TSUKUBA J. MATH.
Vol. 9 No. 2 (1985). 289—297

# PROXIMITY ON FUNCTION SPACES

By

FU Pei-Ren

## Introduction

It has been 30 years since V. A. Efremovič [1] introduced the concept of proximity in 1951. A lot of works have been done and the theory is being perfected. But it is a pity that there is no paper on various proximity on function space.

In this paper we first consider the proximity characterizing the point-wise convergence on function space. Then we investigate the proximity characterizing the uniform convergence on a function space and a condition under which the proximity becomes joint proximally continuous. Last, we study the proximity characterizing the uniform convergence of the function space on a family of subsets and its related properties.

## 1. Proximity of point-wise convergence

For the definitions and notations used in this section are see to [2]. Let $X$ be a set, $(Y, \mathcal{J})$ be a proximity space, and $F\subset Y^{X}$. Let

$$A(x)=\{f(x): f\in A\} \qquad \text{for each} \quad A\subset F,\ x\in X.$$

DEFINITION 1.1. The relative proximity $\underset{x\in X}{\times}\mathcal{J}_{x}|F$ with respect to $F$ is called *proximity of point-wise convergence on* $F$, denote by *$\rho$-proximity on $F$ or* $\mathcal{J}^{X}|F$ for short, where $\underset{x\in X}{\times}\mathcal{J}_{x}$ is the product proximity of the family of proximity spaces $\{(Y_{x}, \mathcal{J}_{x}): x\in X\}$, and $Y_{x}=Y$, $\mathcal{J}_{x}=\mathcal{J}$, for each $x\in X$.

By Definition 1.1, we have the following:

PROPOSITION 1.1. *Let $A$, $B\subset F$. Then $A\mathcal{J}^{X}|FB$ iff for any finite decomposition of $A$ and $B$:*

$$A=\bigcup_{i=1}^{p}A_{i}, \qquad B=\bigcup_{j=1}^{q}B_{j},$$

*there are $i$ and $j$ such that $A_{i}(x)\mathcal{J}B_{j}(x)$ for each $x\in X$.*

PROPOSITION 1.2. *Suppose that the evaluation map $e_{x}: F\rightarrow Y$ is defined by*

Received May 29, 1984. Revised December 19, 1984.

*$e_x(f)=f(x)(f\in F)$ for each $x\in X$, then proximity $\mathcal{J}^X|F$ on $F$ is the coarsest proximity on $F$, which makes every evaluation map $e_x$ proximally continuous for each $x\in X$, in other words, $\mathcal{J}^X|F=\mathrm{Sup}\{e_x^{-1}(\mathcal{J}): x\in X\}$.*

It is known that the topology of a product proximity space is equal to the product of the topology of each coordinate space, and the product topology is the topology of point-wise convergence. Besides, the convergence with respect to the proximity is the convergence with of the topology induced by the same proximity. Thus we have:

THEOREM 1.1. *The net $\{f_\alpha: \alpha\in D\}\mathcal{J}^X|F$-converges to $f\in F$ in a proximity space $(F, \mathcal{J}^X|F)$ iff the net $\{f_\alpha(x): \alpha\in D\}$ $\mathcal{J}$-converges to $f(x)$ in $Y$ for each $x\in X$.*

It is known that a proximity space $(Y, \mathcal{J})$ is compact iff the topological space $(Y, \mathcal{T}_{\mathcal{J}})$ induced by it is compact. By Tychonoff's Theorem, we have the following immediately.

THEOREM 1.2. *Let $(Y, \mathcal{J})$ be a compact proximity space, then $(F, \mathcal{J}^X|F)$ is a compact proximity space iff*

(a) *$f\in Y^X\backslash F$ implies $\{f\}\overline{\mathcal{J}^X}F$.*

If $(Y, \mathcal{J})$ is separated, then the condition (a) is also a necessary condition under which $(F, \mathcal{J}^X|F)$ is a compact proximity space.

