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AUTOMORPHISMS OF CERTAIN ROOT LATTICES

By

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0. Introduction.

Let Δ be a reduced irreducible root system of type X_l in a Euclidean space V, in the sense of Bourbaki [1]. Then Δ generates a lattice Γ of rank l in V. We fix the lattice Γ. Let Δ' be another reduced irreducible root system in V, generating Γ, of type X_l. We investigated whether Δ' coincided with Δ, and found out that only the case of C_4 is exceptional. If X_l is not C_4 then Δ' is equal to Δ. This means that (V, Γ, X_l) determines Δ uniquely unless X_l is C_4. In case X_l is C_4, there are three root systems, generating Γ, of type C_4 in V. As we will explain afterward, these are verified by looking at the list of root systems in Bourbaki [1].

Let W be the Weyl group of Δ, and $O(\Gamma)$ the orthogonal group of Γ. Then $W \subseteqq O(\Gamma)$. Let D be the subgroup of $O(\Gamma)$ generated by all symmetries of the Dynkin diagram of Δ. Put $\tilde{W}=\langle W, D\rangle$, the subgroup of $O(\Gamma)$ generated by W and D. Notice that $-I$ (minus identity) is contained in $\tilde{W}$ (cf. [1], [5]). Then the fact in the previous paragraph can be described as follows. The group index $[O(\Gamma):\tilde{W}]$ is 3 if $X_l=C_4$; 1 otherwise.

In this paper, we will calculate the index $[O(\Gamma):\tilde{W}]$ in the case that Δ is the root system of a Kac-Moody Lie algebra of Euclidean type or of low rank hyperbolic type. Let A be a generalized Cartan matrix of Euclidean type or of hyperbolic type, and B the associated form. Let Δ, Γ and $O(\Gamma)$ be the root system of A, the root lattice of Δ and the orthogonal group of Γ associated with B, respectively. We denote by W (resp. D) the Weyl group (resp. the diagram automorphism group) of A. Put $\tilde{W}=\langle W, D, -I\rangle$. It is known that the index $Ind(A)=[O(\Gamma):\tilde{W}]$ is finite (cf. [1; Chap. 5, §4, Ex. 18], [11]). If A is symmetric, then we get $Ind(A)=1$ as a direct consequence of [7; Prop. 1.6] and [12; Theorem 2]. We will compute $Ind(A)$ explicitely when A is of Euclidean type, of rank 2 hyperbolic type or of rank 3 hyperbolic type. The most interesting case is when $A=\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. In this case, we will observe that a certain

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subgroup of $\widetilde{W}$ acts on the infinite set of all solutions (s, t, u, v) of the following Diophantine equation:

$$\begin{cases} s^2-24t^2=1 & \text{(Pell's equation)} \\ u^2-24v^2=1 & \text{(Pell's equation)} \\ su-24tv=-5 & \end{cases}$$

Furthermore this action is transitive. Using this fact, we can establish $Ind(A)=2$.

In the appendix, we display the list of hyperbolic generalized Cartan matrices of rank $\geqq 3$, which is already known but seems to be published explicitly nowhere. (cf. [1].)

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1. Finite type.

Let Δ denote a reduced irreducible root system in V, in the sense of Bourbaki [1]. Let Π be a base of Δ, and Γ the root lattice. We denote by A a Cartan matrix of Δ. Put $Ind(A)=[O(\Gamma):\widetilde{W}]$. Then we can determine $Ind(A)$ using the list of root systems in [1].

THEOREM 1. *If A is of type C_4, then $Ind(A)=3$. Otherwise $Ind(A)=1$.*

PROOF. To show $Ind(A)=1$, we prove that the elements of Δ are characterized by their lengths among the elements of Γ. If A is symmetric (*i.e.* of type A_n, D_n and E_n), Δ is the set of all the non-zero elements of minimal length in Γ (*e.g.* see [7; Prop. 1.6]). The other cases are similarly proved by direct computation.

To treat the case of type C_4 and to show examples, we give the proof in the case of type F_4 and C_4.

F_4: Δ is

$$\left\{\pm e_i\ (1\leqq i\leqq 4),\ \pm e_i\pm e_j\ (1\leqq i<j\leqq 4),\ \frac{1}{2}(\pm e_1\pm e_2\pm e_3\pm e_4)\right\}$$

and Π is

$$\left\{e_2-e_3,\ e_3-e_4,\ e_4,\ \frac{1}{2}(e_1-e_2-e_3-e_4)\right\} \text{ in } \boldsymbol{R}^4,$$

where $\{e_i\}$ is a standard orthonormal basis. It is easy to see that all elements of Γ of length 1 or 2 are contained in Δ. Therefore $O(\Gamma)$ coincides with the Weyl group W, which implies $Ind(F_4)=1$. In particular, the order of $O(\Gamma)$ is $2^7\cdot 3^2$.

C_4: Δ is

$$\{\pm 2e_i\ (1\leqq i\leqq 4),\ \pm e_i\pm e_j\ (1\leqq i<j\leqq 4)\}$$

and Π is

$$\{e_1-e_2,\ e_2-e_3,\ e_3-e_4,\ 2e_4\}\quad \text{in } \boldsymbol{R}^4.$$

The dual root system $\Delta(F_4)^\vee$ of type F_4 is

$$\{\pm 2e_i,\ \pm e_i\pm e_j,\ \pm e_1\pm e_2\pm e_3\pm e_4\}.$$

Therefore the root lattice Γ of Δ is equal to that of $\Delta(F_4)^\vee$. The $Ind(C_4)=[O(\Gamma):\tilde{W}]=2^7\cdot 3^2/2^7\cdot 3=3$. Q.E.D.

