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# REDUCTION TECHNIQUES FOR HOMOLOGICAL CONJECTURES

By

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Let $A$ be a finite-dimensional $k$-algebra over an algebraically closed field $k$. We denote by mod $A$ the category of finitely generated left $A$-modules. For an $A$-module ${}_AX$ we denote by $\mathrm{pd}_A X$ (resp. $\mathrm{id}_A X$) the projective (resp. injective) dimension of $X$. With $D=\mathrm{Hom}_k(-, k)$ we denote the standard duality with respect to the ground field. Then ${}_AD(A_A)$ is an injective cogenerator for mod $A$. To formulate some of the homological conjectures we need some more notation. Let ${}_A\mathcal{I}\subset \mathrm{mod}\, A$ be the full subcategory containing the finitely generated injective $A$-modules. Let $K^b({}_A\mathcal{I})$ be the homotopy category of bounded complexes over ${}_A\mathcal{I}$. Let $D^b(A)$ be the derived category of bounded complexes over mod $A$. We consider $K^b({}_A\mathcal{I})$ as a full subcategory of $D^b(A)$. We define.

$$K^b({}_A\mathcal{I})^{\perp}=\{X\in D^b(A)\,|\,\mathrm{Hom}(I, X)=0 \text{ for all } I\in K^b({}_A\mathcal{I})\}.$$

We are interested in the following conjectures:

(1) **Finitistic Dimension Conjecture**: $\mathrm{fd}(A)=\sup\{\mathrm{pd}_A X\,|\,\mathrm{pd}_A X<\infty\}$ is finite.

(2) **Vanishing Conjecture**: $K^b({}_A\mathcal{I})^{\perp}=0$.

(3) **Generalized Nakayama Conjecture**: For a simple module ${}_AS$ there is $i\geqq 0$ such that $\mathrm{Ext}_A^i({}_AD(A_A), {}_AS)\neq 0$.

We refer to [**AR**], [**B1**], [**H3**] and [**J**] for some further information about these conjectures.

The aim of this article is to show that using the language of triangulated categories certain reduction techniques can be obtained. To be more precise we will show the following results:

A module $T\in \mathrm{mod}\, A$ is called a (generalized) *tilting module* if the following conditions are satisfied:

(i) $\mathrm{pd}_A T<\infty$

(ii) $\mathrm{Ext}_A^i(T, T)=0$ for all $i>0$

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(iii) There is a long exact sequence $0 \to {}_A A \to T_0 \to \cdots \to T_m \to 0$ with $T_j \in$ add $T$.
In section 2 we will show.

THEOREM 1. *Let $A$ be a finite-dimensional algebra and $T$ a tilting module. Let $B = \mathrm{End}_A T$. Then* $\mathrm{fd}(A) < \infty$ *if and only if* $\mathrm{fd}(B) < \infty$.

Using the notion of recollement introduced by [**BBD**] (see section 3 for more details) we show the following result.

THEOREM 2. *Let $A$ be a finite-dimensional algebra and assume that $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$ for some finite-dimensional algebras $A'$, $A''$. Then* $\mathrm{fd}(A) < \infty$ *if and only if* $\mathrm{fd}(A')$ *and* $\mathrm{fd}(A'') < \infty$.

In section 4 we will see that the generalized Nakayama conjecture is related to a problem about Grothendieck groups of triangulated categories.

In section one we will recall the terminology about complexes which we will have to use and recall the relationship of the conjectures above.

We denote the compositionn of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in a given category $\mathcal{K}$ by $fg$.

## 1. Perpendicular categories.

**1.1.** For the convenience of the reader we recall some of the terminology for complexes which we have to use.

Let $\mathfrak{a}$ be an arbitrary additive subcategory of mod $A$.

A *complex* $X^{\cdot} = (X^i, d_X^i)_{i \in \boldsymbol{Z}}$ over $\mathfrak{a}$ is a collection of objects $X^i$ from $\mathfrak{a}$ and morphisms $d^i = d_X^i\colon X^i \to X^{i+1}$ such that $d^i d^{i+1} = 0$. A complex $X^{\cdot} = (X^i, d_X^i)$ is *bounded below* if $X^i = 0$ for all but finitely many $i < 0$. It is called *bounded above* if $X^i = 0$ for all but finitely many $i > 0$. It is *bounded* if it is bounded below and bounded above. It is said to have *bounded cohomology* if $H^i(X^{\cdot}) = 0$ for all but finitely many $i \in \boldsymbol{Z}$, where by definition $H^i(X^{\cdot}) = \ker d_X^i / \mathrm{im}\, d_X^{i-1}$. Denote by $C(\mathfrak{a})$ the category of complexes over $\mathfrak{a}$, by $C^-(\mathfrak{a})$ (resp. $C^+(\mathfrak{a})$ resp. $C^{-,b}(\mathfrak{a})$ resp. $C^{+,b}(\mathfrak{a})$, resp. $C^b(\mathfrak{a})$) the full subcategories of complexes bounded above (resp. bounded below, resp. bounded above with bounded cohomology, resp. bounded below with bounded cohomology, resp. bounded above and below).

If $X^{\cdot} = (X^i, d_X^i)$ is a complex, then $\mathrm{supp}\, X^{\cdot} = \{i \in \boldsymbol{Z} \mid X^i \neq 0\}$ is called the *support* of $X^{\cdot}$.

If $X^{\cdot}=(X^i, d_X^i)_{i\in\mathbf{Z}}$ and $Y^{\cdot}=(Y^i, d_Y^i)_{i\in\mathbf{Z}}$ are two complexes, a morphism $f^{\cdot}$: $X^{\cdot}\to Y^{\cdot}$ is a sequence of morphisms $f^i$: $X^i\to Y^i$ of $\mathfrak{a}$ such that

$$d_X^i f^{i+1}=f^i d_y^i$$

for all $i\in\mathbf{Z}$. The *translation functor* is defined by

$$(X^{\cdot}[1])^i=X^{i+1}, \qquad (d_{X[1]})^i=-(d_X)^{i+1}.$$

The *mapping cone* $C_{f^{\cdot}}$ of a morphism $f^{\cdot}$: $X^{\cdot}\to Y^{\cdot}$ is the complex

$$C_{f^{\cdot}}=((X^{\cdot}[1])^i\oplus Y^i, d_{C_f}^i)$$

with 'differential'

$$d_{C_f}^i=\begin{pmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^i \end{pmatrix}.$$

We denote by $K^-(\mathfrak{a})$, $K^+(\mathfrak{a})$, $K^{-,b}(\mathfrak{a})$, $K^{+,b}(\mathfrak{a})$ and $K^b(\mathfrak{a})$ the homotopy categories of the categories of complexes introduced above. Note that all these categories are triangulated categories in the sense of [*V*].

