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ON THE NON-ZERO POINTS OF SECTIONS OF A LINE BUNDLE ON AN ABELIAN VARIETY

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In the present paper, we generalize a main theorem given by Koizumi [2] for positive characteristic. The result is given in the following style:

Let (X, L) be a principally polarized abelian variety over an algebraically closed field of characteristic $p\geqq 0$; a, b be two positive integers with $b>a$. Assnme that $K(L^{b})$ has a reduced maxiaml isotropic direct summand $H(b)$, where $K(L^{b})$ is the scheme-theoretic kernel of the homomorphism $\phi_{L^{b}}$ of X to the dual abelian variety $\hat{X}$ defined by $\phi_{L^{b}}(x)=T_{x}^{*}L^{b}\otimes L^{-b}$. Then, for a non-trivial section f in $\Gamma(L^{a})$ and a point x in X, there exists a point y in $H(b)$ such that $f(x+y)\neq 0$.

This fact was first discovered by Koizumi [1] for characteristic zero provided $(a, b)=1$, and immediately thereafter he removed the condition $(a, b)=1$ in [2]. Later, the author ([6], [7]) generalized it for every characteristic under the condition $(a, b)=1$. Here, we shall remove the condition $(a, b)=1$ in the case of positive characteristic, following Koizumi's idea in [2] and using the methods developed in [7].

Moreover, as an application of our result, we shall show the fact:

Let (X, L) be a polarized abelian variety over an algebraically closed field k, and assume that $K(L^{3})$ has a reduced maximal isotropic direct summand. Then the cannonical map

$$\sum_{\alpha\in\hat{X}_{3}(k)}\Gamma(L\otimes P_{\alpha})\otimes\Gamma(L^{2}\otimes P_{-\alpha})\longrightarrow\Gamma(L^{3})$$

is surjective, where P is the Poincaré invertible sheaf on $X\times\hat{X}$ and $P_{\hat{x}}=P|_{X\times\{\hat{x}\}}$ for a point $\hat{x}\in\hat{X}$.

According to the arguments in [8], this fact implies that $\phi(X)$ is ideal-theoretically an intersection of cubics when X is ordinary, even if the characteristic is three. Here $\phi: X\longrightarrow \mathbf{P}(\Gamma(L^{3}))$ is the canonical embedding of X into the projective space $\mathbf{P}(\Gamma(L^{3}))$.

The latter result was obtained by Mr. R. Sasaki independently with his technical but skillful method. During the preparation of this paper the author obtained

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some useful suggestions from conversations with Mr. R. Sasaki, to whom he would like to express his hearty thanks.

We follow the previous papers [6], [7] for notation in this paper.

Throughout the paper, we denote by k an algebraically closed field of characteristic $p\geqq 0$, and by X an abelian variety over k of dimension g. Let a, b be two positive integers; and $\xi: X\times X\longrightarrow X\times X$ be the homomorphism defined by $(x, y)\longmapsto(x-by, x+ay)$. Then for a symmetric invertible sheaf L on X and closed points α, β in $\hat{X}$, we have an isomorphism

$$(1)\qquad \xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))\simeq p_1^*(L^{a+b}\otimes p_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$$

(cf. [6], Prop. 1. 2). Hereafter, we fix a principal symmetric invertible sheaf L (i.e., an ample symmetric invertible sheaf with Euler-Poincaré characeristic 1). Moreover, we assume the condition

(C) $K(L^{a+b})$ has a Göpel decomposition $K(L^{a+b})=H_1(a+b)\oplus H_2(a+b)$ with $H_1(a+b)=H_1(a+b)_{\text{red.}}$.

Obviously this condition is satisfied for any X and a, b with $p\nmid a+b$ and for an ordinary abelian variety X and any a, b. Under the condition (C), we can easily take a maximal isotropic subgroup $H(b(a+b))$ of $K(L^{b(a+b)})$ satisfying the condition

(C′) $bH(b(a+b))=H_1(a+b)$, $H(b(a+b))\cap H_2(a+b)=\{0\}$ and $(a+b)H(b(a+b))$ is a maximal isotropic subgroup $H(b)$ of $K(L^b)$.