2. Euclidean type and hyperbolic type.

An $l\times l$ integral matrix $A=(a_{ij})$ is called a generalized Cartan matrix if $a_{ii}=2\ (1\leqq i\leqq l)$, $a_{ij}\leqq 0\ (1\leqq i\neq j\leqq l)$ and $a_{ij}=0$ whenever $a_{ji}=0$. Cartan matrices arising from root systems in the sense of Bourbaki [1] are generalized Cartan matrices. Such generalized Cartan matrices are called of finite type. A generalized Cartan matrix A is called of Euclidean type if A is singular and possesses the property that removal of any row and the corresponding column leaves a Cartan matrix (i.e. a generalized Cartan matrix of finite type). A generalized Cartan matrix A is called indecomposable (resp. symmetrizable) if A cannot be expressed as $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ under any permutations of indices (resp. if there are positive rational numbers $q_1, \cdots, q_l$ such that $q_i a_{ij}=q_j a_{ji}$ for any $i, j=1, \cdots, l$). The generalized Cartan matrices of Euclidean type are indecomposable and symmetrizable. Of course, Cartan matrices are symmetrizable. A generalized Cartan matrix A is called of hyperbolic type if A is indecomposable, symmetrizable, not of finite type, not of Euclidean type and possesses the property that removal of any row and the corresponding column leaves a union of Cartan matrices and the generalized Cartan matrices of Euclidean type. The generalized Cartan matrices of Euclidean type and the generalized Cartan matrices of hyperbolic type have been classified (cf. Appendix, [1], [2], [6], [10], [13]).

From now on, we suppose that A is a generalized Cartan matrix of Euclidean type or of hyperbolic type. Then the root system $\Delta=\Delta(A)$ of the Kac-Moody Lie algebra associated with A is described as follows. For Kac-Moody Lie algebras, we refer the reader, *e.g.* [8]. Let $\Gamma=\bigoplus_{i=1}^{l}\boldsymbol{Z}\alpha_i$ be a free abelian group with free generators $\alpha_1, \cdots, \alpha_l$. We take an element $w_i\ (1\leqq i\leqq l)$ of $GL(\Gamma)$ defined by $w_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$ for all $j=1, \cdots, l$. The Weyl group of A is defined to be the subgroup W of $GL(\Gamma)$ generated by w_i for all $i=1, \cdots, l$. Let B be a symmetric bilinear form on Γ satisfying $B(\alpha_i, \alpha_j)=q_i a_{ij}$. This form B is W-invariant. Then the root system Δ is a disjoint union of real roots, $\Delta_R=\{w(\alpha_i)\mid w\in W, 1\leqq i\leqq l\}$ and imaginary roots, $\Delta_I=\{\alpha\in\Gamma\mid B(\alpha, \alpha)\leqq 0\}$ (cf. [12]). Let

$$O(\Gamma)=\{g\in GL(\Gamma)\mid B(g\alpha, g\beta)=B(\alpha, \beta) \text{ for all } \alpha, \beta\in\Gamma\},$$

and let D be the subgroup, called the diagram automorphism group of A, of $O(\Gamma)$ generated by all symmetries which are induced by permutations on $\{\alpha_1, \cdots, \alpha_l\}$ preserving the form B. Put $\widetilde{W}=\langle W, D, -I\rangle\subseteqq O(\Gamma)$. We are interested in the index of $\widetilde{W}$ in $O(\Gamma)$; denote it by $Ind(A)$. Let $\Gamma_+=\{\alpha=\sum_{i=1}^{l} a_i\alpha_i\in\Gamma\mid a_i\geqq 0 \text{ for all } i\}$ and $Z=\Gamma_+\cup(-\Gamma_+)$, and $\Pi=\{\alpha_1, \cdots, \alpha_l\}$.

THEOREM 2. *Suppose that A is of Euclidean type. Then $Ind(A)=1$ if $A=X_n^{(1)}$ $(\neq C_4^{(1)})$ or $A_{2n}^{(2)}$; $Ind(A)=2$ if $A=A_{2n-1}^{(2)}$ $(n\neq 4)$; $Ind(A)=3$ if $A=C_4^{(1)}$ or $D_4^{(3)}$; $Ind(A)=4$ if $A=E_6^{(2)}$; $Ind(A)=6$ if $A=A_7^{(2)}$; $Ind(A)=2^{n-1}$ if $A=D_{n+1}^{(2)}$.*

PROOF. We can assume that $A=\left(\begin{array}{c|c} A_0 & * \\ \hline * & 2\end{array}\right)$, where A_0 is of finite type X_n (resp. B_n, C_n, B_n, F_4, G_2) if A is of type $X_n^{(1)}$ (resp. $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$). For the convenience, we assume that α_1 is a short root associated with A_0. As is well-known, $\Delta_I=\{\alpha\in\Gamma\mid B(\alpha, \alpha)=0\}=Rad(B)$ and Δ_I is a free $\boldsymbol{Z}$-module of rank 1. Take a generator ξ of Δ_I, which is called a fundamental null root. Let $\Gamma_0=\bigoplus_{i=1}^{l-1}\boldsymbol{Z}\alpha_i$, then $\Gamma=\Gamma_0\oplus\boldsymbol{Z}\xi$ (orthogonal sum). Take an element $\sigma\in O(\Gamma)$. Since $\sigma(\xi)=\pm\xi$, we can write $\sigma=\left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline * & \pm 1\end{array}\right)$, where $\sigma_0\in O(\Gamma_0)$ and $O(\Gamma_0)$ is embedded in $O(\Gamma)$ by $\sigma_0\mapsto\left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & 1\end{array}\right)$. Therefore $\sigma\equiv\left(\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline * & 1\end{array}\right)$ modulo $O(\Gamma_0)\times\langle -I_\Gamma\rangle$.