Recall that two morphisms $f^{\cdot}$, $g^{\cdot}$: $X^{\cdot}\to Y^{\cdot}$ are called *homotopic*, if there exist morphisms $h^i$: $X^i\to Y^{i-1}$ such that $f^i-g^i=d_X^i h^{i+1}+h^i d_Y^{i-1}$ for all $i\in\mathbf{Z}$.

We denote by ${}_A\mathcal{P}$ (resp. ${}_A\mathcal{I}$) the full subcategory of mod $A$ formed by the projective (resp. injective) $A$-modules. Then we identify the derived category $D^b(A)$ of bounded complexes over mod $A$ with $K^{-,b}({}_A\mathcal{P})$ or with $K^{+,b}({}_A\mathcal{I})$. In case $A$ has finite global dimension this yields the identification of $D^b(A)$ with $K^b({}_A\mathcal{P})$ or with $K^b({}_A\mathcal{I})$, since the natural embedding of $K^b({}_A\mathcal{P})$ into $K^{-,b}({}_A\mathcal{P})$ is an equivalence in this case. We identify the derived category $D^-(A)$ of complexes bounded above over mod $A$ with $K^-({}_A\mathcal{P})$ and we identify the derived category $D^+(A)$ of complexes bounded below over mod $A$ with $K^+({}_A\mathcal{I})$.

**1.2.** We will briefly recall from [***H*3**] the relationship between the conjectures mentioned in the introduction. Recall that we have defined

$$K^b({}_A\mathcal{I})^{\perp}=\{X\in D^b(A)\,|\,\mathrm{Hom}(I, X)=0 \text{ for all } I\in K^b({}_A\mathcal{I})\}.$$

PROPOSITION. *Let $A$ be a finite-dimensional $k$-algebra. Then*

(i) *If* $\mathrm{fd}(A)<\infty$, *then* $K^b({}_A\mathcal{I})^{\perp}=0$.

(ii) *If* $K^b({}_A\mathcal{I})^{\perp}=0$, *then given* $0\neq{}_AX$ *there exists* $i\geq 0$ *such that* $\mathrm{Ext}_A^i(D(A_A), X)\neq 0$.

PROOF. For (i) assume that $K^b({}_A\mathcal{I})^{\perp}\neq 0$. Let $I^{\cdot}\in K^b({}_A\mathcal{I})^{\perp}$. We assume that $I^{\cdot}=(I^i, d_I^i)\neq 0$. Applying the translation functor if necessary we may assume that $I^i=0$ for $i<0$. Next consider the complex

$$\cdots 0 \longrightarrow \mathrm{Hom}(D(A_A), I^0) \longrightarrow \mathrm{Hom}(D(A_A), I^1) \longrightarrow \cdots$$

Observe that $\mathrm{Hom}(D(A_A), I^j) \in {}_A\mathcal{P}$. Since $I^\cdot \in K^b({}_A\mathcal{I})^\perp$ we infer that this complex is acyclic. Note that pd cok $\mathrm{Hom}(D(A_A), d^i) < \infty$. But this contradicts $\mathrm{fd}(A) < \infty$.

For (ii) assume that there exists $0 \neq {}_AX$ with $\mathrm{Ext}_A^i((D(A_A), X) = 0$ for all $i \geqq 0$. Let $I^\cdot$ be a minimal injective resolution of ${}_AX$. We consider $I^\cdot$ as element of $D^b(A)$. We claim that $I^\cdot \in K^b({}_A\mathcal{I})^\perp$.

For this let $J^\cdot = (J^i, d_J^i) \in K^b({}_A\mathcal{I})$. So there exists $r \leqq s$ such that $J^i = 0$ for $i < r$ and $i > s$. The *width* $w(J^\cdot)$ of $J^\cdot$ is by definition $s - r + 1$. The assertion $\mathrm{Hom}(J^\cdot, I^\cdot) = 0$ now follows easily by induction on $w(J^\cdot)$ by considering a triangle as in lemma 1.1 of [**H1**] and applying the cohomological functor $\mathrm{Hom}_{D^b(A)}(J^\cdot, -)$ to this triangle. The start of the induction is just the assumption $\mathrm{Ext}_A^i(D(A_A), X) = 0$ for all $i \geqq 0$.

The proposition above shows that $K^b({}_A\mathcal{I})^\perp = 0$ implies that $A$ satisfies the socalled Nunke condition (see [**J**]), in particular satisfies the generalized Nakayama conjecture.

**1.3.** Let $A$ and $B$ be two finite-dimensional algebras. We say that $A$ and $B$ are *derived equivalent* if $D^b(A)$ the $D^b(B)$ are equivalent as triangulated categories. We refer to [**Ri**] for necessary and sufficient conditions and to 2.1 for specific classes of examples.

PROPOSITION. *Let $A$ and $B$ be derived equivalent finite-dimensional algebras. Then $K^b({}_A\mathcal{I})^\perp = 0$ if and only if $K^b({}_B\mathcal{I})^\perp = 0$.*

PROOF. In fact, let $F$: $D^b(A) \to D^b(B)$ be a triangle equivalence. Then the restriction of $F$ to the subcategory $K^b({}_A\mathcal{I})$ induces a triangle equivalence $K^b({}_A\mathcal{I}) \to K^b({}_B\mathcal{I})$, hence $K^b({}_A\mathcal{I})^\perp \cong K^b({}_B\mathcal{I})^\perp$.

## 2. Tilting invariance.

**2.1.** Let $A$ be a finite-dimensional algebra and ${}_AT$ be a tilting module. We consider also $B = \mathrm{End}_A T$. We refer to [**H1**], [**H2**] and [**Mi**] for an outline of tilting theory. We will need that in this situation $A$ and $B$ are derived equivalent. The functors giving this equivalence are obtained as follows.

