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CURVATURE BOUND AND TRAJECTORIES FOR MAGNETIC FIELDS ON A HADAMARD SURFACE

By

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Introduction.

On a complete oriented Riemannian manifold M, a closed 2-form $\boldsymbol{B}$ is called a *magnetic field*. Let Ω denote the skew symmetric operator $\Omega : \mathrm{TM} \to \mathrm{TM}$ defined by $\langle u, \Omega(v)\rangle = \boldsymbol{B}(u,v)$ for every u, $v \in TM$. We call a smooth curve γ a *trajectory* for $\boldsymbol{B}$ if it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \Omega(\dot\gamma)$. Since Ω is skew symmetric, we find that every trajectory has constant speed and is defined for $-\infty < t < \infty$. We shall call a trajectory *normal* if it is parametrized by its arc length. When γ is a trajectory for $\boldsymbol{B}$, the curve σ defined by $\sigma(t) = \gamma(\lambda t)$ with some constant λ is a trajectory for $\lambda\boldsymbol{B}$. We call the norm $\|\boldsymbol{B}_x\|$ of the operator $\boldsymbol{B}_x : T_xM \times T_xM \to \boldsymbol{R}$ the *strength* of the magnetic field at the point x. For the trivial magnetic field $\boldsymbol{B} = 0$, the case without the force of a magnetic field, trajectories are nothing but geodesics. In term of physics it is a trajectory of a charged particle under the action of the magnetic field. For a classical treatment and physical meaning of magnetic fields see [8].

On a Riemann surface M we can write down $\boldsymbol{B} = f \cdot \mathrm{Vol}_M$ with a smooth function f and the volum form Vol_M on M. When f is a constant function, the case of constant strength, the magnetic field is called *uniform*. On surfaces of constant curvature the feature of trajectories are well-known for every uniform magnetic field $k \cdot \mathrm{Vol}_M$. On a Euclidean plane $\boldsymbol{R}^2$ they are circles (in usual sense of Euclidean geometry) of radius $1/|k|$. On a sphere $S^2(c)$ they are small circles with prime period $2\pi/\sqrt{k^2+c}$. In these cases all trajectories are closed. On a hyperbolic plane $H^2(-c)$ of constant curvature $-c$, the situation is different. In his paper [4] Comtet showed that the feature of trajectories changes according to the strength of a uniform magnetic field $k \cdot \mathrm{Vol}_M$. When the strength $|k|$ is greater than $\sqrt{c}$, normal trajectories are still closed, hence bounded, but if $|k| \le \sqrt{c}$ they are unbounded simple curves, in particular, if $|k| = \sqrt{c}$ they are horocycles. In the preceeding paper [2] we studied trajectories for Kähler magnetic fields $k \cdot \boldsymbol{B}_J$,

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which are scalar multiples of the Kähler form $\boldsymbol{B}_J$, on a manifold of complex space form. On a complex projective plane all trajectories for Kähler magnetic fields are closed. But on a complex hyperbolic space $CH^n(-c)$ of constant holomorphic sectional curvature $-c$, normal trajectories for Kähler magnetic fields have similar properties as of trajectories for uniform magnetic fields on a hyperbolic plane. Their feature depend on the strength of a Kähler magnetic field; trajectories are bounded, horocyclic, or unbounded according to the strength is greater, equal to, or smaller than $\sqrt{c}$. In this context it is quite natural to pose the following problem. Consider a Hadamard manifold, which is a simply connected complete Riemannian manifold of nonpositive curvature $-\beta^2 \leq \mathrm{Riem}_M \leq -\alpha^2$, $\beta \geqq \alpha \geqq 0$. Are they true that all trajectories are unbounded if the strength is smaller than α and that all trajectories are bounded if the strength is greater than β? In this note we shall concerned with this problem on a Hadamard surface.