Set $T=\left\{\left(\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline s_1\cdots s_{l-1} & 1\end{array}\right)\middle| s_i\in\boldsymbol{Z}\right\}$. Then we have $O(\Gamma)=(O(\Gamma_0)\ltimes T)\times\langle -I_\Gamma\rangle$. Let W_0 (resp. W) be the Weyl group of A_0 (resp. A), and let D_0 (resp. D) be the diagram automorphism group of A_0 (resp. A). For each element α of Δ_R, we define an element w_α of $O(\Gamma)$ by $w_\alpha(x)=x-(2B(\alpha, x)/B(\alpha, \alpha))\alpha$ for all $x\in\Gamma$. Set $m_i=min\{m>0\mid\alpha_i+m\xi\in\Delta_R\}$ for $i=1, \cdots, l-1$. For each $i=1, \cdots, l-1$, an element h_i of W is defined to be $w_{2\alpha_i+m_i\xi}w_{\alpha_i}$ if $A=A_{2n}^{(2)}$ and $i=1$; $w_{\alpha_i+m_i\xi}w_{\alpha_i}$ otherwise. Let H be the subgroup of W generated by $h_1, \cdots, h_{l-1}$. Then

$W=W_0\ltimes H$. We note that $\widetilde{W}=(W\rtimes D)\times\langle -I_\Gamma\rangle$ and $\widetilde{W}_0=W_0\rtimes D_0$. Hence we have $[O(\Gamma):\widetilde{W}]=\dfrac{[O(\Gamma_0):\widetilde{W}_0]}{[D:D_0]}[T:H]$. Furthermore $[T:H]=\prod_{i=1}^{l-1} m_i\ (det\ A_0)\kappa$, where $\kappa=\dfrac{1}{2}$ if $A=A_{2n}^{(2)}$; 1 otherwise. Then $[D:D_0]=(det\ A_0)\kappa$ (cf. [9; p. 96]). Therefore $Ind(A)=Ind(A_0)\prod_{i=1}^{l-1} m_i$. By Theorem 1 and the structure of Δ, we can compute the index $Ind(A)$. Q.E.D.

Let A be of rank 2 hyperbolic type. That is, $A=\begin{pmatrix} 2 & -a \\ -b & 2\end{pmatrix}$, $ab>4$. We put $q_1=\dfrac{b}{2}$, $q_2=\dfrac{a}{2}$, so the associated form B is defined by $B(\alpha_1,\alpha_1)=b$, $B(\alpha_2,\alpha_2)=a$ and $B(\alpha_1,\alpha_2)=-\dfrac{ab}{2}$. The Weyl group W is generated by $w_1=\begin{pmatrix} -1 & a \\ 0 & 1\end{pmatrix}$ and $w_2=\begin{pmatrix} 1 & 0 \\ b & -1\end{pmatrix}$. Let $\sigma\in O(\Gamma)$, and choose an element $\beta=n_1\alpha_1+n_2\alpha_2$ of $W\sigma(\alpha_1)$, the W-orbit of $\sigma(\alpha_1)$, which satisfies the condition that $n_1^2+n_2^2$ is minimal in this orbit. Since $w_1(\beta)=(-n_1+an_2)\alpha_1+n_2\alpha_2$ and $w_2(\beta)=n_1\alpha_1+(bn_1-n_2)\alpha_2$, we have $n_2(an_2-2n_1)\geqq 0$ and $n_1(bn_1-2n_2)\geqq 0$ by the condition of β. If $n_1>0$, $n_2>0$ (resp. $n_1<0$, $n_2<0$), then $\dfrac{2}{a}n_1\leqq n_2\leqq\dfrac{b}{2}n_1$ (resp. $\dfrac{2}{a}n_1\geqq n_2\geqq\dfrac{b}{2}n_1$), which means $0\geqq B(\beta,\beta)=B(\alpha_1,\alpha_1)=b$, a contradiction. Thus, $n_1n_2\leqq 0$. On the other hand, $b=bn_1^2-abn_1n_2+an_2^2$ since $B(\beta,\beta)=B(\alpha_1,\alpha_1)=b$. Then $n_1n_2\leqq 0$ implies that $(n_1,n_2)=(\pm 1,0)$, or that $(n_1,n_2)=(0,\pm 1)$ and $a=b$. In the latter, A is symmetric, so we already know $\sigma\in\widetilde{W}$. Therefore we can assume that there is an element $w\in\widetilde{W}$ satisfying $w\sigma(\alpha_1)=\alpha_1$. Write $w\sigma(\alpha_2)=k_1\alpha_1+k_2\alpha_2$. Then $B(w\sigma(\alpha_2),w\sigma(\alpha_2))=bk_1^2-abk_1k_2+ak_2^2=a$ and $B(w\sigma(\alpha_2),\alpha_1)=bk_1-\dfrac{ab}{2}k_2=-\dfrac{ab}{2}$. Hence $(k_1,k_2)=(0,1)$ or $(-a,-1)$. This leads to $w\sigma=I$ or $w\sigma=(-I)w_1$, and $\sigma\in\widetilde{W}$. Thus we have the following.

THEOREM 3. *Suppose that A is of rank 2 hyperbolic type. Then $Ind(A)=1$.*

Next we treat the case that A is of rank 3 hyperbolic type. We use here the classification of the generalized Cartan matrices of this type (cf. Appendix, [1], [2], [13]). Suppose that A is none of $\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2\end{pmatrix}$, $\begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ -2 & -4 & 2\end{pmatrix}$ or $\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2\end{pmatrix}$. Then $O(\Gamma)\Pi\subseteqq Z$, hence in particular $W\sigma(\alpha_i)\subseteqq Z$ for all $\sigma\in O(\Gamma)$, $1\leqq i\leqq 3$. Therefore $O(\Gamma)\Pi=\Delta_R$ (cf. [7]). By [12; Theorem 2], we have $O(\Gamma)=\widetilde{W}$ and $Ind(A)=1$. We shall consider the remaining three cases.

(1) The case when $A=\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$.

