

SOME ALMOST-HOMOGENEOUS COMPLEX STRUCTURES ON $P^2 \times P^2$

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1. Introduction

It is well known that on $P^1(C) \times P^1(C)$ there exists an infinite sequence of different complex structures, namely the Hirzebruch surfaces Σ_{2m} , $m \in N$. These surfaces are of the form $P(\mathcal{O}_{P^1}(m) \oplus \mathcal{O}_{P^1}(-m))$ and are all almost-homogeneous (see [H]). In generalization of this, Brieskorn has studied P^n -bundles over P^1 and has proved that all complex structures on $P^1 \times P^n$ satisfying some supplementary conditions (see [Br], (5.3)) are such P^n -bundles. All these structures are almost-homogeneous.

Motivated by these results, it is natural to consider complex structures on $P^2 \times P^2$ of the form $P(E)$, where E is a topologically trivial holomorphic vector bundle of rank 3 on P^2 . In contrast with the situation on P^1 , a complete classification of such bundles is not known, however Bănică has classified all topologically trivial rank 2 vector bundles on P^2 (see [B], §2). In particular these bundles do not depend only on discrete parameters, but also on "continuous" moduli. Using rank 3 vector bundles on P^2 of the form $E := F \oplus \mathcal{O}_{P^2}$, with F topologically trivial of rank 2, one can easily construct complex structures on $P^2 \times P^2$, depending on "continuous" moduli, which are not almost-homogeneous.

Here we study some examples of almost-homogeneous complex structures on $P^2 \times P^2$ of the form $P(E)$, for homogeneous and almost-homogeneous E . In §2 are studied the cases when E is $T_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-1)$ or its dual (together with $\mathcal{O}_{P^2}^{\oplus 3}$, these are the only topologically trivial homogeneous rank 3 vector bundles on P^2 , see for example [M]). It turns out that the automorphism group of $X_1 := P(T_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-1))$ has an open orbit, whose complement is an irreducible homogeneous hypersurface (hence X_1 gives an example of the manifolds classified by Ahiezer [Ah]), while the automorphism group of $X_2 := P(T_{P^2}(-2) \oplus \mathcal{O}_{P^2}(1))$ has an open orbit, whose complement is irreducible and homogeneous of codimension 2. In §3 we consider the complex manifold $X := P(F \oplus \mathcal{O}_{P^2})$ with F a topologically trivial rank 2 vector bundle on P^2 of generic splitting type $(-1, 1)$ and we prove that the automorphism group of X has an open orbit, whose complement is an irreducible hypersurface, which contains a whole fiber of $P(E)$.

Section 1.

In this section we introduce some notations and preliminary material.

Here a vector bundle is always a holomorphic vector bundle and we often identify vector bundles and locally free sheaves. Following the notations of [OSS] for a vector bundle $E \rightarrow S$ we denote by $E(x)$ the fiber over a point $x \in S$.

Let S be a complex manifold, $E \rightarrow S$ a rank m vector bundle on S , and let $P(E) \rightarrow S$ be the corresponding projective bundle. Set $X := P(E)$. We denote by

- $\text{Aut}(E)$ the group of all biholomorphic maps $E \rightarrow E$, which transform fibers in fibers and are linear on fibers;
- $\text{Aut}(P(E))$ the group of all biholomorphic maps of $P(E)$ into itself, which transform fibers in fibers;
- $\text{Aut}_S(E), \text{Aut}_S(P(E))$ the subgroups of all elements of $\text{Aut}(E)$ and $\text{Aut}(P(E))$ respectively, which induce the identity on S ;
- $\text{PGL}(E)$ the subgroup of all elements of $\text{Aut}_S(P(E))$, which are induced by elements of $\text{Aut}_S(E)$;
- $\text{Aut}(X)$ the group of all biholomorphic maps of X onto itself.

(1.1) PROPOSITION.