THEOREM 1. *Let* $\boldsymbol{B} = f \cdot \mathrm{Vol}_M$ *be a magnetic field with* $|f| \leq \alpha$ *on a Hadamard surface M of curvature* $\mathrm{Riem}_M \leq -\alpha^2$. *Then every normal trajectory for* $\boldsymbol{B}$ *is unbounded for both directions.*

For Hadamard manifolds we have an important notion of ideal boundary. We denote by $\overline{M} = M \cup M(\infty)$ the compactification of a Hadamard surface M with its ideal boundary $M(\infty)$. For a two-sides unbounded curve γ on M, if $\lim_{t\to\infty}\gamma(t)$ and $\lim_{t\to\infty}\gamma(t)$ exist in $\overline{M}$ we denote these points by $\gamma(\infty)$ and $\gamma(-\infty)$ respectively, and call that γ has points of infinity. If we review the Comtet's result from this point of view, it assures the following. On $H^2(-c)$ every trajectory γ for a uniform magnetic field $k \cdot \mathrm{Vol}_{H^2(-c)}$ with $|k| \leq \sqrt{c}$ has points of infinity $\gamma(\infty), \gamma(-\infty)$. When $|k| = \pm\sqrt{c}$ they coincide $\gamma(\infty) = \gamma(-\infty)$, and they are distinct when $|k| < \sqrt{c}$. We show that a similar property holds for general Hadamard surfaces.

THEOREM 2. *Let* $\boldsymbol{B} = f \cdot \mathrm{Vol}_M$ *be a magnetic field with* $|f| \leq \alpha$ *on a Hadamard surface M of curvature* $\mathrm{Riem}_M \leq -\alpha^2 \leq 0$. *Suppose either* $f \leq 0$ *or* $f \geq 0$ *except on a compact subset of M. We then have the following.*

(1) *Every normal trajectory for* $\boldsymbol{B}$ *has points of infinity.*

(2) *If* $|f| < \alpha$ *except on a compact subset of* M, *every normal trajectory has two distinct points at infinity.*

§1. A note on γ-Jacobi fields.

We shall show our theorems by applying the Rauch's comparison theorem. Let $B = f \cdot \mathrm{Vol}_M$ be a magnetic field on a oriented surface M. We denote by Ω_0 the skew symmetric operator associated with the uniform magnetic field Vol_M. Clearly the skew symmetric operator associated with B is of the form $\Omega = f \cdot \Omega_0$. For a normal trajectory γ for B, we denote by $V_t(s)$ the γ-Jacobi field along the geodesic $s \to \sigma(t,s) = \exp_{\gamma(t)} s\Omega_0(\dot\gamma)$ with $V_t(0) = \dot\gamma(t)$. This Jacobi field V_t is perpendicular to $\sigma(t,\cdot)$ and is obtained by the variation $\{\sigma(t+\varepsilon,\cdot)\}_\varepsilon$ of geodesics; $V_t(s) = \frac{\partial}{\partial t}\sigma(t,s)$.

For the sake of reader's convenience, we recall the explicit formula for normal trajectories and γ-Jacobi fields for uniform magnetic fields on surfaces of constant curvature.

EXAMPLE 1. On a Euclidean plane $\boldsymbol{R}^2$, trajectories for the uniform magnetic fields of strength k satisfy the following equation:

$$\gamma(t) = \left(\frac{1}{k}\cos(kt-\theta), \frac{1}{k}\sin(kt-\theta)\right) + (\xi_1, \xi_2).$$

The variation of geodesics is given by

$$\sigma(t,s) = \left(\frac{1}{k}(1-ks)\cos(kt-\theta), \frac{1}{k}(1-ks)\sin(kt-\theta)\right) + (\xi_1,\xi_2)$$

and the γ-Jacobi field is

$$V_t(s) = (1-ks)\dot\gamma(t),$$

hence it vanishes at $s_0 = 1/k$. The point $\sigma(t, 1/k) = (\xi_1, \xi_2)$ is usually called the center of γ.

EXAMPLE 2. On a sphere $S^2(c) = \{x = \{(x_1,x_2,x_3) \in \boldsymbol{R}^3 | \langle x, x\rangle = x_1{}^2 + x_2{}^2 + x_3{}^2 = 1\}$ of sectional curvature c, the trajectory γ for the uniform magnetic field of strength k satisfies the following equation when $\gamma(0) = x \in S^2(c)$, $\dot\gamma(0) = u \in U_x S^2(c) \simeq \{\xi \in \boldsymbol{R}^3 | \langle x,\xi\rangle = 0,\ \langle \xi,\xi\rangle = c\}$:

$$\gamma(t) = \frac{1}{k^2+c}(k^2 + c\cdot\cos\sqrt{k^2+c}t)\cdot x$$
$$+\frac{1}{\sqrt{k^2+c}}\sin\sqrt{k^2+c}t\cdot u + \frac{k}{k^2+c}(1-\cos\sqrt{k^2+c}t)\cdot\Omega_0(u).$$

