$$\begin{aligned} T_{12} &= \sum_{a,b=1}^{4} N^a{}_{b1} N^a{}_{b2} = \sum_{a=1}^{4} (N^a{}_{31} N^a{}_{32} + N^a{}_{41} N^a{}_{42}) \\ &= g(N(e_3,e_1), N(e_3,Je_1)) + g(N(e_4,e_1), N(e_4,Je_1)) \\ &= -g(N(e_3,e_1), JN(e_3,e_1)) - g(N(e_4,e_1), JN(e_4,e_1)) \\ &= 0. \end{aligned}$$

Similarly, we have

$$T_{22} = T_{33} = T_{44} = \frac{1}{4}\|N\|^2,$$

and

$$T_{13} = T_{14} = T_{23} = T_{24} = T_{34} = 0.$$

Consequently, we have

$$T_{ij} = \frac{1}{4}\|N\|^2 g_{ij}.$$

By Proposition 3.1, we see that M is an Einstein and weakly $*$-Einstein manifold. Since c and τ are constant on M, τ^* is also constant by (3.9), that is, M is $*$-Einstein. Then, taking account of the theorem of Sekigawa and Vanhecke [10], we can conclude that M is Kählerian. ■

REMARK. From the result of U. K. Kim, I-B. Kim and J-B. Jun [3], it will be also obtained that M is Einstein and weakly $*$-Einstein.

COROLLARY 3.4. *Let M be a 4-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature satisfying*

$$\rho - \rho^* = \frac{\tau - \tau^*}{4} g$$

Then M is a Kähler manifold.

PROOF. This follows from Theorem 3.2 and Theorem 3.3. ■

COROLLARY 3.5. *Let M be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature. If M satisfies the condition* (b)*, then M is a Kähler manifold.*

PROOF. Under the condition (b), we can see that the function c is constant on M ([6]). Hence this follows immediately from Theorem 3.3. ■

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ON AUTOMORPHISMS OF A CHARACTER RING

Dedicated to Professor Tosiro TSUZUKU

By

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1. Introduction

Throughout this paper $G, Z(G)$ and C denote a finite group, the center of G and the field of complex numbers respectively. For a finite set S, we denote the number of elements in S by $|S|$.

Let $Irr(G)$ be the full set of irreducible C-characters of G and $X(G)$ be the character ring of G. If R is any subring of C, we write $RX(G)$ to denote the R-algebra of R-linear combinations of irreducible C-characters of G.

Suppose G and H are finite groups. Weidman showed that if $X(G)$ is isomorphic to $X(H)$, then G and H have the same character table.

In addition Saksonov proved the following theorem, which is a strengthened version of Weidman's theorem.

THEOREM 1.1. (Saksonov) *Suppose R is the ring of all algebraic integers and there exists an R-algebra isomorphism ϕ from $RX(G)$ onto $RX(H)$. If $Irr(G)=\{\chi_1,\cdots,\chi_h\}$ and $Irr(H)=\{\psi_1,\cdots,\psi_h\}$, then the following holds:*

(i) *The character tables of G and H are the same.*

(ii) $\phi(\chi_i)=\varepsilon_i\psi_{i'}$ $(i=1,\cdots,h)$ *where the ε_i are roots of unity and $i\to i'$ is a permutation.*

From now on we assume that R is the ring of all algebraic integers. Then in this paper we intend to prove the following theorem.

THEOREM 1.2. *Suppose G and H are finite groups. Then we have*

(i) *If u is a central element in G and $\tau_u : RX(G)\to RX(G)$ is the map defined by $\chi\to(\chi(u)/\chi(1))\chi$ where $\chi\in Irr(G)$ and 1 is the identity element of G, then τ_u is an R-automorphism of $RX(G)$. Furthermore the map $u\to\tau_u$ is a group isomorphism of $Z(G)$ onto a subgroup $T=\{\tau_u \mid u\in Z(G)\}$ of $Aut(RX(G))$.*

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(ii) *Every R-isomorphism* $\phi : RX(G) \to RX(H)$ *is the composition of an R-isomorphism* θ *that maps* $Irr(G)$ *onto* $Irr(H)$ *with an automorphism of* $RX(H)$ *of the form* τ_u *for some element u in* $Z(H)$.

(iii) *The full group* $A = Aut(RX(G))$ *is the product of the subgroup T of part* (i) *above, which is normal, with the subgroup P consisting of those automorphisms that map* $Irr(G)$ *onto* $Irr(G)$.

2. Proof of Theorem 1.2

In order to prove Theorem 1.2 we prove a basic lemma concerning the roots of unity which appear in Saksonov's Theorem.

LEMMA 2.1. *Suppose for each character* χ *in* $Irr(G)$, *there is a root of unity* $\varepsilon(\chi)$ *such that each product* $\varepsilon(\chi)\chi\varepsilon(\psi)\psi$ *for* χ, ψ *in* $Irr(G)$ *is a non-negative integer linear combination of* $\varepsilon(\xi)\xi$, *as* ξ *runs over* $Irr(G)$. *Then there exists u in* $Z(G)$ *such that* $\varepsilon(\chi) = \chi(u)/\chi(1)$ *for every character* χ *in* $Irr(G)$.