With the same notations as above, let us suppose S simply connected. Then

$$\text{Aut}_S(P(E)) = \text{PGL}(E).$$

PROOF: Let U be a simply connected open subset of S such that $E|U$ is trivial. An element $\phi \in \text{Aut}_U(P(E|U))$ can be regarded as a holomorphic map $\bar{\phi}: U \rightarrow \text{PGL}(m)$. Since U is simply connected and $\text{SL}(m)$ is a covering space of $\text{PGL}(m)$, the map $\bar{\phi}$ can be lifted to a holomorphic map $\tilde{\phi}: U \rightarrow \text{SL}(m)$, and this gives an element $\Phi \in \text{Aut}_S(E|U)$, which induces ϕ .

Now let $\phi \in \text{Aut}_S(P(E))$ and let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open covering of S such that every U_i is simply connected and $E|U_i$ is trivial. Then for each U_i there exists $\Phi_i \in \text{Aut}_{U_i}(E|U_i)$ with $\det \Phi_i = 1$, which induces ϕ . On $U_i \cap U_j$ the two matrices Φ_i and Φ_j induce the same projective automorphism ϕ , therefore they coincide up to the multiplication by an m -th root of unity. Thus we obtain an element of $H^1(S, \mu_m)$, where μ_m is the locally constant sheaf of m -th roots of unity. Since S is simply connected $H^1(S, \mu_m) = 0$, hence $\Phi_i \cdot \Phi_j^{-1} = \lambda_i \cdot \lambda_j^{-1}$ where λ_i, λ_j are m -th roots of unity. Thus the $\lambda_i^{-1} \cdot \Phi_i$ can be glued together to give an element $\Phi \in \text{Aut}_S(E)$ which induces ϕ .

We recall the following

(1.2) DEFINITION.

Let S be a homogeneous complex manifold and let $E \rightarrow S$ be a vector bundle on S .

We say that E is *homogeneous* if for all $g \in \text{Aut}(S)$ one has $g^*E \simeq E$.

We say that E is *almost-homogeneous* if there exists a subgroup G of $\text{Aut}(S)$, one of whose orbits is a Zariski open dense subset of S , such that $g^*E \simeq E$ for all $g \in G$.

(1.3) COROLLARY.

Let S be a homogeneous simply connected complex manifold, $E \rightarrow S$ a homogeneous vector bundle on S and let $\mathbf{P}(E) \rightarrow S$ be the corresponding projective bundle.

Then there is the exact sequence

$$0 \rightarrow \text{PGL}(E) \rightarrow \text{Aut}(\mathbf{P}(E)) \rightarrow \text{Aut}(S) \rightarrow 0.$$

PROOF: This is an obvious consequence of (1.1).

Section 2.

In this section we show that the complex manifolds $X_1 := \mathbf{P}(T_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2})$ and $X_2 := \mathbf{P}(T_{\mathbf{P}^2}(-3) \oplus \mathcal{O}_{\mathbf{P}^2})$ are almost-homogeneous and we determine the orbits with respect to the action of the group of automorphisms.

It is well known (see [A], th. 3) that in the decomposition $E := E_1 \oplus \dots \oplus E_n$ of a vector bundle over a compact variety into direct sum of indecomposable bundles, the bundles E_i are uniquely determined up to order and isomorphy; however in general the bundles E_i are not uniquely determined as subbundles of E .

(2.1) LEMMA.

Let $E_1 := T_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}$. In this decomposition only the vector bundle $F_1 := T_{\mathbf{P}^2} \oplus 0$ is uniquely determined as subbundle of E_1 .

PROOF: We first observe that the vector bundle $G_1 := 0 \oplus \mathcal{O}_{\mathbf{P}^2}$ is not uniquely determined as subbundle of E , since it is not invariant under an automorphism $\phi \in \text{Aut}_S(E_1)$ of the form

$$\left(\begin{array}{c|c} id_{T_{\mathbf{P}^2}} & \psi \\ \hline 0 & id_{\mathcal{O}_{\mathbf{P}^2}} \end{array} \right)$$

with $\psi \in \text{Hom}(\mathcal{O}_{\mathbf{P}^2}, T_{\mathbf{P}^2})$, $\psi \neq 0$.