Let σ_0 be an endomorphism of Γ defined by

$$\sigma_0(\alpha_1)=-\alpha_1, \quad \sigma_0(\alpha_2)=\alpha_1-\alpha_3, \quad \sigma_0(\alpha_3)=-\alpha_1-\alpha_2.$$

Then σ_0 preserves the form B and $\sigma_0\in O(\Gamma)-\tilde{W}$. Take an element $\sigma\in O(\Gamma)-\tilde{W}$. Since the elements $\alpha\in\Gamma$ satisfying $B(\alpha, \alpha)=B(\alpha_2, \alpha_2)$ and $\alpha\notin Z$ are $\pm(\alpha_1-\alpha_3)$, there is an element $w\in\tilde{W}$ such that $w\sigma(\alpha_2)=\alpha_1-\alpha_3$. (For $\sigma\in O(\Gamma)-\tilde{W}$, there is an element $w'\in\tilde{W}$ such that $w'\sigma(\alpha_2)=\alpha_1-\alpha_3$ or α_2. The latter induces $w'\sigma(\Delta_R)=\Delta_R$. But this leads to a contradiction.)

Therefore to consider $\tilde{W}\backslash O(\Gamma)$ we can assume $\sigma(\alpha_2)=\alpha_1-\alpha_3$. Write $\sigma(\alpha_1)=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3$ and $\sigma(\alpha_3)=l_1\alpha_1+l_2\alpha_2+l_3\alpha_3$. We put $q_1=\frac{1}{2}$, $q_2=\frac{3}{2}$ and $q_3=\frac{1}{2}$. Then we have:

$$(\boldsymbol{E}_1) \qquad \begin{cases} k_1^2+3k_2^2+k_3^2-3k_1k_2-k_1k_3-3k_2k_3=1 \\ l_1^2+3l_2^2+l_3^2-3l_1l_2-l_1l_3-3l_2l_3=1 \\ 2k_1l_1+6k_2l_2+2k_3l_3-3k_1l_2-k_1l_3-3k_2l_1-3k_2l_3-k_3l_1-3kl_2=-1 \\ k_1-k_3=-1 \\ l_1-l_3=-1. \end{cases}$$

Put $s=2k_1-6k_2+1$, $t=k_2$, $u=2l_1-6l_2+1$ and $v=l_2$. Then the Diophantine equation $(\boldsymbol{E}_1)$ implies the Diophantine equation

$$(\boldsymbol{E}_2) \qquad \begin{cases} s^2-24t^2=1 \\ u^2-24v^2=1 \\ su-24tv=-5. \end{cases}$$

Notice that $5+\sqrt{24}$ is the dominant fundamental factor of the Pell's equations $s^2-24t^2=1$ and $u^2-24v^2=1$ (cf. [3; P. 83], [4; P. 110]). Let

$$S=\{(m, n; \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mid m, n\in \boldsymbol{Z}_{\geq 0}, |m-n|=1, \varepsilon_i=\pm 1, \varepsilon_1\varepsilon_3=\varepsilon_2\varepsilon_4=-1\}.$$

Then the set of all solutions of the Diophantine equation $(\boldsymbol{E}_2)$ is parametrized by S. That is,

$$s=\varepsilon_1(\zeta_+^m+\zeta_-^m)/2, \qquad t=\varepsilon_2(\zeta_+^m-\zeta_-^m)/2\sqrt{24},$$
$$u=\varepsilon_3(\zeta_+^n+\zeta_-^n)/2, \qquad v=\varepsilon_4(\zeta_+^n-\zeta_-^n)/2\sqrt{24},$$

where $\zeta_\pm=5\pm\sqrt{24}$. Here we will choose three elements of $\tilde{W}$. Let $\rho_1=w_2$, $\rho_2=w_1w_3w_1$ and $\rho_3=(-I)d$, where $d=[\alpha_1\mapsto\alpha_3, \alpha_2\mapsto\alpha_2, \alpha_3\mapsto\alpha_1]$, a non-trivial diagram automorphism. Then these ρ_i's fix $\alpha_1-\alpha_3$. Thus $\rho_i\sigma$ $(i=1, 2, 3)$ gives a new solution of $(\boldsymbol{E}_1)$. Since $\rho_1\sigma(\alpha_1)=k_1\alpha_1+(k_1-k_2+k_3)\alpha_2+k_3\alpha_3$ and $\rho_1\sigma(\alpha_3)=l_1\alpha_1+(l_1-l_2+l_3)\alpha_2+l_3\alpha_3$, we see that ρ_1 produces a new solution

$$s'=-\varepsilon_1(\zeta_+^{m+1}+\zeta_-^{m+1})/2 \qquad (\text{resp. } s'=-\varepsilon_1(\zeta_+^{m-1}+\zeta_-^{m-1})/2),$$

$$t'=\varepsilon_2(\zeta_+^{m+1}-\zeta_-^{m+1})/2\sqrt{24} \qquad (\text{resp. } t'=\varepsilon_2(\zeta_+^{m-1}-\zeta_-^{m-1})/2\sqrt{24}),$$

$$u'=-\varepsilon_3(\zeta_+^{n+1}+\zeta_-^{n+1})/2 \qquad (\text{resp. } u'=-\varepsilon_3(\zeta_+^{n-1}+\zeta_-^{n-1})/2),$$

$$v'=\varepsilon_4(\zeta_+^{n+1}-\zeta_-^{n+1})/2\sqrt{24} \qquad (\text{resp. } v'=\varepsilon_4(\zeta_+^{n-1}-\zeta_-^{n-1})/2\sqrt{24})$$

of $(\boldsymbol{E}_2)$ from an original solution (s, t, u, v) if $\varepsilon_1\varepsilon_2>0$ (resp. $\varepsilon_1\varepsilon_2<0$). Since

$$\rho_2\sigma(\alpha_1)=(6k_2-k_3)\alpha_1+k_2\alpha_2+(-k_1+6k_2)\alpha_3$$

and

$$\rho_2\sigma(\alpha_3)=(6l_2-l_3)\alpha_1+l_2\alpha_2+(-l_1+6l_2)\alpha_3\,,$$

the element ρ_2 produces a new solution $(-s, t, -u, v)$ from (s, t, u, v). Since

$$\rho_3\sigma(\alpha_1)=-k_3\alpha_1-k_2\alpha_2-k_1\alpha_3$$

and

$$\rho_3\sigma(\alpha_3)=-l_3\alpha_1-l_2\alpha_2-l_1\alpha_3\,,$$

the element ρ_3 produces a new solution $(-s, -t, -u, -v)$ from (s, t, u, v). Hence the subgroup G of $\widetilde{W}$ generated by ρ_1, ρ_2 and ρ_3 transitively acts on the set of all solutions of $(\boldsymbol{E}_2)$. This means $O(\Gamma)-\widetilde{W}=\widetilde{W}\sigma_0$, so $\{1, \sigma_0\}$ is the complete set of representatives of $\widetilde{W}\backslash O(\Gamma)$.