We identify $D^b(A)$ with $K^{+,b}({}_A\mathcal{I})$. Let $\mathfrak{t}$ be the full subcategory of mod $B$ with objects $\mathrm{Hom}_A(T, I)$ for $I \in {}_A\mathcal{I}$. We consider $K^{+,b}(\mathrm{add}\,\mathfrak{t})$ as a full subcategory of $K^{+,b}(\mathrm{mod}\,B)$ and denote by $Q^{+,b}$ the localization functor from $K^{+,b}(\mathrm{mod}\,B)$ to $D^b(B)$. Let

$$F'=\mathrm{Hom}_A(T, -): K^{+,b}({}_A\mathcal{I}) \longrightarrow K^{+,b}(\mathrm{add}\,\mathfrak{t})$$

Then $F=Q^{+,b}F'$ is a triangle equivalence from $D^b(A)$ to $D^b(B)$.

For the inverse we identify $D^b(B)$ with $K^{-,b}({}_B\mathcal{P})$ and consider the full subcategory $\mathfrak{t}'$ of $\mathrm{mod}\,A$ with objects $T\otimes_B P$ for $P\in{}_B\mathcal{P}$. We consider $K^{-,b}(\mathrm{add}\,\mathfrak{t}')$ as a full subcategory of $K^{-,b}(\mathrm{mod}\,A)$ and denote by $Q^{-,b}$ the localization functor from $K^{-,b}(\mathrm{mod}\,A)$ to $D^b(A)$. Let

$$G'=T\otimes_B - : K^{-,b}({}_B\mathcal{P}) \longrightarrow K^{-,b}(\mathrm{add}\,\mathfrak{t}')$$

Then $G=Q^{-,b}G'$ is a triangle equivalence from $D^b(B)$ to $D^b(A)$, which is quasiinverse to $F$.

**2.2.** Let $A$ be a finite-dimensional algebra and ${}_AT$ be a tilting module with $B=\mathrm{End}_A T$. It is known that in this case gl. dim $A<\infty$ if and only if gl. dim $B<\infty$. The next result generalizes this.

THEOREM. $\mathrm{fd}(A)<\infty$ *if and only if* $\mathrm{fd}(B)<\infty$.

PROOF. Let ${}_BM$ be a $B$-module of finite projective dimension and let ${}_BN$ an arbitrary $B$-module. Let $P^\cdot(M)$ and $P^\cdot(N)$ be minimal projective resolutions of $M$ and $N$. Note that by assumption $P^\cdot(M)\in K^b({}_B\mathcal{P})$ and $P^\cdot(N)\in K^{-,b}({}_B\mathcal{P})$. Using the notation of 2.1 we have the following:

$$\begin{aligned}\mathrm{Ext}^t_B(M, N) &= \mathrm{Hom}_{D^b(B)}(M, N[t])\\ &= \mathrm{Hom}_{K^{-b}({}_B\mathcal{P})}(P^\cdot(M), P^\cdot(N)[t])\\ &= \mathrm{Hom}_{D^b(A)}(G(P^\cdot(M)), G(P^\cdot(N)[t]))\\ &= \mathrm{Hom}_{D^b(A)}(G(P^\cdot(M)), G(P^\cdot(N))[t])\end{aligned}$$

for all $t\in\boldsymbol{N}$. We consider the complexes $Q_1^\cdot=(Q_1^i, d^i)=T\otimes_B P^\cdot(M)$ and $Q_2^\cdot=T\otimes_B P^\cdot(N)$. Note that $Q_2^i=0$ for $i>0$ and that $H^{-s}(Q_1^\cdot)=0$ for $s>\mathrm{pd}\,T_B=r<\infty$ since $H^{-s}(Q_1^\cdot)=\mathrm{Tor}^B_s(T, M)$. We consider the complex $X^\cdot\in D^b(A)$ isomorphic to $Q_1^\cdot$

$$X^\cdot= \cdots 0 \longrightarrow \ker d^{-r} \longrightarrow Q_1^{-r} \xrightarrow{d^{-r}} \cdots \longrightarrow Q_1^{-1} \longrightarrow Q_1^0 \longrightarrow 0 \cdots$$

Since $X^\cdot\cong G(P^\cdot(M))\in K^b({}_A\mathcal{P})$ we infer that $\mathrm{pd}_A \ker d^{-r}<\infty$. Therefore $X^\cdot\cong Q^\cdot \in K^b({}_A\mathcal{P})$ with width $w(Q^\cdot)\leqq r+\mathrm{fd}(A)+2$. Thus $Q^\cdot$ and $Q_2^\cdot[t]$ have disjoint support for $t>r+\mathrm{fd}(A)+2$. The calculation above then shows that $\mathrm{Ext}^t_B(M, N)=0$ for $t>r+\mathrm{fd}(A)+2$, so $\mathrm{pd}_B M\leqq r+\mathrm{fd}(A)+2$, in particular $\mathrm{fd}(B)<\infty$.

## 3. Recollement.

**3.1.** Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be triangulated categories. Following [**BBD**] a *recollement* of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is given by

$$\begin{array}{ccccc} & \overset{i^*}{\longleftarrow} & & \overset{j_!}{\longleftarrow} & \\ \mathcal{C}' & \overset{i_*}{\longrightarrow} & \mathcal{C} & \overset{j^!}{\longrightarrow} & \mathcal{C}'' \\ & \underset{i^!}{\overset{i_!}{\longleftarrow}} & & \underset{j_*}{\overset{j^*}{\longleftarrow}} & \end{array}$$

such that

(RI) $(i^*, i_*)$, $(i_!, i^!)$, $(j_!, j^!)$ and $(j^*, j_*)$ are adjoint pairs of exact functors and that $i_*=i_!$, $j^!=j^*$

(RII) $j^*i_*=0$

(RIII) $i^*i_*\cong id$, $id\cong i^!i_!$, $j^*j_*\cong id$ and $id\cong j^!j_!$

(RIV) For $X\in\mathcal{C}$ there are triangles

$$j_!j^!X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow j_!j^!X[1]$$

$$i_!i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow i_!i^!X[1]\,.$$

(The morphisms in (RIII) and (RIV) are the adjunction morphisms.