On the other hand, from the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbf{P}^2}^{\oplus 3} \rightarrow T_{\mathbf{P}^2}(-1) \rightarrow 0,$$

it follows that the vector bundle $T_{\mathbf{P}^2}(-1)$ is generated by global sections, therefore $F_1(-1)$ is the subbundle of $E_1(-1)$ generated by $\Gamma(\mathbf{P}^2, E_1(-1))$ and this characterizes $F_1(-1)$ as subbundle of $E_1(-1)$.

(2.2) COROLLARY.

Let $E_2 := T_{P^2}(-3) \oplus \mathcal{O}_{P^2}$. In this decomposition only the vector bundle $G_2 := 0 \oplus \mathcal{O}_{P^2}$ is uniquely determined as subbundle of E_2 .

PROOF: Since E_2 is the dual bundle of E_1 and G_2 consists of the linear forms on E_1 , which are zero on $F_1 := T_{P^2} \oplus 0$, the assertion follows from (2.1).

(2.3) THEOREM.

Let $E_1 := T_{P^2} \oplus \mathcal{O}_{P^2}$ and let $X_1 := P(E_1)$. The group $\text{Aut}(X_1)$ has exactly two orbits: $A_1 := P(T_{P^2} \oplus 0)$ and $A_2 := X_1 - A_1$.

PROOF: We first prove that A_1 is transformed into itself by all $\phi \in \text{Aut}(X_1)$. By [S], th. A, $\text{Aut}(X_1) = \text{Aut}(P(E_1))$, hence ϕ determines an automorphism $\bar{\phi} \in \text{Aut}(S)$ and an isomorphism $P(E_1) \simeq \bar{\phi}^* P(E_1)$, which induces the identity on P^2 and which can be identified with ϕ . Therefore E_1 is isomorphic to $\bar{\phi}^* E_1 \otimes \mathcal{O}_{P^2}(k)$ and calculating the first Chern classes one sees that $k=0$. Thus ϕ induces an isomorphism $\Phi: E_1 \simeq \bar{\phi}^* E_1$, which must transform $(T_{P^2} \oplus 0)(x)$ into $(\bar{\phi}^*(T_{P^2} \oplus 0))(x) = (T_{P^2} \oplus 0)(\bar{\phi}(x))$ for all $x \in P^2$. Therefore $\phi(P(T_{P^2} \oplus 0)) = P(T_{P^2} \oplus 0)$.

Now we prove that the action of $\text{Aut}(X_1)$ is transitive on both A_1 and A_2 , by showing that for all $x \in P^2$ the subgroup of $\text{Aut}(X_1)$, which fixes the fiber $P(E_1)_x$, acts transitively on $A_1 \cap P(E_1)_x$ and on $A_2 \cap P(E_1)_x$. Let $\xi, \xi' \in A_1 \cap P(E_1)_x$. They correspond to lines r, r' of P^2 through the point x . Let $\alpha \in \text{Aut}(P^2)$ be such that $\alpha(x) = x$ and $\alpha(r) = r'$ and take an element $\phi \in \text{Aut}(P(E_1))$ such that $\bar{\phi} = \alpha$. It is easy to show that $\phi(\xi) = \xi'$.

