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ON HELICES AND PSEUDO-RIEMANNIAN SUBMANIFOLDS

By

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§0. Introduction.

In a Riemannian manifold, a regular curve is called a helix if its first and second curvature is constant and the third curvature is zero. As for helices in a Riemannian submanifold, there is a research of T. Ikawa, who investigated the condition that every helix with curvatures k, l in a Riemannian submanifold is a helix in the ambient space [3]. In a pseudo-Riemannian manifold, helices are defined by almost the same way as the Riemannian case. Recently, T. Ikawa proved the following theorem about helices in a Lorentzian submanifold [4]:

THEOREM A. *Let* M_1 ($\dim M_1 \geqq 3$) *be a Lorentzian submanifold of a pseudo-Riemannian manifold* $\tilde{M}_\beta$. *For any positive constant* k, l, *the following conditions are equivalent:*

(a) *every helix in* M_1 *with* $\langle X, X\rangle = -1$, $\langle \nabla_X X, \nabla_X X\rangle = k^2$ *and* $\langle \nabla_X \nabla_X X, \nabla_X \nabla_X X\rangle = -k^4 + k^2 l^2$ *is a helix in* $\tilde{M}_\beta$,

(b) M_1 *is totally geodesic.*

In this paper, we generalize this theorem to the case of a pseudo-Riemannian submanifold.

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§1. Preliminaries.

Let V_α be an n-dimensional real vector space equipped with an inner product $\langle\,,\rangle$ of index α. A non-zero vector x of V_α is said to be *null* if $\langle x, x\rangle = 0$ and *unit* if $\langle x, x\rangle = +1$ or -1. Concerning multilinear mappings on V_α, we have the following lemmas [1]:

LEMMA 1.1. *For any r-linear mapping* T *on* V_α *to a real vector space* W *and* $\varepsilon_0 = +1$ *or* -1 ($-\alpha \leqq \varepsilon_0 \leqq n-\alpha$), *the following conditions are equivalent:*

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(a) $T(x, \cdots, x)=0$ *for any* $x\in V_\alpha$ *such that* $\langle x, x\rangle=\varepsilon_0$,
(b) $T(x, \cdots, x)=0$ *for any* $x\in V_\alpha$.

LEMMA 1.2. *For any* 2*r-linear mapping* T *on* V_α *to a real vector space* W *and* $\varepsilon_0=+1$ *or* -1, $\varepsilon_1=+1$, -1 *or* 0 $(2-2\alpha\leqq\varepsilon_0+\varepsilon_1\leqq 2n-2\alpha-2)$, *the following conditions are equivalent:*

(a) $\sum_{i=1}^{2r}T(x, \cdots, x, \underset{i}{u}, x, \cdots, x)=0$ *for any orthogonal vectors* $x, u\in V_\alpha$ *such that* $\langle x, x\rangle=\varepsilon_0$ *and* $\langle u, u\rangle=\varepsilon_1$,
(b) *there exists* $w\in W$ *such that* $T(x, \cdots, x)=\langle x, x\rangle^r w$ *for any* $x\in V_\alpha$.

Now let M_α be an n-dimensional pseudo-Riemannian manifold of index α $(0\leqq\alpha\leqq n)$ isometrically immersed into an m-dimensional pseudo-Riemannian manifold $\tilde{M}_\beta$ of index β. Then M_α is called a *pseudo-Riemannian submanifold* of $\tilde{M}_\beta$. We denote the metrics of M_α and $\tilde{M}_\beta$ by the symbol $\langle\ ,\ \rangle$ and the covariant differentiation of M_α (resp. $\tilde{M}_\beta$) by ∇ (resp. $\tilde{\nabla}$). Gauss' formula is

$$\tilde{\nabla}_X Y=\nabla_X Y+B(X, Y),$$

where X and Y are tangent vector fields of M_α and B is the second fundamental form of M_α. Weingarten's formula is

$$\tilde{\nabla}_X\xi=-A_\xi X+\nabla_X^\perp\xi,$$

where X (resp. ξ) is a tangent (resp. normal) vector field of M_α, $\nabla^\perp$ is the covariant differentiation with respect to the induced connection in the normal bundle $N(M_\alpha)$ and A_ξ is the *shape operator* of M_α. We have the following relation:

$$\langle A_\xi X, Y\rangle=\langle B(X, Y), \xi\rangle.$$

For the second fundamental form and the shape operator, we define their covariant derivatives by