(2) The case when $A=\begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix}$.

In this case, we can take an element σ_0 of $O(\Gamma)-\widetilde{W}$ defined by $\sigma_0(\alpha_1)=\alpha_2+\alpha_3$, $\sigma_0(\alpha_2)=\alpha_1-\alpha_3$ and $\sigma_0(\alpha_3)=\alpha_3$. For each element $\sigma\in O(\Gamma)-\widetilde{W}$ there exists an element $w\in\widetilde{W}$ such that $w\sigma(\alpha_2)=\alpha_1-\alpha_3$, since the elements $\alpha\in\Gamma$ satisfying $B(\alpha, \alpha)=B(\alpha_2, \alpha_2)$ and $\alpha\notin Z$ are $\pm(\alpha_1-\alpha_3)$. Then the elements $\tau\in O(\Gamma)-\widetilde{W}$ with the property $\tau(\alpha_2)=\alpha_1-\alpha_3$ are τ_1, τ_2, τ_3 and τ_4, where

$$\tau_1=[\alpha_1\mapsto\alpha_3,\ \alpha_2\mapsto\alpha_1-\alpha_3,\ \alpha_3\mapsto\alpha_2+\alpha_3]\,,$$

$$\tau_2=[\alpha_1\mapsto-\alpha_1,\ \alpha_2\mapsto\alpha_1-\alpha_3,\ \alpha_3\mapsto-\alpha_1-\alpha_2]\,,$$

$$\tau_3=[\alpha_1\mapsto\alpha_2+\alpha_3,\ \alpha_2\mapsto\alpha_1-\alpha_3,\ \alpha_3\mapsto\alpha_3]\,,$$

$$\tau_4=[\alpha_1\mapsto-\alpha_1-\alpha_2,\ \alpha_2\mapsto\alpha_1-\alpha_3,\ \alpha_3\mapsto-\alpha_3]\,.$$

Put $d=[\alpha_1\mapsto\alpha_3,\ \alpha_2\mapsto\alpha_2,\ \alpha_3\mapsto\alpha_1]$, a nontrivial diagram automorphism. Then we have $(-I)d\tau_1=\tau_2$, $(-I)d\tau_3=\tau_4$, $w_2\tau_1=\tau_3$ and $w_2\tau_2=\tau_4$. Therefore $O(\Gamma)-\widetilde{W}=\widetilde{W}\sigma_0$, so $\{1, \sigma_0\}$ is the complete set of representatives of $\widetilde{W}\backslash O(\Gamma)$.

(3) The case when $A=\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$.

In this case, the elements $\alpha\in\Gamma$ satisfying $B(\alpha, \alpha)=B(\alpha_1, \alpha_1)$ and $\alpha\notin Z$ are $\pm(\alpha_2-\alpha_3)$ and $\pm(\alpha_1+\alpha_2-\alpha_3)$. Let σ_0 (resp. τ_0) be the endomorphism of Γ defined by $\sigma_0(\alpha_1)=\alpha_2-\alpha_3$, $\sigma_0(\alpha_2)=-\alpha_2$ and $\sigma_0(\alpha_3)=-\alpha_1-\alpha_2$ (resp. $\tau_0(\alpha_1)=\alpha_1+\alpha_2-\alpha_3$, $\tau_0(\alpha_2)=\alpha_3$ and $\tau_0(\alpha_3)=\alpha_2$). Then they belong to $O(\Gamma)-\widetilde{W}$. For each element $\sigma\in O(\Gamma)-\widetilde{W}$, there is an element $w\in\widetilde{W}$ such that $w\sigma(\alpha_1)=\alpha_2-\alpha_2$ or $\alpha_1+\alpha_2-\alpha_3$. Then the elements τ of $O(\Gamma)-\widetilde{W}$ having the property $\tau(\alpha_1)=\alpha_2-\alpha_3$ are σ_1, σ_2, σ_3 and σ_4, where

$$\sigma_1=[\alpha_1\mapsto\alpha_2-\alpha_3,\ \alpha_2\mapsto\alpha_3,\ \alpha_3\mapsto\alpha_1+\alpha_2],$$

$$\sigma_2=[\alpha_1\mapsto\alpha_2-\alpha_3,\ \alpha_2\mapsto-\alpha_2,\ \alpha_3\mapsto-\alpha_1-\alpha_2],$$

$$\sigma_3=[\alpha_1\mapsto\alpha_2-\alpha_3,\ \alpha_2\mapsto-2\alpha_1-3\alpha_2,\ \alpha_3\mapsto\alpha_1+\alpha_2],$$

$$\sigma_4=[\alpha_1\mapsto\alpha_2-\alpha_3,\ \alpha_2\mapsto 2\alpha_1+2\alpha_2+\alpha_3,\ \alpha_3\mapsto-\alpha_1-\alpha_2],$$

and the elements τ of $O(\Gamma)-\widetilde{W}$ having the property $\tau(\alpha_1)=\alpha_1+\alpha_2-\alpha_3$ are τ_1, τ_2, τ_3 and τ_4, where

$$\tau_1=[\alpha_1\mapsto\alpha_1+\alpha_2-\alpha_3,\ \alpha_2\mapsto\alpha_3,\ \alpha_3\mapsto\alpha_2],$$

$$\tau_2=[\alpha_1\mapsto\alpha_1+\alpha_2-\alpha_3,\ \alpha_2\mapsto-\alpha_1-\alpha_2,\ \alpha_3\mapsto-\alpha_2],$$

$$\tau_3=[\alpha_1\mapsto\alpha_1+\alpha_2-\alpha_3,\ \alpha_2\mapsto-\alpha_1-3\alpha_2,\ \alpha_3\mapsto\alpha_2],$$

$$\tau_4=[\alpha_1\mapsto\alpha_1+\alpha_2-\alpha_3,\ \alpha_2\mapsto 2\alpha_2+\alpha_3,\ \alpha_3\mapsto-\alpha_2].$$