$$\text{Since } \text{End } E_1 = \left(\begin{array}{c|c} \text{End } T_{P^2} & \text{Hom}(\mathcal{O}_{P^2}, T_{P^2}) \\ \hline \text{Hom}(T_{P^2}, \mathcal{O}_{P^2}) & \text{End } \mathcal{O}_{P^2} \end{array} \right)$$

and since $\text{Aut}_{P^2}(T_{P^2}) \simeq \text{Aut}_{P^2}(\mathcal{O}_{P^2}) \simeq C^*$ the action on $P(E_1)_x$ of an element of $\text{PGL}(E_1)$ can be thought as the action on P^2 , with projective coordinates $(x_1: x_2: x_3)$, of a matrix like

$$\begin{pmatrix} \lambda & 0 & a_1 \\ 0 & \lambda & a_2 \\ 0 & 0 & \mu \end{pmatrix} \text{ with } \lambda, \mu \in C^*, a_1, a_2 \in C.$$

In this P^2 , $A_1 \cap P(E_1)_x$ can be identified with the line $x_3 = 0$ and $A_2 \cap P(E_1)_x$ with the complement of such a line; therefore it is clear that $\text{PGL}(E)$ acts transitively on $A_2 \cap P(E_1)$.

(2.4) THEOREM.

Let $F_2 := T_{P^2}(-3) \oplus \mathcal{O}_{P^2}$ and let $X_2 := P(E_2)$. The orbits of X_2 with respect to the action of $\text{Aut}(X_2)$ are exacty $B_1 := P(0 \oplus \mathcal{O}_{P^2})$ and $B_2 := X_2 - B_1$.

PROOF: With an argument similar to the one used in prop. (2.3) and using (2.2), one has that all $\phi \in \text{Aut}(X_2) = \text{Aut}(P(E_2))$ transform B_1 into itself.

Since for all $x \in \mathbf{P}^2$, $B_1 \cap \mathbf{P}(E_2)_x$ consists exactly of one point, we have only to prove that for all $x \in \mathbf{P}^2$ the action of the subgroup Σ of $\text{Aut}(X_2)$, containing all automorphisms, which fix the fiber $\mathbf{P}(E_2)_x$, is transitive on $B_2 \cap \mathbf{P}(E_2)_x$.

Let $(x_1:x_2:x_3)$ and $(y_1:y_2:y_3)$ be two points in $B_2 \cap \mathbf{P}(E_2)_x$. With an argument similar to the one used in proving prop. (2.3), there exists $\phi \in \Sigma$, which transforms $(x_1:x_2:x_3)$ into $(y_1:y_2:y_3)$. Now there exists an element ψ in $\text{PGL}(E_2)$ (whose action on $\mathbf{P}(E_2)_x$ can be thought as the action on \mathbf{P}^2 of a matrix like

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ a_1 & a_2 & \mu \end{pmatrix} \text{ with } \lambda, \mu \in \mathbf{C}^*, a_1, a_2 \in \mathbf{C}, \text{ which transforms } (y_1:y_2:y_3) \text{ in } (y_1:y_2:y_3).$$

Section 3.

In this section we show that the complex manifold $\mathbf{P}(F \oplus \mathcal{O}_{\mathbf{P}^2})$, where F is a rank 2 topologically trivial vector bundle on \mathbf{P}^2 of generic splitting type $(-1, 1)$, is almost-homogeneous.

(3.1) PROPOSITION.

Let F be a rank 2 topologically trivial vector bundle on \mathbf{P}^2 of generic splitting type $(-1, 1)$.

Then

i) there is an exact sequence:

$$(*) \quad 0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(1) \xrightarrow{\alpha} F \xrightarrow{\beta} \mathcal{I}_Z(-1) \rightarrow 0,$$

where Z is a simple point of \mathbf{P}^2 , which determines the bundle F up to isomorphy;

- ii) $F \simeq F^\vee$ (that is F is self-dual);
- iii) F is almost-homogeneous.

PROOF: i) The existence of the exact sequence $(*)$ has been proved by Bănică (see [B], lemma 4).