$$\bar{\nabla}B(X, Y, Z)=\nabla_Z^\perp(B(X, Y))-B(\nabla_Z X, Y)-B(X, \nabla_Z Y),$$

$$\bar{\nabla}^2 B(X, Y, Z, W)=\nabla_W^\perp(\bar{\nabla}B(X, Y, Z))-\bar{\nabla}B(\nabla_W X, Y, Z)$$
$$-\bar{\nabla}B(X, \nabla_W Y, Z)-\bar{\nabla}B(X, Y, \nabla_W Z),$$

$$(\bar{\nabla}_Y A)_\xi X=\nabla_Y(A_\xi X)-A_{\nabla_Y^\perp\xi}X-A_\xi\nabla_Y X,$$

where X, Y, Z, W are tangent vector fields of M_α and ξ is a normal vector field of M_α. The *mean curvature vector field* H of M_α is defined by

$$H:=(1/n)\sum_{i=1}^n\langle e_i, e_i\rangle B(e_i, e_i),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal frame of M_α. H is said to be parallel when $\nabla^{\perp}H=0$ holds. If the second fundamental form B satisfies

$$B(X, Y)=\langle X, Y\rangle H$$

for any tangent vector fields X, Y of M_α, then M_α is said to be *totally umbilic.* A totally umbilical submanifold with the parallel mean curvature vector field is called an *extrinsic sphere.* If the second fundamental form vanishes identically on M_α, then M_α is said to be *totally geodesic.*

§2. Helices in a pseudo-Riemannian manifold.

Let $c=c(t)$ be a regular curve in a pseudo-Riemannian manifold M_α. We denote the tangent vector field $c'(t)$ by the letter X. When $\langle X, X\rangle=+1$ or -1, c is called a *unit speed curve.* In this paper, a unit speed curve c in M_α is said to be a *helix* if and only if there exist constants α, β and vector fields U, V of constant length along c such that X, U, V are orthogonal and the following equations hold:

$$\nabla_X X=U\,,\quad \nabla_X U=\alpha X+V\,,\quad \nabla_X V=\beta U\,. \tag{2.1}$$

Especially, if $V=0$ in this equation, the curve is called a *circle* in [1]. Moreover, if $U=V=0$ in this equation, the curve is a *geodesic.*

LEMMA 2.1. *A unit speed curve c in M_α is a helix if and only if there exists a constant λ such that*

$$\nabla_X\nabla_X\nabla_X X=\lambda\nabla_X X\,. \tag{2.2}$$

PROOF. If c is a helix, by means of (2.1) we get

$$\nabla_X\nabla_X\nabla_X X=\nabla_X\nabla_X U=\nabla_X(\alpha X+V)=\alpha\nabla_X X+\nabla_X V=(\alpha+\beta)U=(\alpha+\beta)\nabla_X X\,.$$

Conversely, we assume the existence of λ which satisfies (2.2). Since $\langle X, X\rangle=+1$ or -1, we obtain the following equations:

$$\langle\nabla_X X, X\rangle=0\,, \tag{2.3}$$

$$\langle\nabla_X\nabla_X X, X\rangle+\langle\nabla_X X, \nabla_X X\rangle=0\,, \tag{2.4}$$

$$\langle\nabla_X\nabla_X\nabla_X X, X\rangle+3\langle\nabla_X\nabla_X X, \nabla_X X\rangle=0\,. \tag{2.5}$$

Substituting (2.2) into (2.5) and using (2.3), we have

$$X(\langle\nabla_X X, \nabla_X X\rangle)=0\,. \tag{2.6}$$

Differentiating this equation by X and using (2.2), we get

(2.7)
$$0=\langle\nabla_X\nabla_X\nabla_X X,\ \nabla_X X\rangle+\langle\nabla_X\nabla_X X,\ \nabla_X\nabla_X X\rangle$$
$$=\lambda\langle\nabla_X X,\ \nabla_X X\rangle+\langle\nabla_X\nabla_X X,\ \nabla_X\nabla_X X\rangle.$$

By (2.4), (2.6) and (2.7), we can see that $\langle\nabla_X X, \nabla_X X\rangle$, $\langle\nabla_X\nabla_X X, X\rangle$ and $\langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle$ are constant. We put as follows:

$$U:=\nabla_X X,\quad \alpha:=\langle X,\ X\rangle\langle\nabla_X U,\ X\rangle,\quad V:=\nabla_X U-\alpha X.$$

Note that X, U, V are orthogonal and α, $\langle U, U\rangle$ are constant. $\langle V, V\rangle$ is also constant because

$$\langle V,\ V\rangle=\langle\nabla_X\nabla_X X,\ \nabla_X\nabla_X X\rangle-2\alpha\langle\nabla_X\nabla_X X,\ X\rangle+\alpha^2\langle X,\ X\rangle.$$

At least, we have

$$\nabla_X V=\nabla_X\nabla_X\nabla_X X-\alpha\nabla_X X=(\lambda-\alpha)\nabla_X X=(\lambda-\alpha)U.$$

Thus we can see that c is a helix. Q.E.D.