PROOF. The fact that $(Y, \mathcal{J})$ is a compact proximity space implies that $(Y, \mathcal{T}_{\mathcal{J}})$ is a compact topological space. By Tychonoff's Theorem $(Y^X, \mathcal{T}_{\mathcal{J}}^X)$ is a compact topological space. The condition (a) shows that $F$ is a $\mathcal{T}_{\mathcal{J}^X}$-closed set of $Y^X$, and $\mathcal{T}_{\mathcal{J}}^X=\mathcal{T}_{\mathcal{J}^X}$, thus $(F, \mathcal{T}_{\mathcal{J}}^X|F)$ is a compact topological space and so is $(F, \mathcal{T}_{\mathcal{J}^X|F})$, which means that $(F, \mathcal{J}^X|F)$ is a compact proximity space.

If $(Y, \mathcal{J})$ is separated, then $(Y^X, \mathcal{J}^X)$ is separated, and hence the topology $\mathcal{T}_{\mathcal{J}^X}$ is $T_2$. Thus the condition (a) is the necessary condition under which $(F, \mathcal{J}^X|F)$ is a compact space.

The condition under which $(Y, \mathcal{J})$ is a compact space may be weakened.

THEOREM 1.3. *Let $(Y, \mathcal{J})$ be a proximity space, then the sufficient condition under which $(F, \mathcal{J}^X|F)$ is a compact proximity space is*

(a) *$f\in Y^X\backslash F$ implies $\{f\}\overline{\mathcal{J}^X}F$.*

(b) *$F(x)$ has a compact closure for each $x\in X$ in the topology $\mathcal{T}_{\mathcal{J}}$ in $Y$.*

Furthermore if $(Y, \mathcal{J})$ is separated, then (a) and (b) are necessary conditions under which $(F, \mathcal{J}^X|F)$ is a compact proximity space.

PROOF. Suppose that $\mathfrak{x}$ is a $\mathcal{J}^X|F$-compressed filter base in $F$. Then $e_x(\mathfrak{x})$ is a $\mathcal{J}$-compressed filter base in $Y$ for each $x \in X$. By (b), $F(x)^-$ (the closure of the set $F(x)$ in the $\mathcal{T}_{\mathcal{J}}$ in $Y$) is $\mathcal{T}_{\mathcal{J}}$-compact, and hence $\mathcal{J}$-compact. Thus $e_x(\mathfrak{x})$ is a $\mathcal{J}$-compressed filter base in $F(x)^-$, so there is a $y \in F(x)^-$ such that $e_x(\mathfrak{x})$ $\mathcal{J}$-converges to $y$ denoted by $f(x)$. Thus we get an $f \in Y^X$, and $\mathfrak{x}$ point-wisely converges to $f$, and hence $\mathfrak{x}$ $\mathcal{J}^X$-converges to $f$. By condition (a), $f \in F$, and hence $\mathfrak{x}$ $\mathcal{J}^X|F$-converges to $f \in \mathfrak{x}$, that is $(F, \mathcal{J}^X|F)$ is a compact proximity space.

For each $x \in X$, since $(F, \mathcal{J}^X|F)$ is a compact proximity space, $F(x) = e_x(F)$ is $\mathcal{T}_{\mathcal{J}}$-compact. Furthermore, since $(Y, \mathcal{J})$ is separated, $(Y, \mathcal{T}_{\mathcal{J}})$ is a $T_2$-space. Thus $F(x)$ is $\mathcal{T}_{\mathcal{J}}$-closed, that is $F(x)^- = F(x)$ is $\mathcal{T}_{\mathcal{J}}$-compact.

## 2. Joint proximal continuity

DEFINITION 2.1. Let $(X, \mathcal{P})$ and $(Y, \mathcal{J})$ be proximity spaces, $F \subset Y^X$, and $\mathcal{R}$ be a proximity on $F$. If $\mathcal{R}$ makes a function $P: F \times X \to Y$, defined by $P(f, x) = f(x)$, $(\mathcal{R} \times \mathcal{P}, \mathcal{J})$-proximally continuous, then we call $\mathcal{R}$ *a joint proximally continuous proximity on* $F$, denoted by J. P. C.-proximity for short. And we denote the family of all $(\mathcal{P}, \mathcal{J})$-proximally continuous functions by $C(X, Y)$.

