor}_p^{U \otimes U^m}(R \otimes R, U) = 0$ for $p > 1$. By (i), we have $H_1^1(B) \cong \text{Ker } \sigma$, $H_0^1(B) \cong \text{Cok } \sigma$. So

$$\begin{aligned} H_1^1(B_{\cdot, q}) &\cong R \otimes E_q \otimes R && \text{if } q > 0 \\ &\cong \text{Ker}(R \otimes E_0 \otimes R \xrightarrow{\sigma} R \otimes R) && \text{if } q = 0 \\ H_0^1(B_{\cdot, q}) &= 0 && \text{if } q > 0. \end{aligned}$$

Hence $H_q^1 H_1^1(B)$ is the homology at $R \otimes E_q \otimes R$ of the complex in the statement and the spectral sequence degenerates to give $H_q^1 H_1^1(B) \cong H_{q+1}(\text{tot } B)$ for $q \geq 0$.

(iii) If $\varepsilon : U \rightarrow R$ is a resolution, then $H_n(\text{tot } B) = 0$ for $n > 0$.

PROOF. By Künneth we have

$$H_q^1(B_{p, \cdot}) = H_q(R \otimes U^{\otimes p} \otimes R) \cong \begin{cases} R^{\otimes(p+2)} & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

and the complex

$$\cdots \rightarrow H_0^1(B_{p, \cdot}) \rightarrow \cdots \rightarrow H_0^1(B_{0, \cdot})$$

is isomorphic to the standard free resolution of the R -bimodule R , which is acyclic. Hence $H_p^1 H_q^1(B) = 0$ unless $p = q = 0$. Then $H_n(\text{tot } B) = 0$ for $n > 0$.

Now Proposition 1.2 follows from (ii) and (iii).

3. Case where R is graded

In this section we state a converse of Proposition 1.2 under certain assumptions. As before, let R be a k -algebra, $U = (T(E), d)$ a free differential graded algebra with augmentation $\varepsilon : U \rightarrow R$. Here we further assume that

- R is a connected graded algebra, that is, $R = \bigoplus_{m \geq 0} R^m$ with $R^0 = k$.
- E has another grading $E = \bigoplus_{m \geq 0} E^m$ compatible with the original one, that is, $E_n = \bigoplus_{m \geq 0} E_n^m$ with $E_n^m = E^m \cap E_n$.
- $E_0^0 = 0$.

Then the upper grading of E induces the grading $U = \bigoplus_{m \geq 0} U^m$ so that U is a doubly graded algebra. The third condition means that the graded algebra U_0 is connected. We finally assume

- $d(U^m) \subset U^m, \varepsilon(U^m) \subset R^m$.

REMARK. For any connected graded algebra R , one can find a resolution $\varepsilon: U \rightarrow R$ satisfying the above conditions. This is easily seen from the construction reviewed in Section 1.

THEOREM 3. *The following are equivalent.*

- (1) $\varepsilon: U \rightarrow R$ is a resolution.
- (2) The complex

$$\cdots \xrightarrow{\partial} R \otimes E_n \otimes R \xrightarrow{\partial} \cdots \xrightarrow{\partial} R \otimes E_0 \otimes R \xrightarrow{\sigma} R \otimes R \xrightarrow{\text{mult}} R \rightarrow 0$$

is exact.

- (3) The complex

$$\cdots \xrightarrow{\bar{\partial}} E_n \otimes R \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} E_0 \otimes R \xrightarrow{\bar{\sigma}} R \xrightarrow{\eta} k \rightarrow 0$$

is exact, where $\bar{\partial} = k \otimes_R \partial$, $\bar{\sigma} = k \otimes_R \sigma$ and η is the projection.

(2) \Rightarrow (3) is obvious, and (3) \Rightarrow (2) follows from a version of Nakayama’s lemma. We shall prove (3) \Rightarrow (1) in the next section.

The map $\bar{\partial}$ on E is given also as $E \xrightarrow{d} E \otimes U \xrightarrow{1 \otimes \varepsilon} E \otimes R$, and $\bar{\sigma} = -\varepsilon$ on E (see (v) of the next section).