Note that it is a consequence from the definition that also $i^!j_*=0$ and $i^*j_!=0$. Also we point out that the functors $i_*$, $j_!$ and $j_*$ are full embeddings.

We refer to [**BBD**] for properties of recollements and to [**Kö**] for necessary and sufficient conditions that $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$ for some finite-dimensional algebras $A$, $A'$, $A''$.

In particular we mention the following result from [**Kö**]. If $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$ for some finite-dimensional algebras $A$, $A'$, $A''$ and one of the algebras $A$, $A'$, $A''$ has finite global dimension then $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$.

**3.2.** For the proof of theorem 2 from the introduction we need some preparation. We have a canonical embedding of mod $A\to D^b(A)$ which sends an $A$-module $X$ to the complex concentrated in degree zero with stalk equal to $X$. We will identify $X$ with the corresponding stalk complex.

We will also need the following facts. For a proof we refer to [**Ri**]. Let $X^\cdot\in D^b(A)$, then

$$(*) H^{-j}(X^{\cdot}) = \mathrm{Hom}_{D^b(A)}({}_AA, X^{\cdot}[j]) \qquad \text{for all } j \in \boldsymbol{Z}$$
$$= \mathrm{Hom}_{D^b(A)}(X^{\cdot}[-j], D({}_AA)) \qquad \text{for all } j \in \boldsymbol{Z}.$$

The subcategories $K^b({}_A\mathcal{P})$ of $D^b(A)$ and $K^b({}_A\mathcal{I})$ of $D^b(A)$ can be characterized as follows:

$$(-) K^b({}_A\mathcal{P}) = \{X^{\cdot} \in D^b(A) \mid \forall Y^{\cdot} \in D^b(A) \exists t_0 \text{ with } \mathrm{Hom}(X^{\cdot}, Y^{\cdot}[t]) = 0 \forall t \geqq t_0\}$$
$$(+) K^b({}_A\mathcal{I}) = \{X^{\cdot} \in D^b(A) \mid \forall Y^{\cdot} \in D^b(A) \exists t_0 \text{ with } \mathrm{Hom}(Y^{\cdot}, X^{\cdot}[t]) = 0 \forall t \geqq t_0\}$$

LEMMA. *Let $B$, $C$ be finite-dimensional algebras and let*

$$D^b(C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} D^b(B)$$

*be exact functors with $(i^*, i_*)$ an adjoint pair. Then there is $r \geqq 0$ such that $H^{-j}(i_*X) = 0$ for all $X \in \mathrm{mod}\, C$ and all $j \geqq r$.*

PROOF. Let $P^{\cdot} = i^*{}_BB$. We claim that $P^{\cdot} \in K^b({}_C\mathcal{P})$. For this let $Y^{\cdot} \in D^b(C)$. Then

$$\mathrm{Hom}(P^{\cdot}, Y^{\cdot}) = \mathrm{Hom}({}_BB, i_*Y^{\cdot}) \text{ by adjointness}.$$

Since $i_*Y^{\cdot} \in D^b(B)$ and ${}_BB \in K^b({}_B\mathcal{P})$ there is $t_0$ such that

$$\mathrm{Hom}({}_BB, i_*Y^{\cdot}[t]) = 0 \text{ for all } t \geqq t_0 \text{ by } (-).$$

So there is $t_0$ such that

$$\mathrm{Hom}(P^{\cdot}, Y^{\cdot}[t]) = 0 \qquad \text{for all } t \geqq t_0,$$

hence $P^{\cdot} \in K^b({}_C\mathcal{P})$ by $(-)$.

Since $P^{\cdot} \in K^b({}_C\mathcal{P})$ there is $r \geqq 0$ such that $P^{\cdot}$ and $X[j]$ have disjoint support for all $j \geqq r$. So the assertion follows from $(*)$.

The following is dual to the previous lemma.

LEMMA. *Let $B$, $C$ be finite-dimensional algebras and let*

$$D^b(C) \underset{j^!}{\overset{j_!}{\rightleftarrows}} D^b(B)$$

*be exact functors with $(j_!, j^!)$ an adjoint pair. Then there is $r \geqq 0$ such that $H^{-j}(j_!X) = 0$ for all $X \in \mathrm{mod}\, C$ and all $j \geqq r$.*

PROOF. Let $I^{\cdot} = j^!D(B_B)$. We claim that $I^{\cdot} \in K^b({}_C\mathcal{I})$. For this let $Y^{\cdot} \in$

$D^b(C)$. Then

$$\mathrm{Hom}(Y^{\cdot}, I^{\cdot})=\mathrm{Hom}(j_!Y^{\cdot}, D(B_B)) \text{ by adjointness.}$$

Since $j_!Y^{\cdot}\in D^b(B)$ and $D(B_B)\in K^b({}_B\mathcal{I})$ there is $i_0$ such that

$$\mathrm{Hom}(i_!Y^{\cdot}, D(B_B)[t])=0 \quad \text{for all } t\geqq t_0 \text{ by } (+).$$

So there is $t_0$ such that

$$\mathrm{Hom}(Y^{\cdot}, I^{\cdot}[t])=0 \quad \text{for all } t\geqq t_0,$$

hence $I^{\cdot}\in K^b({}_C\mathcal{I})$ by (+).

Since $I^{\cdot}\in K^b({}_C\mathcal{I})$ there is $r\geqq 0$ such that $I^{\cdot}$ and $X[-j]$ have disjoint support for all $j\geqq r$. So the assertion follows from (*).

**3.3.** We will now give the proof of theorem 2 from the introduction.

THEOREM. *Let $A$ be a finite-dimensional algebra and assume that $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$ for some finite-dimensional algebras $A'$, $A''$. Then* $\mathrm{fd}(A)<\infty$ *if and only if* $\mathrm{fd}(A')<\infty$ *and* $\mathrm{fd}(A'')<\infty$.

PROOF. We first show that $\mathrm{fd}(A')<\infty$.