PROPOSITION 2.1. *If there is a J. P. C.-proximity on $F$, then $F \subset C(X, Y)$.*

PROOF. Suppose that $\mathcal{R}$ is a J. P. C.-proximity on $F$. For each $f \in F$, $A \subset X$, $B \subset X$ if $A \mathcal{P} B$, then

$$(f \times A) \mathcal{R} \times \mathcal{P} (f \times B) \tag{2.1}$$

In fact, for any finite decomposition of $(f \times A)$ and $(f \times B)$

$$f \times A = \bigcup_{i=1}^{n} f \times A_i \qquad f \times B = \bigcup_{j=1}^{m} f \times B_j,$$

accordingly, we have a finite decomposition of $A$, $B$

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{j=1}^{m} B_j.$$

Thus there are $i$ and $j$ such that $A_i \mathcal{P} B_j$. Furthermore we have

$$p_1(f \times A_i) = \{f\}, \qquad p_1(f \times B_j) = \{f\},$$

and hence $p_1(f \times A_i) \mathcal{R} p_1(f \times B_j)$. Note that

$$p_2(f \times A_i) = A_i, \qquad p_2(f \times B_j) = B_j,$$

thus $p_2(f \times A_i) \mathcal{P} p_2(f \times B_j)$. ($p_1$ and $p_2$ represent the projections on $F$ and on $X$ respectively). And by the definition of product proximity (2.1) holds.

Furthermore we have $P(f\times A)=f(A)$, $P(f\times B)=f(B)$. With the fact that the map $P$ is $(\mathcal{R}\times\mathcal{P}, \mathcal{I})$-proximally continuous, it follows that $f(A)\mathcal{I}f(B)$, which means, $f$ is a $(\mathcal{P}, \mathcal{I})$-proximally continuous map, $f\in C(X, Y)$.

For further discussion of joint proximal continuity, we first obtain a sufficient and necessary condition of proximal continuity, then give the concept of the family of equiproximally continuous functions (The terms and notations used in the following can be seen in [3]).

THEOREM 2.1. *Suppose that* $(X, \mathcal{P})$ *and* $(Y, \mathcal{I})$ *are proximity spaces.* $f: X\to Y$. *Then* $f$ *is* $(\mathcal{P}, \mathcal{I})$*-proximally continuous iff for each* $\mathcal{I}$*-uniform cover* $\mathcal{A}=\{A_i: i=1, 2, \cdots, n\}$ *on* $Y$*, there is a* $\mathcal{P}$*-uniform cover* $\mathcal{B}=\{B_j: j=1, 2, \cdots, m\}$ *on* $X$ *such that for each* $B_j$*, there is an* $A_i$ *satisfying* $f(B_j)\subset A_i$.

PROOF. Suppose that $\mathcal{V}_{\mathcal{P}}$ on $X$ and $\mathcal{V}_{\mathcal{I}}$ on $Y$ stand for totally bounded uniformities which are compatible with $\mathcal{P}$ and $\mathcal{I}$, respectively.

Necessity: Suppose that $\mathcal{A}'=\{A'_k: k=1, 2, \cdots, l\}$ is a star-refinement $\mathcal{I}$-uniform cover of $\mathcal{A}$, then $V'=\bigcup_{k=1}^{l} A'_k\times A'_k\in\mathcal{V}_{\mathcal{I}}$. The fact that $f$ is $(\mathcal{P}, \mathcal{I})$-proximally continuous implies that $f$ is $(\mathcal{V}_{\mathcal{P}}, \mathcal{V}_{\mathcal{I}})$-uniformly continuous. Then there is a $\mathcal{P}$-uniform cover $\mathcal{B}=\{B_j: j=1, 2, \cdots, m\}$ on $X$. Suppose that $f(x_0)\in A'_{k_0}$ for a fixed $x_0\in B_j$, then $(f(x_0), f(x))\in V'$ for each $x\in B_j$. Thus there is an index $k$ such that $(f(x_0), f(x))\in A'_k\times A'_k$, which means $f(x_0)\in A'_{k_0}\times A'_k$, and hence $f(x)\in(A'_{k0})^*$ and $\mathcal{A}'$ is a star-refienment of $\mathcal{A}$, and there is an $A_i\in\mathcal{A}$ such that $(A'_{k_0})^*\subset A_i$, thus $f(B_j)\subset A_i$.