Now let F' be a vector bundle on \mathbf{P}^2 , which makes exact the sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(1) \rightarrow F' \rightarrow \mathcal{I}_Z(-1) \rightarrow 0.$$

Both F and F' correspond to elements $\eta, \eta' \in \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_{\mathbf{P}^2}(1))$, which are not zero, since the trivial extension $\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{I}_Z(-1)$ is not a vector bundle. But, from [B], §2, $\dim \text{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_{\mathbf{P}^2}(1))=1$, therefore $\eta = a\eta'$ with $a \in \mathbf{C}^*$, hence $F \simeq F'$.

- ii) Since F has rank 2, we have $F^\vee \simeq F \otimes \det F^\vee \simeq F$.
- iii) Let $G := \{g \in \text{Aut}(\mathbf{P}^2) | g(Z) = Z\}$, and let $g \in G$.

The vector bundle g^*F makes exact the sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow g^*F \rightarrow \mathcal{I}_Z(-1) \rightarrow 0$$

and with the same argument used in (i), $g^*F \simeq F$.

(3.2) LEMMA.

Let $E := F \oplus \mathcal{O}_{P^2}$, with F as in (3.1), and let $\mathcal{V}_1 := \alpha(\mathcal{O}_{P^2}(1)) \oplus 0$, $\mathcal{V} := \alpha(\mathcal{O}_{P^2}(1)) \oplus \mathcal{O}_{P^2}$. The filtration $\mathcal{V}_1 \subset \mathcal{V} \subset E$ is invariant with respect to $\text{Aut}(E)$.

PROOF: From the exact sequence (*), one has $\Gamma(P^2, F) = \Gamma(P^2, \alpha(\mathcal{O}_{P^2}(1)))$. It follows $\mathcal{V} = \mathcal{O}_{P^2} \cdot \Gamma(P^2, E)$. In the same way, from the exact sequence

$$0 \rightarrow \mathcal{O}_{P^2} \xrightarrow{\alpha} F(-1) \xrightarrow{\beta} \mathcal{I}_Z(-2) \rightarrow 0$$

one has $\Gamma(P^2, F(-1)) = \Gamma(P^2, \alpha'(\mathcal{O}_{P^2}))$, hence $\mathcal{V}_1(-1) = \mathcal{O}_{P^2} \cdot \Gamma(P^2, E(-1))$.

(3.3) THEOREM.

Let $E := F \oplus \mathcal{O}_{P^2}$, with F as in (3.1), and let $X := P(E)$. The action of $\text{Aut}(X)$ on X has an open orbit, whose complement is an irreducible hypersurface $H \subset X$, which can be described as follows: let V be the subbundle of $E|_{P^2 - Z}$ defined by $V := (\alpha(\mathcal{O}_{P^2}(1)) \oplus \mathcal{O}_{P^2})|_{P^2 - Z}$. Then $H = P(V) \cup P(E)_Z$.

PROOF: We first observe that, from the fact that Z is characterized by the property that every non-zero section of $E(-1)$ vanishes exactly on Z (see the proof of (3.2)), it follows that every $\phi \in \text{Aut}(X)$ ($= \text{Aut}(P(E))$) by [S], th. A) transforms $P(E)_Z$ into itself. By lemma (3.2), with an argument similar to the one used in proving prop. (2.3), one has that every $\phi \in \text{Aut}(X)$ transforms also $P(V)$ into itself.

Now we show that $\text{Aut}(X)$ acts transitively on $A := X - (P(V) \cup P(E)_Z)$, by proving that for all $x \in P^2 - Z$ the subgroup $\text{PGL}(E)$ of $\text{Aut}(X)$ acts transitively on $A \cap P(E)_x$.

We observe that $\text{End } E = \left(\begin{array}{c|c} \text{End } F & \text{Hom}(\mathcal{O}_{P^2}, F) \\ \hline \text{Hom}(F, \mathcal{O}_{P^2}) & \text{End } \mathcal{O}_{P^2} \end{array} \right)$.