Since the variation of geodesics is given by

$$\sigma(t,s)=\gamma(t)\cos\sqrt{c}s+\Omega_0(\dot\gamma(t))\cdot\frac{1}{\sqrt{c}}\sin\sqrt{c}s$$

hence

$$V_t(s)=\dot\gamma(t)(\cos\sqrt{c}s-\frac{k}{\sqrt{c}}\sin\sqrt{c}s).$$

Therefore it vanishes at $s_0=\frac{1}{\sqrt{c}}\tan^{-1}\sqrt{c}/k$. The point $\sigma(t,s_0)$ and the trajectory γ can be regard as a pole and a latitude line of this sphere.

EXAMPLE 3. On the hyperbolic plane $H^2(-c)=\{x=(x_0,x_1,x_2)\in \boldsymbol{R}^3 | \langle\langle x,x\rangle\rangle = -x_0{}^2+x_1{}^2+x_2{}^2=-1, x_0\geq 1\}$ of constant sectional curvature $-c$, the trajectory of the uniform magnetic field of strength k satisfies the following equation if $\gamma(0)=x$ and $\dot\gamma(0)=u\in U_xH^2(-c)\simeq\{\xi\in\boldsymbol{R}^3 | \langle\langle x,\xi\rangle\rangle=0,\langle\langle\xi,\xi\rangle\rangle=c\}$:

$$\gamma(t)=\frac{1}{c-k^2}(-k^2+c\cdot\cosh\sqrt{c-k^2}t)\cdot x+\frac{1}{\sqrt{c-k^2}}\sinh\sqrt{c-k^2}t\cdot u$$
$$+\frac{k}{c-k^2}(-1+\cosh\sqrt{c-k^2}t)\cdot\Omega_0(u),\quad \text{when } 0\leq k<\sqrt{c},$$

$$\gamma(t)=(1+\frac{ct^2}{2})x+tu+\frac{\sqrt{c}t^2}{2}\Omega_0(u),\quad \text{when } k=\sqrt{c},$$

$$\gamma(t)=\frac{1}{k^2-c}(k^2-c\cdot\cos\sqrt{k^2-c}t)\cdot x+\frac{1}{\sqrt{k^2-c}}\sin\sqrt{k^2-c}t\cdot u$$
$$+\frac{k}{k^2-c}(1-\cos\sqrt{k^2-c}t)\cdot\Omega_0(u),\quad \text{when } k>\sqrt{c}.$$

The variation of geodesics is given by

$$\sigma(t,s)=\gamma(t)\cosh\sqrt{c}s+\Omega_0(\dot\gamma(t))\cdot\frac{1}{\sqrt{c}}\sinh\sqrt{c}s$$

hence

$$V_t(s)=\dot\gamma(t)(\cosh\sqrt{c}s-\frac{k}{\sqrt{c}}\sinh\sqrt{c}s).$$

Therefore if $|k|>\sqrt{c}$ the γ-Jacobi field vanishes at $s_0=\frac{1}{\sqrt{c}}\tanh^{-1}\sqrt{c}/k=\frac{1}{2\sqrt{c}}\log\frac{k+\sqrt{c}}{k-\sqrt{c}}$. If $|k|\leq\sqrt{c}$ it does not vanish. When $k=\sqrt{c}$, the case that γ is a horocycle, the point $\gamma(\infty)=\gamma(-\infty)$ on the ideal boundary can be regard as the vanishing point of the γ-Jacobi field; $\lim_{s\to\infty}V_t(s)=0$.

§2. Proofs.

We are now in the position to prove theorems. Let γ be a trajectory for the magnetic field $f\cdot \mathrm{Vol}_M$ with $|f|\le\alpha$ on a Hadamard surface M of curvature $\mathrm{Riem}_M \le -\alpha^2$. We compare the norm of the γ-Jacobi field V_t with the norm of γ-Jacobi fields for uniform magnetic fields on a hyperbolic space. Since we have

$$\nabla_{\frac{\partial\sigma}{\partial s}} V_t(0) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\sigma(t,s)\Big|_{s=0} = \frac{\partial}{\partial t}\Omega_0(\dot\gamma(t)) = -f(\gamma(t))\dot\gamma(t),$$

we get the following estimate by the Rauch's comparison theorem;

$$\| V_t(s) \| \ge \cosh\alpha s - \frac{1}{\alpha} f(\gamma(t)) \sinh\alpha s .$$