Sufficiency: Suppose $C$, $D\subset X$ and $C\mathcal{P}D$. By the hypothesis, for any $\mathcal{I}$-uniform cover $\mathcal{A}=\{A_i: i=1, 2, \cdots, n\}$, on $Y$, there is a $\mathcal{P}$-uniform cover $\mathcal{B}=\{B_j: j=1, 2, \cdots, m\}$ on $X$ and there is an $A_i$ such that $f(B_j)\subset A_i$ for each $B_j$. Since $C\mathcal{P}D$, there is a $B_j$ such that $C\cap B_j\neq\emptyset$ and $D\cap B_j\neq\emptyset$. Therefore, $f(C)\cap A_i\neq\emptyset$ and $f(D)\cap A_i\neq\emptyset$. This means $f(C)\mathcal{I}f(D)$, and hence $f$ is a $(\mathcal{P}, \mathcal{I})$-proximally continuous map.

DEFINITION 2.2. Suppose that $(X, \mathcal{P})$ and $(Y, \mathcal{I})$ are proximity spaces, $F\subset Y^X$. If for each $\mathcal{I}$-uniform cover $\mathcal{A}=\{A_i: i=1, 2, \cdots, n\}$ on $Y$, there is a $\mathcal{P}$-uniform cover $\mathcal{B}=\{B_j: j=1, 2, \cdots, m\}$ on $X$ such that for each $f\in F$, $B_j\in\mathcal{B}$ there is an $A_i\in\mathcal{A}$ such that $f(B_j)\subset A_i$, then $F$ is called a *family of* $(\mathcal{P}, \mathcal{I})$*-equiproximally continuous functions.*

We obtain the following from Theorem 2.1 and Definition 2.2.

PROPOSITION 2.2. *If* $F$ *is a family of* $(\mathcal{P}, \mathcal{I})$*-equiproximally continuous functions, then* $F\subset C(X, Y)$.

## 3. Uniformly convergent proximity

Let $X$ be a set, $(Y, \mathcal{V})$ be a uniform space, and $F \subset Y^X$. For $V \in \mathcal{V}$, let

$$W(V) = \{(f, g) \in F \times F : (f(x), g(x)) \in V, \text{ for each } x \in X\}$$

We have known that the uniformity with $\{W(V) : V \in \mathcal{V}\}$ as a base is called the uniformity of uniform convergence, denoted by u. c.-uniformity on $F$.

DEFINITION 3.1. The proximity on $F$ generated by a u. c.-uniformity on $F$ is called the *uniformly convergent proximity*, denoted by u. c.-proximity, or is called u. c.-proximity induced by $\mathcal{V}$.

PROPOSITION 3.1. *Suppose that $\mathfrak{R}$ is a u. c.-proximity on $F$, $A \subset F$ and $B \subset F$, then $A \mathfrak{R} B$ iff for each $V \in \mathcal{V}$, there are $f \in A$ and $g \in B$ such that $(f(x), g(x)) \in V$ for each $x \in X$.*

THEOREM 3.1. *Let $\mathfrak{R}$ be a u. c.-proximity on $F$, then the net $\{f_\alpha : \alpha \in D\}$ in $F$ $\mathfrak{R}$-converges to $f \in F$ iff the net $\mathcal{V}$-uniformly converges to $f$ on $X$.*

PROOF. Suppose $G \subset F$ and $\{f\} \bar{\mathfrak{R}} F \backslash G$. Thus there is a $V \in \mathcal{V}$ such that for each $g \in F \backslash G$ there is an $x \in X$ such that

$$(f(x), g(x)) \notin V \tag{3.1}$$

We choose a symmetric element $U \in \mathcal{V}$ such that $U \circ U \subset V$. Since the net $\{f_\alpha : \alpha \in D\}$ uniformly converges to $f$ on $X$, there is an $\alpha_0 \in D$ such that

$$(f(x), f_\alpha(x)) \in U \qquad \text{for each} \quad \alpha \geqq \alpha_0,\ x \in X. \tag{3.2}$$