Let ${}_{A'}S$ be a simple $A'$-module and let $Y^{\cdot}(S)=i_*S$. Since $Y^{\cdot}(S)\in D^b(A)$ there is $t_0\geqq 0$ with $Y^t(S)=0$ for all $t\geqq t_0$ and all simple $A'$-modules $S$. Let $X\in$ mod $A'$ with $\mathrm{pd}_{A'}X<\infty$. Then

$$\begin{aligned}\mathrm{Ext}^t_{A'}(X, S)&=\mathrm{Hom}_{D^b(A')}(X, S[t])\\&=\mathrm{Hom}_{D^b(A)}(i_*X, Y^{\cdot}(S)[t]).\end{aligned}$$

We claim that $i_*X\in K^b({}_A\mathcal{P})$. For this let $Y^{\cdot}\in D^b(A)$. By (−) there is $m_0$ such that for all $m\geqq m_0$ we have that

$$\mathrm{Hom}_{D^b(A')}(X, i^!Y^{\cdot}[m])=0.$$

But again by adjointness we have that

$$\mathrm{Hom}_{D^b(A)}(i_*X, Y^{\cdot}[m])=\mathrm{Hom}_{D^b(A')}(X, i^!Y^{\cdot}[m]).$$

Thus using (−) again we infer that $i_*X\in K^b({}_A\mathcal{P})$. So

$$i_*X=P^*=\cdots 0\to P^{-s}\to\cdots\to P^0\to\cdots\to P^{s'}\to 0\cdots.$$

Let $r\geqq 0$ be an integer satisfying the assertion in 3.2. If $s\leqq r$, then for $t>t_0+r$ we infer that $i_*X$ and $Y^{\cdot}(S)[t]$ have disjoint support for all simple $A'$-modules $S$. If $s>r$, then by 3.2 we infer that $\mathrm{pd}_A \ker d_P^{-r}<\infty$, hence

$\mathrm{pd}_A \ker d_P^{-r} \leqq \mathrm{fd}(A) < \infty$. Hence for $t > t_0 + r + \mathrm{fd}(A)$ we infer that $i_* X$ and $Y^{\cdot}(S)[t]$ have disjoint support for all simple $A'$-modules $S$. Hence $\mathrm{Ext}^t_{A'}(X, S) = 0$ for all $t > t_0 + r + \mathrm{fd}(A)$ and all simple $A'$-modules $S$. In particular, $\mathrm{fd}(A') < \infty$.

A similar proof shows that $\mathrm{fd}(A'') < \infty$. For the convenience of the reader we supply the details.

Let ${}_{A''}S$ be a simple $A''$-module and let $Y^{\cdot}(S) = j_! S$. Since $Y^{\cdot}(S) \in D^b(A)$ there is $t_0 \geqq 0$ with $Y^t(S) = 0$ for all $t \geqq t_0$ and all simple $A'$-modules $S$. Let $X \in \mathrm{mod}\, A''$ with $\mathrm{pd}_{A''} X < \infty$. Then

$$\begin{aligned}\mathrm{Ext}^t_{A''}(X, S) &= \mathrm{Hom}_{D^b(A'')}(X, S[t]) \\ &= \mathrm{Hom}_{D^b(A)}(j_! X, Y^{\cdot}(S)[t]).\end{aligned}$$

We claim that $j_! X \in K^b({}_A\mathcal{P})$. For this let $Y^{\cdot} \in D^b(A)$. By (—) there is $m_0$ such that for all $m \geqq m_0$ we have that

$$\mathrm{Hom}_{D^b(A'')}(X, j^! Y^{\cdot}[m]) = 0.$$

But again by adjointness we have that

$$\mathrm{Hom}_{D^b(A)}(j_! X, Y^{\cdot}[m]) = \mathrm{Hom}_{D^b(A'')}(X, j^! Y^{\cdot}[m]).$$

Thus using (—) again we infer that $j_! X \in K^b({}_A\mathcal{P})$. So

$$j_! X = P^{\cdot} = \cdots 0 \longrightarrow P^{-s} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^{s'} \longrightarrow 0 \cdots.$$

Let $r \geqq 0$ be an integer satisfying the assertion of the second lemma in 3.2. If $s \leqq r$, then for $t > t_0 + r$ we infer that $j_! X$ and $Y^{\cdot}(S)[t]$ have disjoint support for all simple $A''$-modules $S$. If $s > r$, then by 3.2 we infer that $\mathrm{pd}_A \ker d_P^{-r} < \infty$, hence $\mathrm{pd}_A \ker d_P^{-r} \leqq \mathrm{fd}(A) < \infty$. Hence for $t > t_0 + r + \mathrm{fd}(A)$ we infer that $j_! X$ and $Y^{\cdot}(S)[t]$ have disjoint support for all simple $A''$-modules $S$. Hence $\mathrm{Ext}^t_{A''}(X, S) = 0$ for all $t > t_0 + r + \mathrm{fd}(A)$ and all simple $A''$-modules $S$. In particular, $\mathrm{fd}(A'') < \infty$.

We now show the converse.

Let $X \in \mathrm{mod}\, A$ with $\mathrm{pd}_A X < \infty$ and $S$ a simple $A$-module. By (RIV) there are triangles:

$$j_! j^! X \longrightarrow X \longrightarrow i^* i^* X \longrightarrow j_! j^! X[1]$$

and

$$i_! i^! S \longrightarrow S \longrightarrow j_* j^* S \longrightarrow i_! i^! S[1].$$

For abbreviation we set: $X' = j_! j^! X$, $X'' = i_* i^* X$ and $S' = i_! i^! S$, $S'' = j_* j^* S$. Apply $\mathrm{Hom}_{D^b(A)}(X, -)$ to the second triangle. This yields a long exact sequence, where we use ${}^m(-, -) = \mathrm{Hom}_{D^b(A)}(-, -[m])$ as an abbreviation.

$$\cdots \longrightarrow {}^m(X, S') \longrightarrow {}^m(X, S) \longrightarrow {}^m(X, S'') \longrightarrow {}^{m+1}(X, S') \longrightarrow \cdots.$$

Applying $\mathrm{Hom}_{D^b(A)}(-, S')$ to the first triangle yields a long exact sequence

$$\cdots \longrightarrow {}^{m}(X'', S') \longrightarrow {}^{m}(X, S') \longrightarrow {}^{m}(X', S') \longrightarrow {}^{m+1}(X'', S') \longrightarrow \cdots.$$

Applying $\mathrm{Hom}_{D^b(A)}(-, S'')$ to the first triangle yields a long exact sequence

$$\cdots \longrightarrow {}^{m}(X'', S'') \longrightarrow {}^{m}(X, S'') \longrightarrow {}^{m}(X', S'') \longrightarrow {}^{m+1}(X'', S'') \longrightarrow \cdots.$$