4. Proof of Theorem 3

Let \mathfrak{A} be the set of finite sequences $\nu = (\nu_1, \dots, \nu_r)$ of non-negative integers. Write $|\nu| = \nu_1 + \dots + \nu_r$, $l(\nu) = r$. For $\mu = (\mu_1, \dots, \mu_q)$ and $\nu = (\nu_1, \dots, \nu_r)$, define

$$\mu\nu = (\mu_1, \dots, \mu_q, \nu_1, \dots, \nu_r).$$

We write also $0^k = (0, \dots, 0)$, $0^k 1 = (0, \dots, 0, 1)$ (k is the number of 0).

Define a partial order $<$ on \mathfrak{A} as follows. For μ and ν as above we set $\mu < \nu$ if $|\mu| = |\nu|$ and $\mu_1 = \nu_1, \dots, \mu_{h-1} = \nu_{h-1}, \mu_h < \nu_h$ for some $h \leq q, r$. Note that $\mu 0^i < \nu 0^j$ if and only if $\mu < \nu$. Let $\mathfrak{A}' = \{(\nu_1, \dots, \nu_r) \in \mathfrak{A} \mid r > 0, \nu_r > 0\}$. Then $<$ is a total order on the subset $\{\nu \in \mathfrak{A}' \mid |\nu| = n\}$ for each $n > 0$.

For $\nu = (\nu_1, \dots, \nu_r) \in \mathfrak{A}'$, define $\nu_- = (\nu_1, \dots, \nu_{r-1}, \nu_r - 1)$.

For $\nu \in \mathfrak{A}$, we set $E_\nu = E_{\nu_1} \otimes \dots \otimes E_{\nu_r}$. Then

$$U = T(E) = \bigoplus_{\nu \in \mathfrak{A}} E_\nu, \quad U_0 = T(E_0) = \bigoplus_{k \geq 0} E_{0^k}, \quad U_+ = \bigoplus_{\nu \in \mathfrak{A}'} E_\nu U_0.$$

For $\nu \in \mathfrak{A}$, let

$$\text{pr}_\nu : U \rightarrow E_\nu, \quad \pi_\nu : U \rightarrow \bigoplus_{k \geq 0} E_{\nu 0^k} = E_\nu U_0$$

be the projections with respect to the above decomposition for U .

(i) $d(E) \subset EU$.

PROOF. Clearly $d(E_n) \subset U_{n-1} \subset EU$ if $n > 1$. And

$$d(E_1) \subset \text{Ker}(\varepsilon : U_0 \rightarrow R) \subset U_0^+ = E_0U_0$$

by the assumption $U_0^0 = k$.

We define a map $d_* : U_+ \rightarrow U$ as follows. First, d_* on E_+ is the composite

$$E_+ \xrightarrow{d} E \otimes U \xrightarrow{1 \otimes \pi_\varnothing} E \otimes U_0,$$

where π_\varnothing is the projection onto U_0 . As $U_+ = U \otimes E_+ \otimes U_0$, we can then define d_* on U_+ by

$$d_*(xyz) = (-1)^p x d_*(y)z$$

for $x \in U_p, y \in E_+, z \in U_0$. Clearly d_* is right U_0 -linear and left skew U -linear. Also $d_*(E_\nu U_0) \subset E_{\nu_-} U_0$ for $\nu \in \mathfrak{A}'$.

(ii) For $x \in E_\nu U_0$ with $\nu \in \mathfrak{A}'$ we have

$$d(x) - d_*(x) \in \bigoplus_{\substack{\mu \in \mathfrak{A}' \\ \mu < \nu_-}} E_\mu U_0$$

and in particular $d_*(x) = \pi_{\nu_-} d(x)$.

PROOF. Let $\nu = (\nu_1, \dots, \nu_r)$. As d and d_* are right U_0 -linear, we may assume $x = x_1 \cdots x_r$ with $x_1 \in E_{\nu_1}, \dots, x_r \in E_{\nu_r}$. Then

$$\begin{aligned} d(x) &= \sum_{i=1}^r \pm x_1 \cdots x_{i-1} d(x_i) x_{i+1} \cdots x_r \\ &= \sum_{i=1}^r \sum_{\lambda} \pm x_1 \cdots x_{i-1} \text{pr}_\lambda d(x_i) x_{i+1} \cdots x_r, \end{aligned}$$

where λ runs over elements of \mathfrak{A} such that $|\lambda| = \nu_i - 1, l(\lambda) \geq 1$ (by (i)). If $i < r$ or if $i = r$ and $\lambda_1 < \nu_r - 1$, then

$$(\nu_1, \dots, \nu_{i-1}) \lambda (\nu_{i+1}, \dots, \nu_r) < \nu_-.$$

If $i = r$ and $\lambda_1 = \nu_r - 1$, then $\lambda = (\nu_r - 1, 0, \dots, 0)$. The sum of the terms for such i, λ is

$$\pm x_1 \cdots x_{r-1} \sum_{k \geq 0} \text{pr}_{(\nu_r - 1)0^k} d(x_r) = d_*(x).$$

Thus $d(x) - d_*(x) \in \bigoplus_{\mu < \nu_-} E_\mu$.