Furthermore $w_1w_2w_1\sigma_1=\sigma_4$, $w_1w_2w_1\sigma_2=\sigma_3$ and $w_1\sigma_i=\tau_i$ $(1\leqq i\leqq 4)$. Therefore $O(\Gamma)-\widetilde{W}=\widetilde{W}\sigma_0\cup\widetilde{W}\tau_0$. On the other hand, $\sigma_0\tau_0^{-1}(\alpha_1)=\alpha_1+\alpha_2-\alpha_3\notin\Delta_R$, so $\sigma_0\tau_0^{-1}\notin W$. This means that $\{1, \sigma_0, \tau_0\}$ is the complete set of representatives of $\widetilde{W}\backslash O(\Gamma)$.

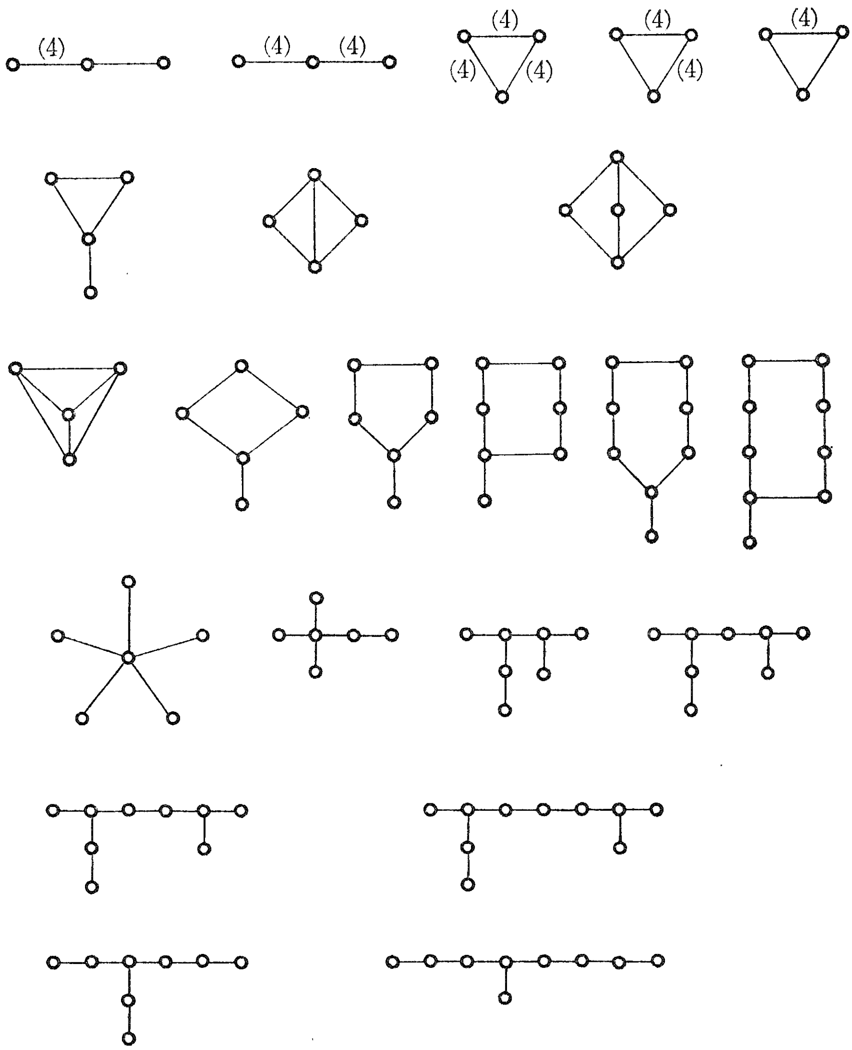
THEOREM 4. *Suppose that A is of rank 3 hyperbolic type. Then $Ind(A)=2$ if $A=\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix}$; $Ind(A)=3$ if $A=\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$; $Ind(A)=1$ otherwise.*

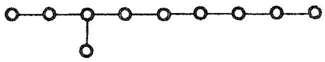
Appendix

Hyperbolic generalized Cartan matrices of rank$\geqq 3$

a_{ij}	a_{ji}	i j	a_{ij}	a_{ji}	i j
0	0	○ ○	-1	-3	○ (3) → ○
-1	-1	○—○	-1	-4	○ (4) → ○
-1	-2	○ (2) → ○	-2	-2	○ (4) ○