From the exact sequence (*) we get the exact sequence

$$(**) \quad 0 \rightarrow \text{Hom}(F, \mathcal{O}_{P^2}(1)) \xrightarrow{\sigma} \text{End } F \xrightarrow{\tau} \text{Hom}(F, \mathcal{I}_Z(-1)) \rightarrow \dots$$

Since the endomorphisms of F , which are in $\text{Im } \sigma = \text{Ker } \tau$, cannot be surjective, $\text{id}_F \notin \text{Ker } \tau$, hence $\dim(\text{Hom}(F, \mathcal{I}_Z(-1))) \geq 1$. On the other hand, from (*) we have also the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{I}_Z(-1), \mathcal{I}_Z(-1)) \rightarrow \text{Hom}(F, \mathcal{I}_Z(-1)) \rightarrow \text{Hom}(\mathcal{O}_{P^2}(1), \mathcal{I}_Z(-1)) \rightarrow \dots$$

where $\text{Hom}(\mathcal{O}_{P^2}(1), \mathcal{I}_Z(-1)) = \text{Hom}(\mathcal{O}_{P^2}(1), \mathcal{O}_{P^2}(-1)) = 0$. Therefore $\text{Hom}(F, \mathcal{I}_Z(-1)) \simeq \text{Hom}(\mathcal{I}_Z(-1), \mathcal{I}_Z(-1)) = \text{Hom}(\mathcal{I}_Z(-1), \mathcal{O}_{P^2}(-1))$ and the last, by Riemann's extension theorem, is isomorphic to $\text{Hom}(\mathcal{O}_{P^2}(-1), \mathcal{O}_{P^2}(-1)) \simeq \mathbb{C}$. Therefore, $\text{Hom}(F, \mathcal{I}_Z(-1)) \simeq \mathbb{C}$, and, since $\tau(\text{id}_F) \neq 0$, the morphism τ of (**) is surjective and (**) becomes

$$(**)' \quad 0 \rightarrow \text{Hom}(F, \mathcal{O}_{P^2}(1)) \xrightarrow{\sigma} \text{End } F \xrightarrow{\tau} \text{Hom}(F, \mathcal{I}_Z(-1)) \rightarrow 0.$$

Again from (*), we have the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{G}_Z(-1), \mathcal{O}_{\mathbf{P}^2}(1)) \rightarrow \text{Hom}(F, \mathcal{O}_{\mathbf{P}^2}(1)) \rightarrow \text{End}(\mathcal{O}_{\mathbf{P}^2}(1)) \rightarrow 0,$$

since $H^1(\text{Hom}(\mathcal{G}_Z(-1), \mathcal{O}_{\mathbf{P}^2}(1))) \simeq H^1(\mathcal{O}_{\mathbf{P}^2}(2)) = 0$

as $\text{codim } Z = 2$. Moreover, since $\text{Hom}(\mathcal{G}_Z(-1), \mathcal{O}_{\mathbf{P}^2}(1)) \simeq \text{Hom}(\mathcal{O}_{\mathbf{P}^2}(-1), \mathcal{O}_{\mathbf{P}^2}(1)) \simeq \mathcal{O}_{\mathbf{P}^2}(2)$ and $\text{End}(\mathcal{O}_{\mathbf{P}^2}(1)) \simeq \mathcal{O}_{\mathbf{P}^2}$ are globally generated, for any point x in $\mathbf{P}^2 - Z$ every homomorphism of $F(x)$ into $(\mathcal{O}_{\mathbf{P}^2}(1))(x)$ is induced by an element of $\text{Hom}(F, \mathcal{O}_{\mathbf{P}^2}(1))$. Now we fix a point x in $\mathbf{P}^2 - Z$ and a base v_1, v_2 of $F(x)$ such that $v_1 \in \alpha(\mathcal{O}_{\mathbf{P}^2}(1))(x)$. With respect to such a base the automorphisms of $F(x)$ induced by global automorphisms of F can be represented by matrices of the type $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, c \in \mathbf{C}^*, b \in \mathbf{C}$. From (*) we have the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}(1)) \rightarrow \text{Hom}(\mathcal{O}_{\mathbf{P}^2}, F) \rightarrow \text{Hom}(\mathcal{O}_{\mathbf{P}^2}, \mathcal{G}_Z(-1)) = 0,$$