Let $K=\text{Ker}\ \xi=\{(by, y)|y\in X_{a+b}\}$; let K^* be the level subgroup in $\mathcal{G}(p_1^*(L^{a+b}\otimes P_{a+b})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$ of K corresponding to the isomorphism (1); and let $\mathcal{G}^*$ be the centralizer of K^*. Then we have a canonical exact sequence

$$1\longrightarrow K^*\longrightarrow \mathcal{G}^*\xrightarrow{\mathcal{G}(\xi)}\mathcal{G}(p_1^*(L^a\otimes P_2^*(L^b\otimes P_\beta))\longrightarrow 1\,.$$

Obviously,

$$(2)\qquad K(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))\supset K^\perp\supset\{(by, y)|y\in X_{b(a+b)}\}\,.$$

So we put $H_i(a+b)^\Delta=\{(by, y)|y\in H_i(a+b)\}$ $(i=1, 2)$ and $H(b(a+b))^\Delta=\{(by, y)|y\in H(b(a+b))\}$. Then we have the inclusions $H_1(a+b)\times H(b(a+b))\supset H_1(a+b)^\Delta$, $H_2(a+b)\times H_2(a+b)\supset H_2(a+b)^\Delta$, and an equality $K=H_1(a+b)^\Delta\oplus H_2(a+b)^\Delta$. Since K is lifted up to the level subgroup K^*, $H_i(a+b)^\Delta$ $(i=1, 2)$ are also automatically lifted up to level subgroups $H_i(a+b)^{\Delta*}$ $(i=1, 2)$ and we have a decomposition $K^*=H_1(a+b)^{\Delta*}\oplus H_2(a+b)^{\Delta*}$. Moreover, by the above inclusions, $H_1(a+b)\times H(b(a+b))$ (resp. $H_2(a+b)\times H_2(a+b)$) can be lifted up to a level subgroup $(H_1(a+b)\times H(b(a+b)))^*$ (resp. $(H_2(a+b)\times H_2(a+b))^*$) which contains $H_1(a+b)^{\Delta*}$ (resp. $H_2(a+b)^\Delta$). So, the group $H(b(a+b))^\Delta$ is also automatically lifted up to a level subgroup $H(b(a+b))^{\Delta*}$. The relation (2) implies that $\mathcal{G}^*$ contains $H(b(a+b))^{\Delta*}$, and $\mathcal{G}(\xi)(H(b(a+b))^{\Delta*})$ is a level subgroup of $\{0\}\times H(b)$, which we denote by $\bar{H}(b)^*$. Let

$$\tau:\quad \mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})\times \mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})\longrightarrow \mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$$

be the canonical homomorphism. Then, for suitable level subgroups $H(b(a+b))^*$, $H_1(a+b)^{**}$, $H_2(a+b)^{**}$ and $H_2(a+b)^*$, we have the isomorphisms

$$\text{rest. of } \tau:\quad H_1(a+b)^{**}\times H(b(a+b))^*\simeq(H_1(a+b)\times H(b(a+b)))^*$$

and

$$\text{rest. of } \tau:\quad H_2(a+b)^{**}\times H_2(a+b)\simeq(H_2(a+b)\times H_2(a+b))^*\,.$$

Note that $H(b)$ is automatically lifted up to a level subgroup $H(b)^*$ in $H(b(a+b))^*$. By these isomorphisms, the level subgroup $H(b(a+b))^{\Delta*}$ (resp. $H_2(a+b)^{\Delta*}$) defines a homomorphism ϕ_1: $H(b(a+b))^*\longrightarrow H_1(a+b)^{**}$ (resp. ϕ_2: $H_2(a+b)^*\longrightarrow H_2(a+b)^{**}$) with $\tau(\phi_1(\lambda),\ \lambda)\in H(b(a+b))^{\Delta*}$ (resp. $\tau(\phi_2(\mu),\ \mu)\in H_2(a+b)^{\Delta*}$). Here we take the sections t and v such that $\langle t\rangle=\Gamma(L^{a+b}\otimes P_{\alpha+\beta})^{H_2(a+b)^{**}}$ and $\langle v\rangle=\Gamma(L^b\otimes P_\beta)^{\bar{H}(b)^*}$. Under this notation we have

LEMMA 1. *For a non-zero section u of $\Gamma(L^a\otimes P_\alpha)$,*

$$\xi^*(u\otimes v)=\sum_{\lambda\in\mathrm{Rep}(H(b(a+b))^*/H(b)^*)}U_{\phi_1\lambda}t\otimes U_\lambda\theta \tag{3}$$

with some non-zero section θ of $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})^{H(b)^*+H_2(a+b)^*}$. Here, for a group G and its subgroup H, Rep(G/H) *denotes a complete set of representatives of the quotient G/H.*

PROOF. By our assumption, $\langle\{U_\lambda t\}_{\lambda\in H_1(a+b)^{**}}\rangle=\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$.