(iii) $\text{pr}_{\lambda\mu} d(xy) = (-1)^{|\lambda|} x \text{pr}_\mu d(y)$ for $\lambda, \mu \in \mathfrak{A}, x \in E_\lambda, y \in U$.

PROOF. Expanding $d(x)$ as above, we see $\text{pr}_{\lambda\mu}(d(x)y) = 0$.

(iv) Let $\mu \in \mathfrak{A}$, $\nu \in \mathfrak{A}'$, $\mu < \nu$. If $\pi_{\nu}d(E_{\mu}) \neq 0$, then $\mu = \nu_0^i 10^j$ for some $i, j \geq 0$.

PROOF. We have $\text{pr}_{\lambda}d(E_{\mu}) \neq 0$ for some $\lambda = \nu_0^k, k \geq 0$. Put $r = l(\nu), q = l(\mu)$, $p = l(\lambda) = r + k$. Then $q \leq p$ by (i), and there exists $h \leq q$ such that

$$\begin{aligned} \mu_1 &= \lambda_1, \dots, \mu_{h-1} = \lambda_{h-1}, \\ \mu_h &= \lambda_h + \dots + \lambda_{h+p-q} + 1, \\ \mu_{h+1} &= \lambda_{h+p-q+1}, \dots, \mu_q = \lambda_p. \end{aligned}$$

If $h < r$, then $\mu > \nu$, a contradiction. If $h = r$, then $\mu = \nu 0^{q-r}$, which is also impossible. Hence $h > r$ and $\mu = \nu_0^i 10^j$ for some i, j .

Let $\eta: R \rightarrow k$ be the projection map. $E \otimes R \cong k \otimes_R (R \otimes E \otimes R)$ becomes a complex with differential $\bar{\partial} = k \otimes_R \partial$ and augmentation $\bar{\sigma} = k \otimes_R \sigma: E \otimes R \rightarrow R$.

(v) We have

$$\bar{\partial}|E: E \xrightarrow{d} E \otimes U \xrightarrow{1 \otimes \varepsilon} E \otimes R, \quad \bar{\sigma}|E = -\varepsilon.$$

PROOF. By the definition of ρ and the fact $\eta\varepsilon(E) = 0$, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{p|U} & R \otimes E \otimes R \\ \cup & & \downarrow \eta \otimes 1 \otimes 1 \\ E \otimes U & \xrightarrow{1 \otimes \varepsilon} & E \otimes R. \end{array}$$

From this and (i), the first assertion follows. The second is clear.

(vi) The following diagram is commutative.

$$\begin{array}{ccc} E_n \otimes U_0 & \xrightarrow{1 \otimes \varepsilon} & E_n \otimes R \\ d_* \downarrow & & \downarrow \bar{\partial} \\ E_{n-1} \otimes U_0 & \xrightarrow{1 \otimes \varepsilon} & E_{n-1} \otimes R. \end{array}$$

PROOF. Follows from (v) and the definition of d_* .

From now on we assume (3) of Theorem 3.

(vii) $\varepsilon: U \rightarrow R$ is onto.

PROOF. The exactness of $E_0 \otimes R \xrightarrow{\bar{\sigma}} R \xrightarrow{\eta} k$ implies $R^+ = \varepsilon(E_0)R$. So R is generated by $\varepsilon(E_0)$ as a k -algebra.

(viii) $U_1 \xrightarrow{d} U_0 \xrightarrow{\varepsilon} R$ is exact.