§3. Helices and pseudo-Riemannian submanifolds.

In this section, we prove the following theorems:

THEOREM 3.1. *Let M_α be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\tilde{M}_\beta$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2=+1$ or -1 $(-2\alpha+3\leqq\varepsilon_0+\varepsilon_1+\varepsilon_2\leqq 2n-2\alpha-3)$. For any positive constants k, l, the following conditions are equivalent:*

(a) *every helix in M_α with $\langle X, X\rangle=\varepsilon_0$, $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ and $\langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle=\varepsilon_0 k^4+\varepsilon_2 k^2 l^2$ is a helix in $\tilde{M}_\beta$,*

(b) *M_α is a totally geodesic submanifold.*

THEOREM 3.2. *Let M_α be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\tilde{M}_\beta$ and $\varepsilon_0, \varepsilon_1=+1$ or -1 $(-2\alpha+4\leqq\varepsilon_0+\varepsilon_1\leqq 2n-2\alpha-4)$. For any positive constant k, the following conditions are equivalent:*

(a) *every helix in M_α with $\langle X, X\rangle=\varepsilon_0$, $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ and $\langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle=\varepsilon_0 k^4$ is a helix in $\tilde{M}_\beta$,*

(b) *M_α is an extrinsic sphere.*

PROOF. In order to prove these theorems simultaneously, we suppose $\varepsilon_2=+1$, -1 or 0 and put the following assumption at first:
every helix in M_α with $\langle X, X\rangle=\varepsilon_0$, $\langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2$ and $\langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle=\varepsilon_0 k^4+\varepsilon_2 k^2 l^2$ is a helix in $\tilde{M}_\beta$.

Note that this condition is reduced to the condition (a) of Theorem 3.2 when $\varepsilon_2=0$. Let x, u, v be any mutually orthogonal vectors of M_α at p such that

$$\langle x, x\rangle=\varepsilon_0, \quad \langle u, u\rangle=\varepsilon_1, \quad \text{and} \quad \langle v, v\rangle=\varepsilon_2.$$

There exists a helix c of M_α such that

$$c(0)=p, \quad X(p)=x, \quad (\nabla_X X)(p)=ku \quad \text{and } (\nabla_X\nabla_X X)(p)=-\varepsilon_0\varepsilon_1 k^2 x+klv, \tag{3.1}$$

where $X:=c'(t)$. By Lemma 2.1, there exists a constant λ such that

$$\nabla_X\nabla_X\nabla_X X=\lambda\nabla_X X.$$

Since (2.7) holds in this situation, λ is calculated as

$$\begin{aligned}\lambda&=\varepsilon_1 k^{-2}\lambda\langle\nabla_X X, \nabla_X X\rangle(p)\\ &=-\varepsilon_1 k^{-2}\langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle(p)\\ &=-\varepsilon_0\varepsilon_1 k^2-\varepsilon_1\varepsilon_2 l^2.\end{aligned}$$

Thus we obtain

$$\nabla_X\nabla_X\nabla_X X=(-\varepsilon_0\varepsilon_1 k^2-\varepsilon_1\varepsilon_2 l^2)\nabla_X X. \tag{3.2}$$

Since c is a helix in $\tilde{M}_\beta$ by the assumption, there exists a constant $\tilde{\lambda}$ such that

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=\tilde{\lambda}\tilde{\nabla}_X X$$

because of Lemma 2.1. Since the constant $\tilde{\lambda}$ depends on the initial vectors x, u, v, we rewrite the above equation as

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=\tilde{\lambda}(x, u, v)\tilde{\nabla}_X X. \tag{3.3}$$