Therefore, $\xi^*(u\otimes v)$ can be written in the form

$$\xi^*(u\otimes v)=\sum_{\lambda\in H_1(a+b)^{**}}U_\lambda t\otimes\theta_\lambda \qquad \text{with} \quad \theta_\lambda\in\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})\,. \tag{4}$$

For a point μ in $H(b(a+b))^*$,

$$\xi^*(u\otimes v)=\sum_{\lambda\in H_1(a+b)^{**}}U_{\lambda+\phi_1\mu}t\otimes U_\mu\theta_\lambda\,.$$

Hence we have $U_\mu\theta_\lambda=\theta_{\lambda+\phi_1\mu}$ for $\lambda\in H_1(a+b)^*$ and $\mu\in H(b(a+b))^*$. In particular, $U_\mu\theta_0=\theta_{\phi_1\mu}$ for $\mu\in H(b(a+b))^*$ and $U_\mu\theta_\lambda=\theta_\lambda$ for $\lambda\in H_1(a+b)^*$ and $\mu\in H(b)^*$; i.e., $\theta_\lambda\in\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})^{H(b)^*}$. Therefore the equality (4) can be replaced by

$$\xi^*(u\otimes v)=\sum_{\lambda\in\mathrm{Rep}(H(b(a+b))^*/H(b)^*)}U_{\phi_1\lambda}t\otimes U_\lambda\theta\,,$$

where $\theta=\theta_0$. Moreover, for a point $\mu\in H_1(a+b)^*$,

$$\xi^*(u\otimes v)=\sum_{\lambda\in \mathrm{Rep}(H(b(a+b))^*/H(b)^*)} U_{\phi_2\mu}U_{\phi_1\lambda}t\otimes U_\mu U_\lambda\theta$$

$$=\sum_\lambda e^{L^{a+b}}(\phi_2\mu,\ \phi_1\lambda)U_{\phi_1\lambda}t\otimes U_\mu U_\lambda\theta\,.$$

Hence we have

$$U_\mu U_\lambda\theta=e^{L^{a+b}}(\phi_1\lambda,\ \phi_2\mu)U_\lambda\theta\,.$$

In particular, $U_\mu\theta=\theta$ for any $\mu\in H_2(a+b)^*$; i.e.,

$\theta\in\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})^{H(b)^*+H_2(a+b)^*}$. Q.E.D.

Under these preliminaries, we shall show the following theorem which is a generalization of Theorem 1.1 in [2] including the case of positive characteristic. Moreover, it is also a partial generalization of Theorem 1.4 in [7].

THEOREM 2. *Let (Y, M) be a principally polarized abelian variety of dimension g. Let c, d be two positive integers with $c>d$. Assume that $K(M^c)$ has a Göpel decomposition*

$$K(M^c)=H_1(M^c)\oplus H_2(M^c)\quad \text{with}\quad H_1(M^c)=H_1(M^c)_{\mathrm{red.}}\,.$$

Let $H_i(M^c)^*$ be a level subgroup in $G(M^{cd})$ of $H_i(M^c)$ for each $i=1,2$; and $\{\theta_1,\cdots,\theta_n\}$ be a basis of $\Gamma(M^{cd})^{H_2(M^c)^*}$, where $n=d^g$. Then, for any closed point $y\in Y$, we have

$$\mathrm{rank}\,[(U_\lambda\theta_i)(y)]_{(\lambda,i)\in H_1(M^c)^*\times\{1,\cdots,n\}}=d^g\,.$$

PROOF. Obviously, we have only to show our theorem for a suitable lifting of $H_2(M^c)$ and for a suitable basis $\{\theta_i\}$ of $\Gamma(M^{cd})^{H_2(M^c)^*}$. We put $d=a$ and $c-d=b$. In the same way as in the proof of Lemma 1.1 in [6], there exist an abelian variety X, an isogeny π: $X\longrightarrow Y$ and a principal symmetric invertible sheaf L on X satisfying the following conditions:

(a) $\pi^*(M^{cd})\simeq L^{bcd}\otimes P_\gamma$ for some $\gamma\in\hat{X}$,

(b) Ker π is a maximal isotropic subgroup $H(b)$ of $K(L^b)$,

(c) there exist a Göpel decomposition $K(L^{a+b})=H_1(a+b)\oplus H_2(a+b)$

with $H_1(a+b)=H_1(a+b)_{\mathrm{red.}}$ and a maximal isotropic subgroup $H(b(a+b))$ of $K(L^{b(a+b)})$ satisfying the condition (C'),

(d) $\pi(H(b(a+b))=H_1(M^c)$ and $\pi(H_2(a+b))=H_2(M^c)$.

Therefore, if we choose two closed points α, β in $\hat{X}$ such that $a\beta-b\alpha=cd\gamma$, Lemma 1 is applicable for (X, L) and the isotropic subgroups given in (c) and (d). Moreover, suitably choosing the two points α, β, we may assume that the level subgroup $H(b)^*$ given in Lemma 1 is defined by the isomorphism in (a). Hence $\Gamma(M^{cd})\xrightarrow[\sim]{\pi^*}\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})^{H(b)^*}$. Let $\{u_i\}_{i=1,\cdots,n}$ be a basis of $\Gamma(L^a\otimes P_\alpha)$. Then, by Lemma

1, we have

$$\xi^*(u_i\otimes v)=\sum_{\lambda\in \mathrm{Rep}(H(b(a+b))^*/H(b)^*)} U_{\phi_1\lambda}t\otimes U_\lambda\theta_i \quad (i=1,\cdots,n),$$

i.e.,

$$(5)\qquad (u_i(x-by)v(x+ay))_{i\in\{1,\cdots,n\}}$$
$$=[(U_{\phi_1\lambda}t)(x)]_{\lambda\in \mathrm{Rep}(H(b(a+b))^*/H(b)^*)}\cdot[(U_\lambda\theta_i)(y)]_{(\lambda,i)},$$

with $\theta_i\in\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})^{H(b)^*+H_2(a+b)^*}$. Since the components of the row vector in the left side are linearly independent for any fixed y,

$$(6)\qquad \mathrm{rank}\,[(U_\lambda\theta_i)(y)]_{(\lambda,i)}=n \qquad \text{for any } y\in X.$$

Therefore $\{\theta_i\}_{i=1,\cdots,n}$ forms a basis of $\Gamma(L^{ab(+ab)}\otimes P_{a\beta-b\alpha})^{H(b)^*+H_2(a+b)^*}$. On the other hand, by virtue of (d), $\mathcal{G}(\pi)(H(b(a+b))^*)$ and $\mathcal{G}(\pi)(H_2(a+b)^*)$ are level subgroups of $H_1(M^c)$ and $H_2(M^c)$, which we denote by $H_i(M^c)^*$ $(i=1,2)$, respectively. Hence, identifying $\Gamma(M^{cd})$ and $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})^{H(b)^*}$ by π^*, $\{\theta_i\}_{i=1,\cdots,n}$ forms a basis of $\Gamma(M^{cd})^{H_2(M^c)^*}$ and (6) implies our assertion. *Q.E.D.*

In the same way as in the proofs of corollary 2.5.1 in [1] and of Corollary 1.3 in [2], we can deduce the following theorem from the above theorem.

THEOREM 3. *Let (X, L) be a principally polarized abelian variety; and a, b be two positive integers. Assume that $K(L^b)$ has a Göpel decomposition $K(L^b)=H_1(b)\oplus H_2(b)$ with $H_1(b)=H_1(b)_{\mathrm{red.}}$. Let x be any fixed point in X.*

(i) *If $b>a$, for a non-zero section f in $\Gamma(L^a)$, there exists a k-valued point y of $H_1(b)$ such that $f(x+y)\neq 0$.*

(ii) *If $b+a$, and f is an eigenvector in $\Gamma(L^a)$ for a level subgroup $H(a)^*$ of a maximal isotropic subgroup $H(a)$ in $K(L^a)$, there exists a k-valued point y in $H_1(b)$ such that $f(x+y)\neq 0$.*

In view of this theorem, we can show the following proposition, which in the case of $p=3$ asserts, as mentioned in the beginning, that if (X, L) is a polarized ordinary abelian variety over k of characteristic 3, $\phi(X)$ is ideal-theoretically an intersection of cubics, where ϕ: $X\longrightarrow \boldsymbol{P}(\Gamma(L^3))$ is the canonical embedding of X (cf. [8]).