Next we observe that

$$\begin{aligned} {}^{m}(X', S') &= {}^{m}(j_!j^!X, i^!i^!S) \\ &= {}^{m}(j_!j^!X, i_*i^!S) \\ &= {}^{m}(i^*j_!j^!X, i^!S) \quad \text{(by adjointness)} \\ &= 0 \qquad\qquad \text{(since } i_*j_! = 0) \end{aligned}$$

and that

$$\begin{aligned} {}^{m}(X'', S'') &= {}^{m}(i_*i^*X, j_*j^*S) \\ &= {}^{m}(i_!i^*X, j_*j^*S) \\ &= {}^{m}(i^*X, i^!j_*j^*S) \quad \text{(by adjointness)} \\ &= 0 \qquad\qquad \text{(since } i^!j_* = 0). \end{aligned}$$

We claim that there is $m_0 \geqq 0$ such that ${}^{m}(X'', S') = 0$ for all $m \geqq m_0$. Note that

$$\begin{aligned} {}^{m}(X'', S') &= {}^{m}(i_*i^*X, i_!i^!S) \\ &= \mathrm{Hom}_{D^b(A')}(i^*X, i^!S[m]) \end{aligned}$$

Let $Y^{\cdot}(S) = i^!S$. So there is $t_0 \geqq 0$ such that $Y^t(S) = 0$ for all $t \geqq t_0$. As in 3.2 we may show that $i^*X \in K^b({}_{A'}\mathcal{P})$. So

$$i^*X = P^{\cdot} = \cdots 0 \longrightarrow P^{-s} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^{s'} \longrightarrow 0 \cdots.$$

We may apply the second lemma in 3.2 to the pair $(i^*, i_*)$. So there is $r \geqq 0$ such that $H^{-j}(i_*X) = 0$ for all $X \in \mathrm{mod}\, A$ and all $j \geqq r$. If $s \leqq r$, then for $m > t_0 + r$ we infer that $i^*X$ and $Y^{\cdot}(S)[m]$ have disjoint support for all simple $A$-modules $S$. If $s > r$, then by 3.2 we infer that $\mathrm{pd}_{A'} \ker d_{\bar{P}}^{r} < \infty$, hence $\mathrm{pd}_{A'} \ker d_{\bar{P}} \leqq \mathrm{fd}(A') < \infty$. Hence for $m > t_0 + r + \mathrm{fd}(A')$ we infer that $i^*X$ and $Y^{\cdot}(S)[m]$ have disjoint support for all simple $A$-modules $S$. Hence there is $m_0 \geqq 0$ such that ${}^{m}(X'', S') = 0$ for all $m \geqq m_0$.

We claim that there is $n_0 \geqq 0$ such that ${}^{n}(X', S'') = 0$ for all $n \geqq n_0$. Note that

$$\begin{aligned} {}^{n}(X', S'') &= {}^{n}(j_!j^!X, j_*j^*S) \\ &= \mathrm{Hom}_{D^b(A'')}(j^!X, j^*S[m]) \end{aligned}$$

Let $Y^{\cdot}(S) = j^*S$. So there is $t_0 \geqq 0$ such that $Y^t(S) = 0$ for all $t \geqq t_0$. We show first that $j^!X=j^*X\in K^b({}_{A''}\mathscr{P})$. For this let $Y^\cdot\in D^b(A'')$. Then

$$\mathrm{Hom}(j_*X,\ Y^\cdot)=\mathrm{Hom}(X,\ j_*Y^\cdot) \quad \text{by adjointness.}$$

Since $j_*Y^\cdot\in D^b(A)$ and $X\in K^b({}_B\mathscr{P})$ there is $p_0$ such that

$$\mathrm{Hom}(X,\ j_*Y^\cdot[t])=0 \qquad \text{for all} \quad p\geqq p_0 \text{ by } (-).$$

So there is $p_0$ such that

$$\mathrm{Hom}(j^*X,\ Y^\cdot[p])=0 \qquad \text{for all} \quad p\geqq p_0,$$

hence $j^*X\in K^b({}_{A''}\mathscr{P})$ by $(-)$. So

$$j^!X=P^\cdot=\cdots 0\longrightarrow P^{-s}\longrightarrow\cdots\longrightarrow P^0\longrightarrow\cdots\longrightarrow P^{s'}\longrightarrow 0\cdots.$$

We may apply the first lemma in 3.2 to the pair $(j_!,\ j^!)$. So there is $r\geqq 0$ such that $H^{-j}(j^!X)=0$ for all $X\in\mathrm{mod}\,A$ and all $j\geqq r$. If $s\leqq r$, then for $n>t_0+r$ we infer that $j^!X$ and $Y^\cdot(S)[n]$ have disjoint support for all simple $A$-modules $S$. If $s>r$, then by 3.2 we infer that $\mathrm{pd}_{A''}\ker d_P^{-r}<\infty$, hence $\mathrm{pd}_{A''}\ker d_P^{-r}\leqq \mathrm{fd}(A'')<\infty$. Hence for $n>t_0+r+\mathrm{fd}(A'')$ we infer that $j^!X$ and $Y^\cdot(S)[m]$ have disjoint support for all simple $A$-modules $S$. Hence there is $n_0\geqq 0$ such that ${}^n(X',\ S'')=0$ for all $n\geqq n_0$.

Let $s_0=\max(m_0,\ n_0)$.

The previous considerations show that ${}^s(X,\ S')={}^s(X,\ S'')=0$ for all $s\geqq s_0$. The first long exact sequence then shows that ${}^s(X,\ S)=\mathrm{Ext}^s_A(X,\ S)=0$ for all $s\geqq s_0$ and all simple $A$-modules $S$. In particular, $\mathrm{fd}(A)<\infty$.

Note that the theorem above generalizes a result of [**W**] where under the same assumptions it was shown that gl. dim $A<\infty$ if and only if gl. dim $A'<\infty$ and gl. dim $A''<\infty$.

**3.4.** We give two examples to which this theorem may be applied. We stress that there exist proofs of these results avoiding the use of triangulated categories (see for example [**FS**]).