On the other hand, by Gauss' formula we have

$$\tilde{\nabla}_X X=\nabla_X X+B(X, X). \tag{3.4}$$

Differentiating with respect to X and using Gauss' formula and Weingarten's formula, we get

$$\tilde{\nabla}_X\tilde{\nabla}_X X=\nabla_X\nabla_X X-A_{B(X,X)}X+3B(X, \nabla_X X)+\bar{\nabla}B(X, X, X).$$

Differentiating again and using Gauss' formula and Weingarten's formula, we obtain

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=&\nabla_X\nabla_X\nabla_X X-2A_{\bar{\nabla}B(X,X,X)}X-5A_{B(X,\nabla_X X)}X\\ &-(\bar{\nabla}_X A)_{B(X,X)}X-A_{B(X,X)}\nabla_X X\\ &-B(X, A_{B(X,X)}X)+4B(X, \nabla_X\nabla_X X)+3B(\nabla_X X, \nabla_X X)\\ &+5\bar{\nabla}B(X, \nabla_X X, X)+\bar{\nabla}B(X, X, \nabla_X X)+\bar{\nabla}^2 B(X, X, X, X).\end{aligned} \tag{3.5}$$

Substituting (3.2) and (3.3) into (3.5), we have

$$\begin{aligned}&\tilde{\lambda}(x, u, v)(\nabla_X X+B(X, X))\\&=(-\varepsilon_0\varepsilon_1 k^2-\varepsilon_1\varepsilon_2 l^2)\nabla_X X-2A_{\bar{\nabla}B(X,X,X)}X-5A_{B(X,\nabla_X X)}X\\&\quad-(\bar{\nabla}_X A)_{B(X,X)}X-A_{B(X,X)}\nabla_X X\\&\quad-B(X, A_{B(X,X)}X)+4B(X, \nabla_X\nabla_X X)+3B(\nabla_X X, \nabla_X X)\\&\quad+5\bar{\nabla}B(X, \nabla_X X, X)+\bar{\nabla}B(X, X, \nabla_X X)+\bar{\nabla}^2 B(X, X, X, X).\end{aligned}$$

Taking tangent and normal parts at p and making use of (3.1), we get

$$\tilde{\lambda}(x, u, v)ku=(-\varepsilon_0\varepsilon_1 k^3-\varepsilon_1\varepsilon_2 kl^2)u-2A_{\bar{\nabla}B(x,x,x)}x-5kA_{B(x,u)}x-(\bar{\nabla}_x A)_{B(x,x)}x-kA_{B(x,x)}u, \tag{3.6}$$

$$\begin{aligned}\tilde{\lambda}(x, u, v)B(x, x)=&-B(x, A_{B(x,x)}x)-4\varepsilon_0\varepsilon_1 k^2 B(x, x)\\&+4klB(x, v)+3k^2 B(u, u)+5k\bar{\nabla}B(x, u, x)\\&+k\bar{\nabla}B(x, x, u)+\bar{\nabla}^2 B(x, x, x, x).\end{aligned} \tag{3.7}$$

Note that these equations hold for any mutually orthogonal vectors $x, u, v \in T_p(M_\alpha)$ such that $\langle x, x\rangle=\varepsilon_0$, $\langle u, u\rangle=\varepsilon_1$ and $\langle v, v\rangle=\varepsilon_2$. If we add (3.6) to the equation obtained by changing u into $-u$ in (3.6), we have

$$\{-\tilde{\lambda}(x, -u, v)+\tilde{\lambda}(x, u, v)\}ku=-4A_{\bar{\nabla}B(x,x,x)}x-2(\bar{\nabla}_x A)_{B(x,x)}x. \tag{3.8}$$

Subtracting (3.6) from the equation obtained by changing v into $-v$ in (3.6), we find

$$\tilde{\lambda}(x, u, -v)=\tilde{\lambda}(x, u, v). \tag{3.9}$$

By subtracting (3.7) from the equation obtained by changing u into $-u$ in (3.7), we have

$$\{\tilde{\lambda}(x, -u, v)-\tilde{\lambda}(x, u, v)\}B(x, x)=-10k\bar{\nabla}B(x, u, x)-2k\bar{\nabla}B(x, x, u). \tag{3.10}$$

If we subtract (3.7) from the equation obtained by changing v into $-v$ in (3.7), we get

$$\{\tilde{\lambda}(x, u, -v)-\tilde{\lambda}(x, u, v)\}B(x, x)=-8klB(x, v). \tag{3.11}$$