PROPOSITION 4. *Let k be an algebraically closed field of characteristic $p>0$; X be an ordinary abelian variety over k of dimension g; and L be an ample invertible sheaf on X. Let α, β be two closed points in $\hat{X}$. Then the canonical map*

$$\sum_{\gamma\in\hat{X}_p(k)}\Gamma(L^{p-1}\otimes P_{\alpha+\gamma})\otimes\Gamma(L\otimes P_{\beta-\gamma})\longrightarrow\Gamma(L^p\otimes P_{\alpha+\beta})$$

is surjective.

PROOF. As in the proof of Theorem 2.2 in [6], we can easily reduce our theorem to the case where L is principal. Moreover, without loss of generality we assume L is symmetric.

Let ξ: $X\times X\longrightarrow X\times X$ be the homomorphism defined by $(x, y)\longmapsto(x-y, x+(p-1)y)$. Then

$$\xi^*(p_1{}^*(L^{p-1}\otimes P_\alpha)\otimes p_2{}^*(L\otimes P_\beta))\simeq p_1{}^*(L^p\otimes P_{\alpha+\beta})\otimes p_2{}^*(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})$$

and

$$\Gamma(L^{p-1}\otimes P_\alpha)\otimes\Gamma(L\otimes P_\beta)\xrightarrow{\xi^*}\Gamma(L^p\otimes P_{\alpha+\beta})\otimes\Gamma(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha}).$$

Here we take a Göpel decomposition of $K(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})$:

$$K(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})=H_1((p-1)p)\oplus H_2((p-1)p)$$

with $H_1((p-1)p)=H_1((p-1)p)_{\mathrm{red.}}$, and we put $pH_i((p-1)p)=H_i(p-1)$ and $(p-1)H_i((p-1)p)=H_i(p)$ for $i=1, 2$. Let $H_i(p)^*\subset\mathcal{G}(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})$ and $H_i(p)^{**}\subset\mathcal{G}(L^p\otimes P_{\alpha+\beta})$ are level subgroups given in Lemma 1 for $i=1, 2$. Moreover, let $H_i(p-1)^*$ be a level subgroup in $\mathcal{G}(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})$ of $H_i(p-1)$ for each $i=1, 2$. Then, obviously $\{0\}\times X_{p-1}\subset\mathrm{Ker}(\xi)^\perp$ and $\mathcal{G}(\xi)(\{0\}\times H_i(p-1)^*)$ is a level subgroup of $H_i(p-1)\times\{0\}$ for each $i=1, 2$. We put $\mathcal{G}(\xi)(\{0\}\times H_i(p-1)^*)=\bar{H}_i(p-1)^*$. Let u, v and t be the sections such that $\langle u\rangle=\Gamma(L^{p-1}\otimes P_\alpha)^{\bar{H}_2(p-1)^*}$, $\langle v\rangle=\Gamma(L\otimes P_\beta)$ and $\langle t\rangle=\Gamma(L^p\otimes P_{\alpha+\beta})^{H_2(p)^{**}}$. Then, by Lemma 1,

$$\xi^*(u\otimes v)=\sum_{\lambda\in H_1(p)^*}U_{\phi_1\lambda}t\otimes U_\lambda\theta \tag{7}$$

with $\theta\in\Gamma(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})^{H_2(p)^*+H_2(p-1)^*}$. Moreover, for any $\nu\in H_1(p-1)^*$,

$$\xi^*(U_{\mathcal{G}(\xi)\tau(0,\nu)}u\otimes v)=\sum_{\lambda\in H_1(p)^*}U_{\phi_1\lambda}t\otimes U_{\lambda+\nu}\theta. \tag{8}$$