Let $A'$, $A''$ be finite-dimensional algebras and let ${}_{A'}M_{A''}$ be a bimodule. Consider the triangular matrix algebra $A$ of the form

$$A=\begin{pmatrix} A' & M \\ 0 & A'' \end{pmatrix}$$

with multiplication

$$\begin{pmatrix} a' & m \\ 0 & a'' \end{pmatrix}\begin{pmatrix} b' & m' \\ 0 & b'' \end{pmatrix}=\begin{pmatrix} a'b' & a'm'+mb'' \\ 0 & a''b'' \end{pmatrix}$$

where $a'$, $b'\in A'$, $m$, $m'\in M$ and $a''$, $b''\in A''$.

It is easy to see that the reslt in [**Kö**] may be applied. So $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$.

Asume that $A'$ or $A''$ has finite global dimension. Then $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$. In particular, $\mathrm{fd}(A)<\infty$ if $\mathrm{fd}(A')<\infty$ and $\mathrm{fd}(A'')<\infty$.

We refer to [**Kö**] for other conditions that $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$.

In the next example we will use the concept of perpendicular categories as introduced in [**GL**], see also [**H4**].

Let $X\in \mathrm{mod}\, A$ with $\mathrm{pd}_A X\leqq 1$. We define the right *perpendicular category* $X^\perp$ to be the full subcategory of $\mathrm{mod}\, A$ whose objects $Z$ satisfy

$$\mathrm{Hom}_A(X, Z)=0=\mathrm{Ext}_A^1(X, Z).$$

It is straightforward to see that $X^\perp$ is an abelian category, which is closed under extensions and that the inclusion functor $X^\perp \hookrightarrow \mathrm{mod}\, A$ is exact. The next result states some useful properties of $X^\perp$ under additional assumptions. For the proof we refer to [**GL**] or [**H4**].

THEOREM. *Let $X\in \mathrm{mod}\, A$ such that $\mathrm{pd}_A X\leqq 1$ and $\mathrm{Ext}_A^1(X, X)=0$, then there exists ${}_AQ\in X^\perp$ such that $X^\perp\cong \mathrm{mod}\, A_0$, with $A_0=\mathrm{End}_A Q$. If $X$ is indecomposable, then $rk\, K_0(A_0)=rk\, K_0(A)-1$, where $K_0(A)$ denotes the Grothendieck group of $A$.*

Now assume that $A$ admits a simple $A$-module $S$ with $\mathrm{pd}_A S=1$. Note that we clearly have $\mathrm{Ext}_A^1(S, S)=0$. Again using [**Kö**] we see that $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$, where $A'=\mathrm{End}_A Q$ for a projective generator ${}_AQ$ of $S^\perp$ and $A''=k$. Since gl. dim $k=0$ we infer that $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$. In particular, $\mathrm{fd}(A)<\infty$ if $\mathrm{fd}(A')<\infty$. We refer to [**H4**] and [**Kö**] for other situations in which $D^b(A)$ admits a recollement.

## 4. Grothendieck groups.

**4.1.** The generalized Nakayama conjecture is related to a problem about Grothendieck groups of triangulated categories. First we need a reformulation of the generalized Nakayama conjecture which is due to [**AR**].

(3′): Let ${}_AM$ be a cogenerator for $\mathrm{mod}\, A$ with $\mathrm{Ext}_A^i(M, M)=0$ for all $i>0$, then ${}_AM$ is injective.

The following is shown in [**AR**]. The generalized Nakayama conjecture holds for all finite-dimensional algebras if and only if the conjecture (3′) holds for all finite-dimensional algebras.

Let us indicate one direction.

The following notation seems to be useful. If ${}_AM$ is an $A$-module we may decompose ${}_AM=\bigoplus_{i=1}^{s} M_i^{n_i}$ with $M_i$ indecomposable, $M_i \not\cong M_j$ for $i\neq j$ and $n_i>0$. In this case we denote the number $s$ of non-isomorphic indecomposable direct summands of $M$ by $\delta(M)$.

We assume that (3′) holds for all finite-dimensional algebras with $rk\, K_0(A)=n-1$ and we claim that the generalized Nakayama conjecture holds for all finite-dimensional algebras with $rk\, K_0(A)=n$. In fact, let $A$ be an algebra with $rk\, K_0(A)=n$ and let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow D(A_A) \longrightarrow 0$$

be a minimal projective resolution of $D(A_A)$. Assume that there is a simple $A$-module $S$ with $\mathrm{Ext}_A^i(D(A_A), S)=0$ for all $i$. Let $P(S)$ be the projective cover of $S$. Then $P(S)$ is not a direct summand of $P_i$ for all $i$. Let ${}_AA=P\oplus P(S)^r$ such that $P(S)$ is not a summand of $P$. Let $B=\mathrm{End}_A P$. Then it follows from [**Ri**] that we have a full exact embedding of triangulated categories $D^-(B)\rightarrow D^-(A)$. By the choice of $P$ we infer that $K^b({}_A\mathcal{I})$ is contained in $D^-(B)$. Using the obvious identifications we may consider $D(A_A)$ as $B$-module. But $\delta(D(A_A))=n$ and $\mathrm{Ext}_B^i(D(A_A), D(A_A))=0$ for all $i>0$ yields a contradiction to (3′).

We recall now the definition of the Grothendieck group of a triangulated category [**Gr**]. For this let $\mathcal{C}$ be a triangulated category. Let $\mathcal{F}$ be the free abelian group on the isomorphism classes of objects in $\mathcal{C}$. The isomorphism class of an object $X\in\mathcal{C}$ is denoted by $[X]$. Let $\mathcal{F}'$ be the subgroup of $\mathcal{F}$ generated by $[X]+[Z]-[Y]$ for all triangles $X\rightarrow Y\rightarrow Z\rightarrow X[1]$ in $\mathcal{C}$. Then by definition the *Grothendiek group* of $\mathcal{C}$ is $K_0(\mathcal{C})=\mathcal{F}/\mathcal{F}'$.

Let $F$: $\mathcal{C}'\rightarrow\mathcal{C}$ be an exact functor of triangulated categories. Then there is an induced map $K_0(F)$: $K_0(\mathcal{C}')\rightarrow K_0(\mathcal{C})$.