It follows from (3.9) and (3.11) that

$$B(x, v)=0.$$

Since this equation holds for any mutually orthogonal vectors $x, v\in T_p(M_\alpha)$ such that $\langle x, x\rangle=\varepsilon_0$ and $\langle v, v\rangle=\varepsilon_2$, by means of Lemma 1.2, there exists a normal vector w such that

$$B(x, x)=\langle x, x\rangle w$$

for any $x\in T_p(M_\alpha)$. Thus M_α is totally umbilic and w is the mean curvature vector H at each point of M_α. Now we have

$$B(x, y)=\langle x, y\rangle H, \quad A_\xi x=\langle H, \xi\rangle x, \tag{3.12}$$

for any $x, y\in T_p(M_\alpha)$ and $\xi\in N_p(M_\alpha)$. By these equations, (3.8) is reduced to

$$\{-\tilde{\lambda}(x, -u, v)+\tilde{\lambda}(x, u, v)\}ku=-6\varepsilon_0\langle H, \nabla_x^\perp H\rangle x.$$

Taking the inner product with u, we find that

$$\tilde{\lambda}(x, -u, v)=\tilde{\lambda}(x, u, v).$$

By this equation and (3.12), (3.10) is reduced to

$$\nabla_u^\perp H=0.$$

Since this equation holds for any $u\in T_p(M_\alpha)$ such that $\langle u, u\rangle=\varepsilon_1$, making use of Lemma 1.1, we get

$$\nabla^\perp H=0. \tag{3.13}$$

Thus we have proved that (a) implies (b) in Theorem 3.2. Moreover, by (3.12), (3.13) and (3.6), we have

$$\tilde{\lambda}(x, u, v)=-\varepsilon_0\varepsilon_1 k^2-\varepsilon_1\varepsilon_2 l^2-\varepsilon_0\langle H, H\rangle.$$

On the other hand, by (3.12), (3.13) and (3.7), we obtain

$$\varepsilon_0\tilde{\lambda}(x, u, v)H=-\langle H, H\rangle H-\varepsilon_1 k^2 H.$$

Thus we get

$$\varepsilon_2 H=0,$$

which means that M_α is totally geodesic if ε_2 is not zero. Now we have seen that (a) implies (b) in Theorem 3.1.

Since it is clear that (b) implies (a) in Theorem 3.1, all we have to do next is to derive (a) from (b) in Theorem 3.2. Let c be any helix in M_α such that

$$\langle X, X\rangle=\varepsilon_0, \quad \langle\nabla_X X, \nabla_X X\rangle=\varepsilon_1 k^2 \text{ and } \langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle=\varepsilon_0 k^4, \tag{3.14}$$

where $X:=c'(t)$. By Lemma 3.1, there exists a constant λ such that

$$\nabla_X\nabla_X\nabla_X X=\lambda\nabla_X X.$$

By means of (2.7), λ is calculated as

$$\lambda=\varepsilon_1 k^{-2}\lambda\langle\nabla_X X, \nabla_X X\rangle=-\varepsilon_1 k^{-2}\langle\nabla_X\nabla_X X, \nabla_X\nabla_X X\rangle=-\varepsilon_0\varepsilon_1 k^2.$$

Consequently, we have

$$\nabla_X\nabla_X\nabla_X X=-\varepsilon_0\varepsilon_1 k^2\nabla_X X. \tag{3.15}$$

By the condition (b), we have

$$B(x, y)=\langle x, y\rangle H, \quad A_{\xi}x=\langle H, \xi\rangle x$$

for any $x, y\in T_p(M_\alpha)$, $\xi\in N_p(M_\alpha)$ and

$$\bar{\nabla}B=0, \quad \bar{\nabla}A=0.$$

Since (3.5) holds for any curve, by making use of the above equations and (3.14), we have

$$\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X=\nabla_X\nabla_X\nabla_X X-\varepsilon_0\langle H, H\rangle\nabla_X X-\langle H, H\rangle H-\varepsilon_1 k^2 H.$$

Substituting (3.15) into this equation, we find

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_X\tilde{\nabla}_X X&=(-\varepsilon_0\varepsilon_1 k^2-\varepsilon_0\langle H, H\rangle)(\nabla_X X+\varepsilon_0 H)\\&=(-\varepsilon_0\varepsilon_1 k^2-\varepsilon_0\langle H, H\rangle)\tilde{\nabla}_X X,\end{aligned}$$

which means that c is a helix in $\tilde{M}_\beta$ by Lemma 2.1. Q.E.D.

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