PROOF. By (vi) we have a commutative diagram

$$\begin{array}{ccccc}
 U_1 & \supset & E_1 \otimes U_0 & \xrightarrow{1 \otimes \varepsilon} & E_1 \otimes R \\
 d \downarrow & & \downarrow d_* & & \downarrow \bar{d} \\
 U_0 & \supset & E_0 \otimes U_0 & \xrightarrow{1 \otimes \varepsilon} & E_0 \otimes R \\
 \varepsilon \downarrow & & & & \downarrow -\bar{\varepsilon} \\
 R & = & & & R.
 \end{array}$$

Consider the ideal $I = \text{Ker}(\varepsilon: U_0 \rightarrow R) \subset U_0^+ = E_0 \otimes U_0$. By the exactness of the right column of the diagram, we have $I \subset d_*(E_1 \otimes U_0) + \text{Ker}(1 \otimes \varepsilon)$, hence $I \subset d(U_1) + E_0 I$. Since $d(U_1)$ is an ideal of U_0 , we have $I = d(U_1)$ by Nakayama's lemma.

(ix) $U_{n+1} \xrightarrow{d} U_n \xrightarrow{d} U_{n-1}$ is exact for $n > 0$.

PROOF. Fix $p \geq 0$. Let us show the exactness in upper degree p . Firstly we note that the set $\{v \in \mathfrak{A}' \mid \pi_v(U_n^p) \neq 0\}$ is finite. Indeed, such v must satisfy $|v| = n$ and $\#\{i \mid v_i = 0\} \leq p$ because $E_0^0 = 0$. So $l(v) \leq n + p$.

For $0 \neq x \in U_n^p$, let $H(x)$ be the greatest element of $\{v \in \mathfrak{A}' \mid \pi_v(x) \neq 0\}$ with respect to the order $<$. We shall show that if $0 \neq x \in \text{Ker}(d: U_n^p \rightarrow U_{n-1}^p)$, then there exists $y \in U_{n+1}^p$ such that $H(x - d(y)) < H(x)$ or $x - d(y) = 0$. Then the exactness will follow by induction.

Put $H(x) = v = (v_1, \dots, v_r)$ and $v_r = m > 0, v' = (v_1, \dots, v_{r-1})$. Also put $x_\mu = \pi_\mu(x)$ for $\mu \in \mathfrak{A}'$. Then $x = x_v + \sum_{\mu < v} x_\mu$. We have

$$0 = \pi_{v_-} d(x) = \pi_{v_-} d(x_v) + \sum_{\mu < v} \pi_{v_-} d(x_\mu).$$

By (ii), $\pi_{v_-} d(x_v) = d_*(x_v)$. If $\mu < v$ and $\pi_{v_-} d(x_\mu) \neq 0$, then, by (iv), $\mu^0 k = v_- 0^i 1 0^j$ for some i, j, k . Hence $\mu = v_- 0^i 1$ as $\mu \in \mathfrak{A}'$. So $x_\mu \in E_{v_-} \otimes U_1$. Then, by (iii), $\pi_{v_-} d(x_\mu) \in E_{v_-} \otimes d(U_1)$. Thus we know $d_*(x_v) \in E_{v_-} \otimes d(U_1)$. Hence $(1 \otimes \varepsilon) d_*(x_v) = 0 (*)$.

By (vi), the diagram

$$\begin{array}{ccc}
 E_{v'} \otimes E_{m+1} \otimes U_0 & \xrightarrow{1 \otimes 1 \otimes \varepsilon} & E_{v'} \otimes E_{m+1} \otimes R \\
 d_* \downarrow & & \downarrow 1 \otimes \bar{d} \\
 E_{v'} \otimes E_m \otimes U_0 & \xrightarrow{1 \otimes 1 \otimes \varepsilon} & E_{v'} \otimes E_m \otimes R \\
 d_* \downarrow & & \downarrow 1 \otimes \bar{d} \\
 E_{v'} \otimes E_{m-1} \otimes U_0 & \xrightarrow{1 \otimes 1 \otimes \varepsilon} & E_{v'} \otimes E_{m-1} \otimes R
 \end{array}$$

commutes up to sign. By (*), (vii) and the exactness of the right column, there

exists $z \in (E_{v'} \otimes E_{m+1} \otimes U_0)^p$ such that

$$x_v - d_*(z) \in \text{Ker}(E_v \otimes U_0 \xrightarrow{1 \otimes \varepsilon} E_v \otimes R) = E_v \otimes d(U_1).$$

Since d_* operates as $\pm 1 \otimes d$ on $E_v \otimes U_1$, we have $x_v - d_*(z) = d_*(u)$ for some $u \in (E_v \otimes U_1)^p$.