hence $\text{Hom}(\mathcal{O}_{\mathbf{P}^2}, F) \simeq \text{Hom}(\mathcal{O}_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}(1))$, that is every homomorphism of $\mathcal{O}_{\mathbf{P}^2}$ into F has values in $\alpha(\mathcal{O}_{\mathbf{P}^2}(1))$. From (*) we have also

$$0 \rightarrow \text{Hom}(\mathcal{G}_Z(-1), \mathcal{O}_{\mathbf{P}^2}) \rightarrow \text{Hom}(F, \mathcal{O}_{\mathbf{P}^2}) \rightarrow \text{Hom}(\mathcal{O}_{\mathbf{P}^2}(1), \mathcal{O}_{\mathbf{P}^2}) = \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-1)) = 0,$$

hence $\text{Hom}(F, \mathcal{O}_{\mathbf{P}^2}) \simeq \text{Hom}(\mathcal{G}_Z(-1), \mathcal{O}_{\mathbf{P}^2})$, that is every homomorphism of F into $\mathcal{O}_{\mathbf{P}^2}$ is zero on $\alpha(\mathcal{O}_{\mathbf{P}^2}(1))$. Now we complete the base v_1, v_2 of $F(x) \simeq (F \oplus 0)(x)$ to a base $v_1, v_2, v_3 \in E(x)$ by adding a vector $v_3 \in (0 \oplus \mathcal{O}_{\mathbf{P}^2})(x)$. With respect to such a base the automorphisms of $E(x)$ induced by global automorphisms of E can be represented by a matrix of the type

$$\begin{pmatrix} a & b & d \\ 0 & c & 0 \\ 0 & e & f \end{pmatrix} \text{ with } a, c, f \in \mathbf{C}^*, b, d, e \in \mathbf{C}.$$

Since $A \cap \mathbf{P}(E)_x$ can be identified with the complement of the line $x_2 = 0$, an easy computation shows that the action of $\text{PGL}(E)$ on $A \cap \mathbf{P}(E)_x$ is transitive.

Now we prove that $H := \mathbf{P}(V) \cup \mathbf{P}(E)_Z$ is an irreducible hypersurface in X , by showing that $H = \overline{\mathbf{P}(V)}$. Let U be an open neighbourhood of Z , over which the bundles F and $\mathcal{O}_{\mathbf{P}^2}(1)$ are trivial and let (e_1, e_2) and e bases of $F|U$ and $\mathcal{O}_{\mathbf{P}^2}(1)|U$ respectively. With respect to these bases, the morphism $\alpha: \mathcal{O}_{\mathbf{P}^2}(1) \rightarrow F$ can be described as $\alpha(e) = f_1 e_1 + f_2 e_2$, where f_1, f_2 are holomorphic functions on U , which have exactly one common zero in the point Z . In $\mathbf{P}(E)|U \simeq U \times \mathbf{P}^2$ we have

$$\mathbf{P}(V)|U - Z = \{(p; t_1:t_2:t_3) \in (U - Z) \times \mathbf{P}^2 \mid t_1 f_2(p) - t_2 f_1(p) = 0\},$$

hence

$$\begin{aligned} \overline{\mathbf{P}(V)}|U - Z &= \{(p; t_1:t_2:t_3) \in U \times \mathbf{P}^2 \mid t_1 f_2(p) - t_2 f_1(p) = 0\} \\ &= \mathbf{P}(V)|(U - Z) \cup \mathbf{P}(E)_Z. \end{aligned}$$

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