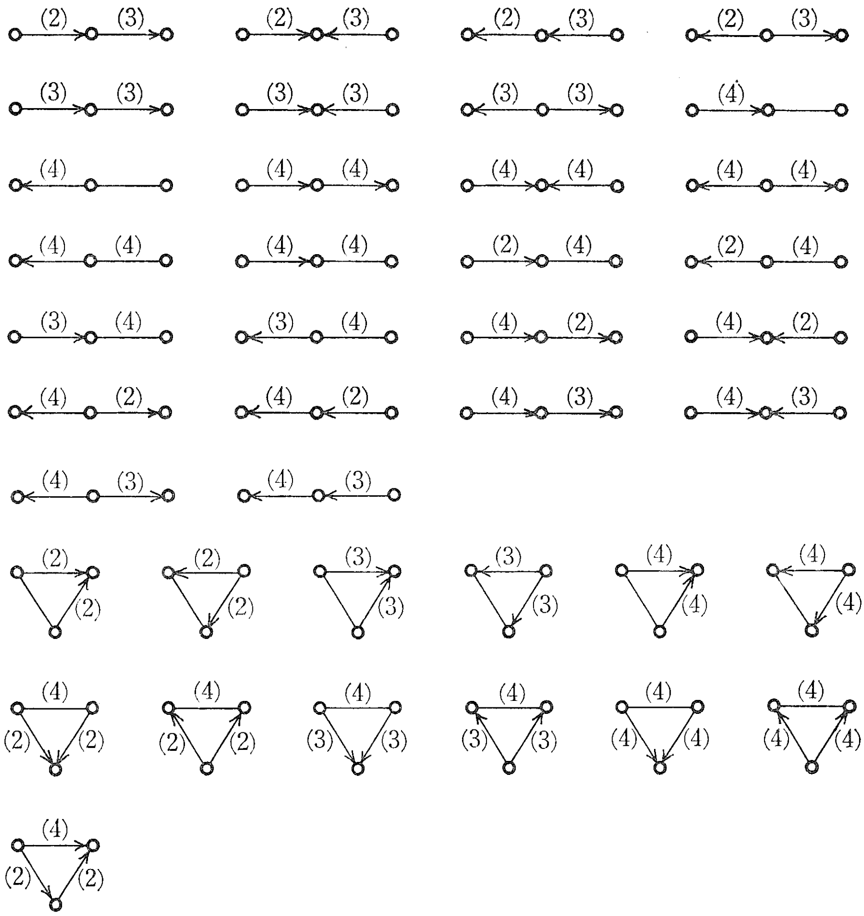
Symmetric case.



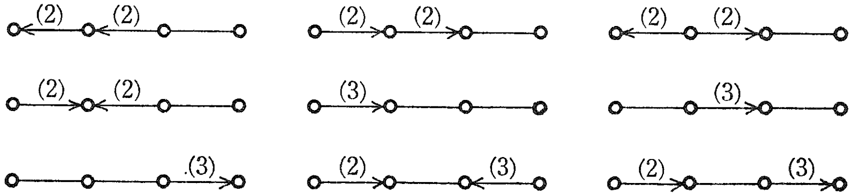


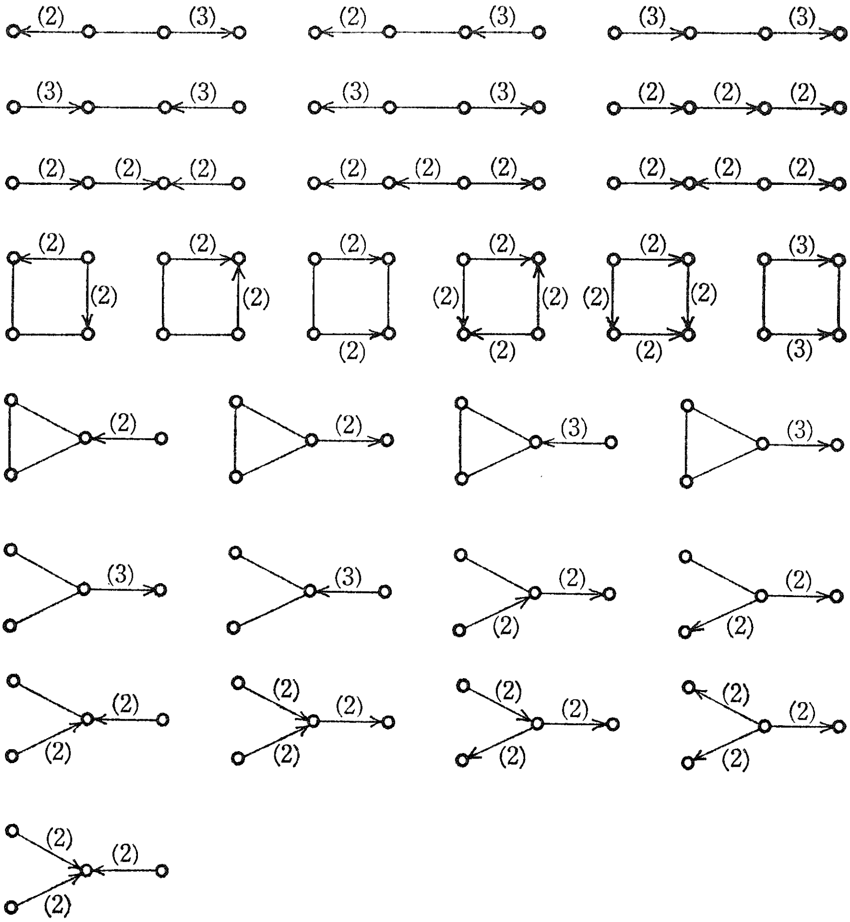
NON-SYMMETRIC CASE.