But by (ii),

$$\begin{aligned} d(z) - d_*(z) &\in \bigoplus_{\mu < v} E_\mu U_0 \\ d(u) - d_*(u) &\in \bigoplus_{\mu < v} E_\mu U_0, \end{aligned}$$

where $\mu \in \mathfrak{A}'$. Hence

$$x - d(z+u) = x_v - d(z+u) + \sum_{\mu < v} x_\mu \in \bigoplus_{\mu < v} E_\mu U_0$$

as required.

5. Examples of resolutions

We shall first give a free dga resolution of a Koszul algebra. Let R be a connected graded algebra generated by elements of degree 1 with defining relations of degree 2. So we can write as $R = T(V)/(I)$ where $I \subset V \otimes V$. Put

$$I^{(n)} = \bigcap_{i+j=n-2} V^{\otimes i} \otimes I \otimes V^{\otimes j} \subset V^{\otimes n}$$

for $n \geq 0$. We understand $I^{(0)} = k$, $I^{(1)} = V$. R is called a Koszul algebra if the following complex of right R -modules is exact.

$$\rightarrow I^{(n)} \otimes R \rightarrow I^{(n-1)} \otimes R \rightarrow \dots \rightarrow R \rightarrow k \rightarrow 0.$$

Here the differential is induced by the inclusion maps

$$I^{(n)} \subset I^{(n-1)} \otimes V \subset I^{(n-1)} \otimes R.$$

For equivalent definitions of Koszul algebras, see [1], [4].

Let $\Delta_{p,q} : I^{(p+q)} \rightarrow I^{(p)} \otimes I^{(q)}$ be the inclusion map for $p, q > 0$. Let $E = \bigoplus_{n \geq 1} I^{(n)}$ with bigrading $E_n = E^{n+1} = I^{(n+1)}$. Put $U = T(E)$. Let $d : U \rightarrow U$ be the derivation such that

$$d(x) = \sum_{\substack{p+q=n+1 \\ p,q > 0}} (-1)^p \Delta_{p,q}(x) \quad \text{for } x \in E_n.$$

Let $\varepsilon : U \rightarrow R$ be the algebra map such that $\varepsilon(x) = x$ for $x \in E_0 = V$ and $\varepsilon(x) = 0$ for $x \in E_+$.

PROPOSITION 5.1. $\varepsilon : (U, d) \rightarrow R$ is a free dga resolution.

PROOF. $d^2 = 0$ follows from the coassociativity of $\Delta_{p,q}$. $\varepsilon d = 0$ is clear. Let $\bar{\partial}$ be as in Theorem 3. For $x \in E_n$ we have

$$\bar{\partial}(x) = (1 \otimes \varepsilon) \sum_{p+q=n+1} (-1)^p \Delta_{p,q}(x) = (-1)^n \Delta_{n,1}(x).$$

Hence, up to sign, $\bar{\partial}$ coincides with the differential of the above free resolution of the R -module k . So $\varepsilon : U \rightarrow R$ is a resolution.

We next introduce a generalization of a Koszul algebra, for which we give a free dga resolution. Fix an integer $e \geq 2$. Let $R = T(V)/(I)$ with $I \subset V^{\otimes e}$. As before, define

$$I^{(n)} = \bigcap_{i+j=n-e} V^{\otimes i} \otimes I \otimes V^{\otimes j} \subset V^{\otimes n}.$$

Consider the complex of right R -modules

$$\rightarrow I^{(e+1)} \otimes R \rightarrow I^{(e)} \otimes R \rightarrow \dots \rightarrow I^{(1)} \otimes R \rightarrow I^{(0)} \otimes R \rightarrow k \rightarrow 0$$

where the differential is induced by the inclusion maps

$$\begin{aligned} I^{(e+1)} &\subset I^{(e)} \otimes V \subset I^{(e)} \otimes R \\ I^{(e)} &\subset I^{(e(n-1)+1)} \otimes V^{\otimes(e-1)} \subset I^{(e(n-1)+1)} \otimes R. \end{aligned}$$

We say R is an e -Koszul algebra if the above complex is exact.

REMARK. (i) 2-Koszul just means Koszul. (ii) $k[x]/(x^e)$ is e -Koszul. (iii) Let $J \subset V \otimes V$. If $T(V)/(J)$ is Koszul, then $T(V)/(J^{(e)})$ is e -Koszul. We omit the proof.