For example consider the embedding of $K^b({}_A\mathcal{P})$ into $D^b(A)$. Then the induced map on the level of Grothendieck groups turns out to be the Cartan map (see [**B2**] for a definition). In particular we see that $K_0(F)$ need not to be injective, if $F$ is an embedding, since it is easy to construct examples of finite-dimensional algebras $A$ such that the determinant of the Cartan map vanishes.

In the following proof we will identify an object $X$ with its isomorphism class $[X]$.

PROPOSITION *Let $\mu$; $D^b(A)\rightarrow D^-(A)$ be the canonical embedding. Then*

$K_0(\mu)=0$.

PROOF. Let $X^{\cdot}=(X^i, d_X^i)\in D^b(A)$ be a complex. Then $X^{\cdot}=\sum(-1)^i X^i$ in $K_0(D^b(A))$. Therefore it is enough to show that $K_0(\mu)(X)=0$ for $X\in \bmod A$, which we identify with the complex of $D^b(A)$ concentrated in degree zero with stalk equal to $X$. For $X\in \bmod A$ we consider the following complexes. Let $Y^{\cdot}=\coprod_{i\leq 0} X[i]$ and $Z^{\cdot}=\coprod_{i<0} X[i]$. We have a triangle in $D^-(A)$ of the form $X\to Y^{\cdot}\to Z^{\cdot}\to X[1]$. Thus $X+Z^{\cdot}-Y^{\cdot}=0$ in $K_0(D^-(A))$. Also consider $Y_1^{\cdot}=Z_1^{\cdot}=\coprod_{i\leq 0} X[2i-1]$, $Y_2^{\cdot}=\coprod_{i\leq 0} X[2i]$ and $Z_2^{\cdot}=\coprod_{i<0} X[2i]$. We have triangles in $D^-(A)$:

$$Z_1^{\cdot}\longrightarrow Z^{\cdot}\longrightarrow Z_2^{\cdot}\longrightarrow Z_1^{\cdot}[1]$$

and

$$Y_1^{\cdot}\longrightarrow Y^{\cdot}\longrightarrow Y_2^{\cdot}\longrightarrow Y_1^{\cdot}[1].$$

In particular, $Z_1^{\cdot}+Z_2^{\cdot}-Z^{\cdot}=0$ and $Y_1^{\cdot}+Y_2^{\cdot}-Y^{\cdot}=0$ in $K_0(D^-(A))$.

Let $\tilde{X}^{\cdot}=(\tilde{X}^i, d^i)$ with $\tilde{X}^i=X$ for $i=0, -1$ and $\tilde{X}^i=0$ for $-1\neq i\neq 0$, and $d^{-1}=id_X$ and $d^i=0$ for $i\neq -1$. Let $K_1^{\cdot}=\coprod_{i\leq 0}\tilde{X}[2i]$ and $K_2^{\cdot}=\coprod_{i\leq 0}\tilde{X}[2i-1]$. Note that $K_1^{\cdot}=K_2^{\cdot}$ in $D^-(A)$, since they are homotopy-equivalent to the zero complex in $K^-(\bmod A)$. Clearly we have triangles

$$Y_2^{\cdot}\longrightarrow K_1^{\cdot}\longrightarrow Y_1^{\cdot}\longrightarrow Y_2^{\cdot}[1]$$

and

$$Z_2^{\cdot}\longrightarrow K_2^{\cdot}\longrightarrow Z_1^{\cdot}\longrightarrow Z_2^{\cdot}[1]$$

Thus $Y_1^{\cdot}+Y_2^{\cdot}-K_1^{\cdot}=0$ and $Z_1^{\cdot}+Z_2^{\cdot}-K_2^{\cdot}=0$ in $K_0(D^-(A))$. Summarizing we obtain in $K_0(D^-(A))$

$$\begin{aligned} X&=Y^{\cdot}-Z^{\cdot}\\ &=Y_1^{\cdot}+Y_2^{\cdot}-Z_1^{\cdot}-Z_2^{\cdot}\\ &=K_1^{\cdot}-K_2^{\cdot}\\ &=0. \end{aligned}$$

It was shown in [**Gr**] that $K_0(D^b(A))\cong K_0(A)$, which is isomorphic to $\mathbf{Z}^n$, with $n=\delta({}_AA)$.

**4.2.** Following [**V**] we call a full triangulated subcategory $C'$ of a triangulated subcategory $C$ an *épaisse subcategory*, if $C'$ is closed under direct summands. We consider the following condition:

(3″): Let $C$ be an épaisce subcategory of $D^b(A)$ such that $K_0(A)$ is finitely generated. Then $rk\, K_0(C)\leq n$.

REMARK. If (3″) holds then the generalized Nakayama conjecture holds.

PROOF. By the mentioned result of [**AR**] it is enongh to verify condition (3′). Let ${}_AM$ be a generator which satisfies $\mathrm{Ext}_A^i(M, M)=0$ for all $i>0$. Then it is easy to see that we obtain a full embedding $K^b$ (add $M$)$\to D^b(A)$ (compare [**H1**]). So we may consider $K^b$(add $M$) as an épaisse subcategory of $D^b(A)$. A straightforward calculation shows that $K_0(K^b(\mathrm{add}\, M))\cong \boldsymbol{Z}^{\delta(M)}$. So $\delta(M)\leqq n$ by (3″). Since $M$ is a generator we know that ${}_AA$ is a direct summand of $M$, hence $M$ is projective.

Note that the proof actually shows that for an arbitrary $A$-module $M$ which satisfies $\mathrm{Ext}_A^i(M, M)=0$ for all $i>0$ the number $\delta(M)\leqq n$ in case (3″) holds.

It is easy to construct counterexamples to the condition (3″) if we leave out the assumptions that $\mathcal{C}$ is épaisse or that $K_0(\mathcal{C})$ is finitely generated. For instance let $A$ be a finite-dimensional tame hereditary algebra and let $\mathcal{C}=D^b(\mathfrak{R})$ be the derived category of the abelian subcategory $\mathfrak{R}\subset \mathrm{mod}\, A$ of regular $A$-modules. Clearly $\mathcal{C}$ is an épaisse subcategory of $D^b(A)$, but $K_0(\mathcal{C})$ is not finitely generated. We thank H. Lenzing for this example, which led to a reformulation of a more optimistic version of (3″).

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