(1) rank 3

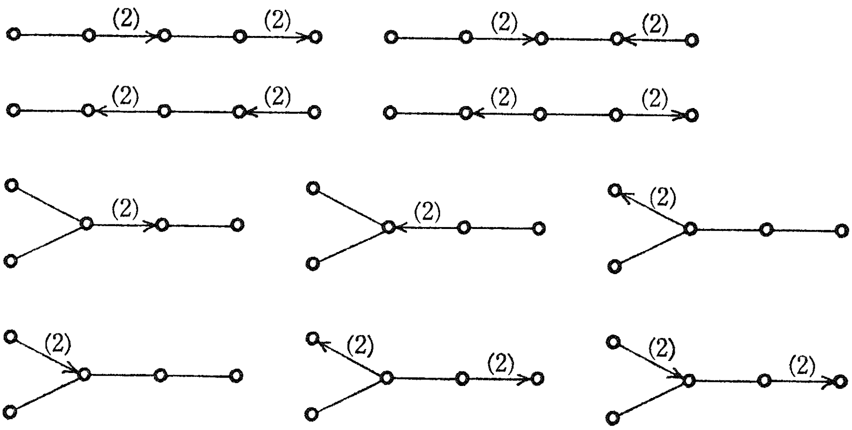


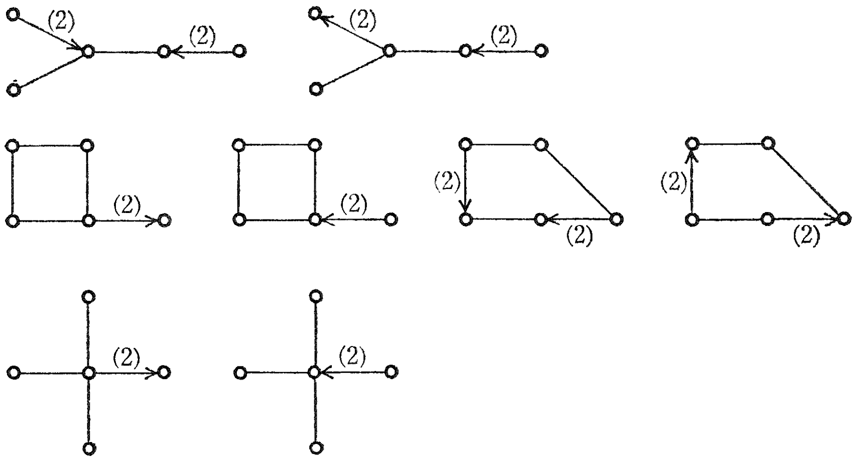
(2) rank 4



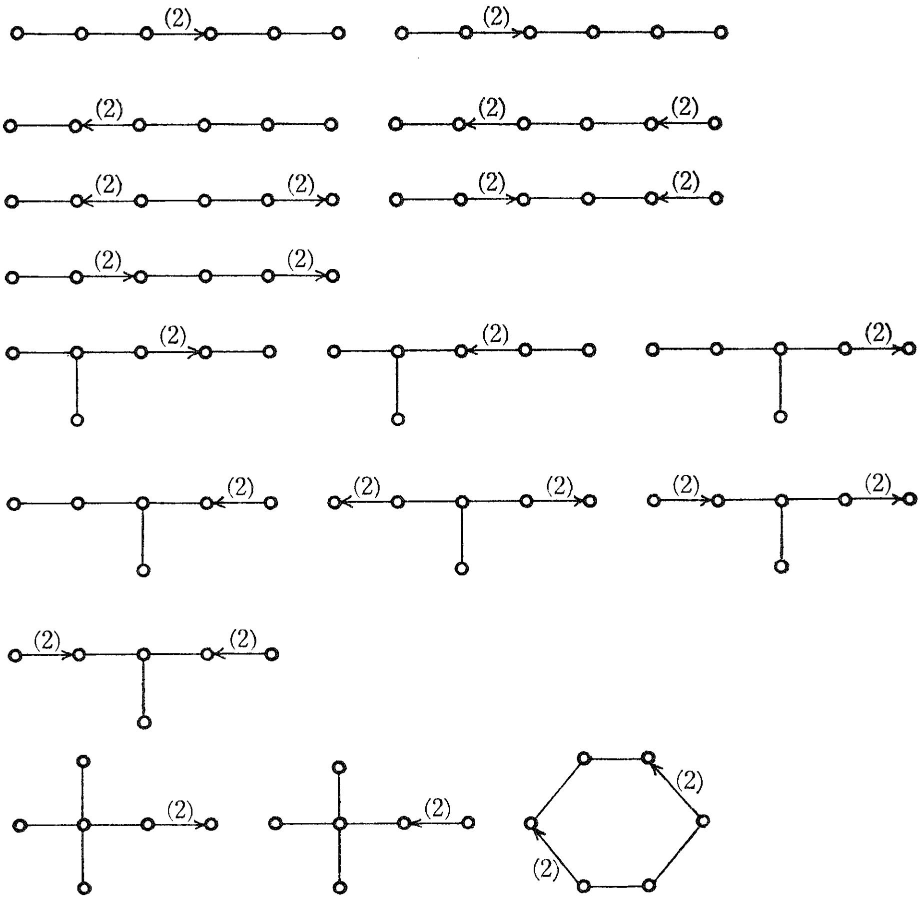


(3) rank 5

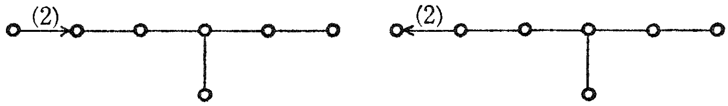




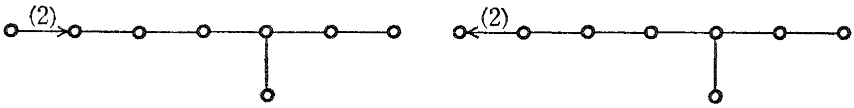
(4) rank 6



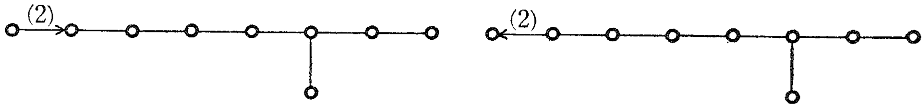
(5) rank 7



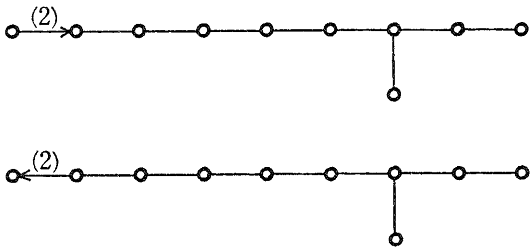
(6) rank 8



(7) rank 9



(8) rank 10



NOTE. *The rank of a hyperbolic type generalized Cartan matrix is at most* 10 (*cf.* [1]).

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