This gaurantees that if $|f(\gamma(t))|\le\alpha$ then V_t does not vanish anywhere and $\liminf_{s\to\pm\infty}\exp(-\alpha s)\cdot\| V_t(s)\| \ge \frac{1}{2}(1-|f(\gamma(t))|/\alpha)$ for every t. Since M is diffeomorphic to an Euclidean plane, we find that the geodesic $\sigma(t_1,\cdot)$ and $\sigma(t_2,\cdot)$ do not intersect each other if $t_1\ne t_2$.

Let $S_r(p)$ denote the geodesic circle $\{x\in M \mid d(x,p)=r\}$ of radius r centered at p. If we suppose $\gamma|_{[0,\infty)}$ is tangent to a geodesic circle $S_r(\gamma(0))$ at $\gamma(t_0)$, then $\sigma(t_0,\cdot)$ passes $\gamma(0)$, which is a contradiction. We therefore have

PROPOSITION. *The trajectory rays $\gamma|_{[0,\infty)}$ and $\gamma|_{(-\infty,0]}$ cross only once to every geodesic circle $S_r(\gamma(0))$.*

This proposition leads us to Theorem 1. In order to see Theorem 2, we denote by u_t for $t\ne 0$ the unit tangent vector at $p=\gamma(0)$ such that the geodesic emanating from p with the initial speed u_t joins p and $\gamma(t)$. We set $u_0=\dot\gamma(0)$. Since γ is unbounded in both directions, we may treat the case that f is nonpositive (or nonnegative) on M. We then find the smooth curve $(u_t)_{t\in[0,\infty)}$ on $U_pM \simeq S^1$ rotates counterclockwisely if $f\ge 0$ and rotates clockwisely if $f\le 0$. If we suppose $u_{t_0}=\pm\Omega_0(u_0)$ for some t_0, then $\sigma(0,\cdot)$ passes $\gamma(t_0)$. Hence we find that $\{u_t\}_t \subset U_pM\setminus\{\pm\Omega_0(u_0)\}$ and the limit $u_\infty=\lim_{t\to\infty}u_t$ exists. Similarly, we find that the limit $u_{-\infty}=\lim_{t\to-\infty}u_t$ exists. We therefore get that γ has points at infinity;

$$\gamma(\infty)=\rho_{u_\infty}(\infty) \quad\text{and}\quad \gamma(-\infty)=\rho_{u_{-\infty}}(\infty),$$

where ρ_v denote the geodesic with $\dot\rho(0)=v$. Now we suppose that γ has a single point at infinity: $\gamma(\infty)=\gamma(-\infty)$. This means $u_\infty=u_{-\infty}$, hence $\gamma(\infty)=\sigma(t,\infty)$ for every t. This can not occur when $f<\alpha$. We get the conclusion of Theorem 2.

In view of our proof, we can conclude the following.

REMARK. Consider a magnetic field $B = f \cdot \mathrm{Vol}_M, |f| \leq \alpha$, on a Hadamard surface M of curvature $\mathrm{Riem}_M \leq -\alpha^2 < 0$.

(1) A trajectory γ for B has a single point at infinity $\gamma(\infty) = \gamma(-\infty)$ if and only if all the geodesic $\sigma(t,\cdot)$ converges to that point $\sigma(t,\infty) = \gamma(\infty)$.

(2) If a trajectory γ has a single point at infinity, then the magnetic angle at that point is $\pi/2$. Here the magnetic angle means the angle between the outer tangent vector of γ and the outer tangent vector of geodesics ρ with $\rho(\infty) = \gamma(\infty)$ (c.f.[2]).

REMARK. Let $B = k \cdot \mathrm{Vol}_M, |k| < \alpha$ be a uniform magnetic field on a Hadamard surface M of bounded negative curvature $-\beta^2 \leq \mathrm{Riem}_M \leq -\alpha^2 < 0$. We have a positive ε such that the angle $\sphericalangle(\dot\gamma(0), \dot\rho(0))$ between a trajectory γ for B and a geodesic ρ with $\gamma(0) = \rho(0)$ and $\gamma(\infty) = \rho(\infty)$ is always not greater than $\pi - \varepsilon$.

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