PROOF. If we are given χ and ψ in $Irr(G)$, then we assume that

$$\chi\psi = \sum_{\xi \in Irr(G)} m_\xi \xi \text{ and } \varepsilon(\chi)\chi\varepsilon(\psi)\psi = \sum_{\xi \in Irr(G)} n_\xi \varepsilon(\xi)\xi$$

where the coefficients m_ξ and n_ξ are non-negative integers. Then it follows easily that $m_\xi = n_\xi$ for all characters ξ in $Irr(G)$ and thus the map $\phi : \chi \to \varepsilon(\chi)\chi$ defines an automorphism of the algebra $CX(G)$. In particular the map ϕ permutes the primitive idempotents of this C-algebra (See the proof of Lemma 2.3 in [3]) and so it carries the characteristic class function of the identity to the characteristic class function of some other conjugacy class, say the class K. Therefore we have

$$(1/|G|)\sum_{\chi \in Irr(G)} \varepsilon(\chi)\chi(1)\chi = (1/|C_G(v)|)\sum_{\chi \in Irr(G)} \overline{\chi(v)}\chi$$

where v is an element in K. It follows that for each irreducible character χ in $Irr(G)$ we have $\chi(1)\varepsilon(\chi) = |K|\chi(u)$ where $u = v^{-1}$. Applying this where χ is the principal character yields that $|K|$ is a root of unity and so u is a central element in G. Thus for every character χ in $Irr(G)$, $\varepsilon(\chi) = \chi(u)/\chi(1)$ for some element u in $Z(G)$, as claimed. Q.E.D.

PROOF OF THEOREM 1.2. (i) Suppose u is a central element in G. Then for each character χ in $Irr(G)$ we denote by $\varepsilon(\chi)$ and $T(\chi)$ the root of unity given by $\chi(u)/\chi(1)$ and the irreducible matrix representation of G which affords χ respectively. We assume further that for χ, ψ in $Irr(G)$, $\chi\psi = \sum_{\xi \in Irr(G)} m_\xi \xi$ where the m_ξ are non-negative integers. Then we show $\varepsilon(\xi)=\varepsilon(\chi)\varepsilon(\psi)$ for $m_\xi \neq 0$.

Indeed $T(\chi)(u)=diag(\varepsilon(\chi),\cdots,\varepsilon(\chi))$ and $T(\psi)(u)=diag(\varepsilon(\psi),\cdots,\varepsilon(\psi))$ which have diagonals of lengths $\chi(1)$ and $\psi(1)$ respectively. Hence

$$T(\chi)(u)\otimes T(\psi)(u)=diag(\varepsilon(\chi)\varepsilon(\psi),\cdots,\varepsilon(\chi)\varepsilon(\psi))$$

where $T(\chi)\otimes T(\psi)$ is the Kronecker product of $T(\chi)$ and $T(\psi)$. Since $T(\chi)\otimes T(\psi)$ is the representation of G which affords $\chi\psi$, we have $\varepsilon(\xi)=\varepsilon(\chi)$ $\varepsilon(\psi)$ for $m_\xi \neq 0$, as claimed. Therefore we have $\varepsilon(\chi)\chi\varepsilon(\psi)\psi=\sum_{\xi\in Irr(G)} m_\xi\varepsilon(\xi)\xi$.

Thus the map τ_u defined by $\chi\to\varepsilon(\chi)\chi$ is an R-automorphism of $RX(G)$.

The fact that $Z(G)\cong T$ is easy to prove and so we omit its proof.

(ii) Now we can easily observe that Saksonov's result guarantees that the image of $Irr(G)$ under ϕ satisfies the hypotheses of Lemma 2.1 for H. Hence we may write $\phi(\chi_i)=\varepsilon(\psi_{i'})\psi_{i'}$, $\varepsilon(\psi_{i'})=\psi_{i'}(u)/\psi_{i'}(1)$ for some element u in $Z(H)$, $(i=1,\cdots,h)$ where $Irr(G)=\{\chi_1,\cdots,\chi_h\}$, $Irr(H)=\{\psi_1,\cdots,\psi_h\}$ and $i\to i'$ is a permutation.

Therefore the map τ_u defined by $\psi\to\varepsilon(\psi)\psi$ is an R-automorphism of $RX(H)$ from fact (i) above. If we put $\theta=\tau_u^{-1}\phi$, then $\theta(\chi_i)=\tau_u^{-1}(\phi(\chi_i))=\psi_{i'}$, $(i=1,\cdots,h)$ and so θ maps $Irr(G)$ onto $Irr(H)$. Hence we have $\phi=\tau_u\theta$, as required.

(iii) Fact (iii) follows since fact (ii) tells us that $A=TP$ and it is clear from fact (ii) that A induces a permutation action on $Irr(G)$ and T is the kernel of this action. This completes the proof of the theorem. Q.E.D.

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