Let us give a free dga resolution of an e -Koszul algebra $R = T(V)/(I)$. Let

$$\begin{aligned} E &= \bigoplus_{n \geq 0} I^{(e+1)} \oplus \bigoplus_{n \geq 1} I^{(en)} \\ E_{2n} &= E^{en+1} = I^{(e+1)}, \quad E_{2n-1} = E^{en} = I^{(en)}. \end{aligned}$$

Put $E_{\text{cv}} = \bigoplus_{n \geq 0} E_{2n}$, $E_{\text{od}} = \bigoplus_{n \geq 1} E_{2n-1}$. Let

$$\begin{aligned} \delta_{11} : E_{\text{od}} &\rightarrow E_{\text{od}} \otimes E_{\text{od}} \\ \delta_0 : E_{\text{od}} &\rightarrow E_{\text{cv}}^{\otimes e} \\ \delta_{10} : E_{\text{cv}} &\rightarrow E_{\text{od}} \otimes E_{\text{cv}} \\ \delta_{01} : E_{\text{cv}} &\rightarrow E_{\text{cv}} \otimes E_{\text{od}} \end{aligned}$$

be the maps whose components are respectively the inclusion maps

$$\begin{aligned}
 I^{(e(i+j))} &\rightarrow I^{(ei)} \otimes I^{(ej)} \\
 I^{(e(i_1+\dots+i_r+1))} &\rightarrow I^{(ei_1+1)} \otimes \dots \otimes I^{(ei_r+1)} \\
 I^{(e(i+j)+1)} &\rightarrow I^{(ei)} \otimes I^{(ej+1)} \\
 I^{(e(i+j)+1)} &\rightarrow I^{(ei+1)} \otimes I^{(ej)}.
 \end{aligned}$$

Put $U = T(E)$. Let $d : U \rightarrow U$ be the derivation such that

$$E = \bigoplus_{n \geq 0} d = \begin{cases} \delta_0 - \delta_{11} & \text{on } E_{\text{od}} \\ \delta_{01} - \delta_{10} & \text{on } E_{\text{ev}}. \end{cases}$$

Let $\varepsilon : U \rightarrow R$ be the algebra map which is the identity on $E_0 = V$ and vanishes on E_+ .

PROPOSITION 5.2. $\varepsilon : (U, d) \rightarrow R$ is a free dga resolution.

PROOF. Again $d^2 = 0$ is a consequence of the coassociativity of the maps $\delta_{11}, \delta_0, \delta_{10}, \delta_{01}$. Recall the description of $\bar{\partial}$ after Theorem 3. We have the equalities of maps

$$\begin{aligned}
 &(E_{2n-1} \xrightarrow{\delta_0} (E \otimes U)_{2n-2} \xrightarrow{1 \otimes \varepsilon} E_{2n-2} \otimes R) \\
 &= (E_{2n-1} \hookrightarrow E_{2n-2} \otimes E_0^{\otimes(e-1)} \hookrightarrow E_{2n-2} \otimes R), \\
 &(E_{2n} \xrightarrow{\delta_{10}} (E \otimes E)_{2n-1} \xrightarrow{1 \otimes \varepsilon} E_{2n-1} \otimes R) \\
 &= (E_{2n} \hookrightarrow E_{2n-1} \otimes E_0 \hookrightarrow E_{2n-1} \otimes R), \\
 &(1 \otimes \varepsilon)\delta_{01} = 0, \\
 &(1 \otimes \varepsilon)\delta_{11} = 0.
 \end{aligned}$$

Hence $\bar{\partial}$ equals the differential of the free resolution of the R -module k up to sign. So by Theorem 3 $\varepsilon : U \rightarrow R$ is a resolution.

References

- [1] J. Backelin and R. Fröberg, Koszul Algebras, Veronese subrings and rings with linear resolutions, Rev. Roumaine Math. Pures, Appl. **30** (1985) 85–97.
- [2] H. J. Baues, Algebraic Homotopy, Cambridge University Press, Cambridge, 1989.
- [3] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, New Jersey, 1956.
- [4] S. Priddy, Koszul resolutions, Trans. Amer. Math. Soc. **152** (1970) 39–60.
- [5] M. U. Shukla, Cohomologie des algèbres associatives, Ann. Scient. Éc. Norm. Sup. **78** (1961) 163–209.

Department of Mathematics
Hirosaki University
Hirosaki 036,
Japan