On the other hand, for $x\in X_p$, the diagram:

$$\begin{array}{ccc} X\times X & \xrightarrow{\quad\xi\quad} & X\times X \\ {\scriptstyle T_{(0,x)}}\big\downarrow & & \big\downarrow{\scriptstyle T_{(-x,(p-1)x)}} \\ X\times X & \xrightarrow{\quad\xi\quad} & X\times X \end{array}$$

commutes. Hence we have an isomorphism

$$T^*_{(0,x)}\xi^*(p_1{}^*(L^{p-1}\otimes P_\alpha)\otimes p_2{}^*(L\otimes P_\beta))\simeq\xi^*(T^*_{(-x,(p-1)x)}(p_1{}^*(L^{p-1}\otimes P_\alpha)\otimes p_2{}^*(L\otimes P_\beta))$$

$$\simeq \xi^*(p_1^*(L^{p-1}\otimes P_{\alpha-(p-1)\phi_L(x)})\otimes p_2^*(L\otimes P_{\beta+(p-1)\phi_L(x)}))\,.$$

Therefore, for $\mu\in H_1(p)^*$, we obtain a commutative diagram:

$$(9)\qquad \begin{array}{ccc} \Gamma(L^{p-1}\otimes P_\alpha)\otimes\Gamma(L\otimes P_\beta) & \xrightarrow{\ \xi^*\ } & \Gamma(L_p\otimes P_{\alpha+\beta})\otimes\Gamma(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha}) \\ \downarrow U_\mu' & & \downarrow 1\otimes U_\mu \\ \Gamma(L^{p-1}\otimes P_{\alpha-\gamma})\otimes\Gamma(L\otimes P_{\beta+\gamma}) & \xrightarrow{\ \xi^*\ } & \Gamma(L^p\otimes P_{\alpha+\beta})\otimes\Gamma(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha}) \end{array}$$

where U_μ' is a suitable isomorphism and γ is a suitable point in $\hat{X}_p$. Applying this commutative diagram to the equality (8), we obtain

$$(10)\qquad [\xi^*(U_\mu'(U_{\mathcal{G}(\xi)\tau(0,\nu)}u\otimes v))]_{\mu\in H_1(p)^*} = (U_\lambda t)_{\lambda\in H_1(p)}[U_{\lambda+\mu+\nu}\theta]_{(\lambda,\mu)\in H_1(p)^*\times H_1(p)^*}\,.$$

By easy calculation, we have

$$\det[U_{\lambda+\mu+\nu}\theta]_{(\lambda,\mu)\in H_1(p)^*\times H_1(p)^*} = \pm[U_\nu(\sum_{\lambda\in H_1(p)^*} U_p\theta)]^{p^g}\,.$$

Here we notice that this equality was first pointed out by R. Sasaki. Obviously we have

$$\sum_{\lambda\in H_1(p)^*} U_\lambda\theta\in\Gamma(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})^{H_2(p-1)^*+H_1(p)^*}\,.$$

Let $Y=X/H_2(p-1)$; and π: $X\longrightarrow Y$ be the canonical map. Then there exists a principal invertible sheaf M on Y such that

$$\pi^*M^p\simeq L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha}$$

and

$$\Gamma(M^p)\xrightarrow[\sim]{\pi^*}\Gamma(L^{(p-1)p}\otimes P_{(p-1)\beta-\alpha})^{H_2(p-1)^*}\,.$$

Hence $\sum_{\lambda\in H_1(p)^*} U_\lambda\theta$ can be considered as an element of $\Gamma(M^p)^{\mathcal{G}(\pi)(H_1(p)^*)}$. On the other hand, $\mathcal{G}(\pi)(H_1(p)^*)$ is a level subgroup of $K(M^p)_{\text{red.}}$, and $\pi(H_1(p-1))$ is an isotropic direct summand of $K(M^{p-1})$. Therefore, by Theorem 3, (ii), there exists an element x of $H_1(p-1)$ such that

$$\sum_{\lambda\in H_1(p)^*}(U_\lambda\theta)(x)\neq 0\,.$$

Therefore, for the point $\nu\in H_1(p-1)^*$ corresponding to this x,

$$[U_\nu(\sum_{\lambda\in H_1(p)^*} U_\lambda\theta)](0)\neq 0\,.$$

This inequality and (10) imply the surjectivity of our map. *Q.E.D.*

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