If $\{f_\alpha : \alpha \geqq \alpha_0, \alpha \in D\} \mathfrak{R} F \backslash G$, then there is an $\alpha$, $\alpha \geqq \alpha_0$ and $\alpha \in D$, and a $g \in F \backslash G$, such that

$$(f_\alpha(x), g(x)) \in U \qquad \text{for each} \quad x \in X. \tag{3.3}$$

By (3.2) and (3.3), $(f(x), g(x)) \in U \circ U \subset V$ for any $x \in X$, which contradicts (3.1), hence $\{f_\alpha : \alpha \geqq \alpha_0, \alpha \in D\} \bar{\mathfrak{R}} F \backslash G$. Which means the net $\{f_\alpha : \alpha \in D\}$ $\mathfrak{R}$-converges to $f$.

Let $G = W(V)[f]$ for each symmetric element $V \in \mathcal{V}$, then $\{f\} \bar{\mathfrak{R}} F \backslash G$. Otherwise, there must be a $g \notin G$ such that $(f(x), g(x)) \in V$ for each $x \in X$, that means $g \in W(V)[f]$, a desired contradiction.

Since the net $\{f_\alpha : \alpha \in D\}$ $\mathfrak{R}$-converges to $f$, there is an $\alpha_0 \in D$ such that

$$\{f_\alpha : \alpha \geqq \alpha_0, \alpha \in D\} \bar{\mathfrak{R}} F \backslash G.$$

Hence $f_\alpha \in G$ for each $\alpha \geqq \alpha_0$, which implies $(f(x), f_\alpha(x)) \in V$ for each $x \in X$. This means $\{f_\alpha : \alpha \in D\}$ $\mathcal{V}$-uniformly converges to an $f$ on $X$.

THEOREM 3.2. *Let $(X, \mathcal{P})$ and $(Y, \mathcal{J})$ be proximity spaces and $F$ be a family of $(\mathcal{P}, \mathcal{J})$-equiproximally continuous functions. Then the u. c.-proximity $\mathcal{R}$ on $F$ induced by $\mathcal{V}_{\mathcal{J}}$, a totally bounded uniformity and is compatible with $\mathcal{J}$, is J. P. C.-proximity.*

PROOF. Take arbitrary $C \subset F \times X$, $D \subset F \times X$, such that $C \mathcal{R} \times \mathcal{P} D$, and a $\mathcal{J}$-uniform cover $\mathcal{A} = \{A_i : i = 1, 2, \cdots, n\}$ on $Y$, we have a $\mathcal{J}$-uniform cover $\mathcal{A}' = \{A'_i : i = 1, 2, \cdots, n'\}$ on $Y$, which is a star-refinement of $\mathcal{A}$. By equicontinuity of $F$, there is a $\mathcal{P}$-uniform cover $\mathcal{B} = \{B_j : j = 1, 2, \cdots, m\}$ on $X$ such that for each $f \in F$, $B_j \in \mathcal{B}$, there is an $A'_i \in \mathcal{A}'$ such that $f(B_j) \subset A'_i$. We take another $\mathcal{P}$-uniform cover $\mathcal{B}'$ on $X$, as a star-refinement of $\mathcal{B}$.

Let $F_{ij} = \{f \in F : f(B'_j) \subset A'_i\}$,

$$C_{ij} = C \cap (F_{ij} \times B'_j),$$

$$D_{ij} = D \cap (F_{ij} \times B'_j).$$

We obtain finite decompositions of $C$ and $D$: $C = \cup C_{ij}$, $D = \cup D_{ij}$. There are $(i, j)$ and $(i', j')$, such that $p_1(C_{ij}) \mathcal{R} p_1(D_{i'j'})$ and $p_2(C_{ij}) \mathcal{P} p_2(D_{i'j'})$. Hence there are $f \in p_1(C_{ij})$ and $g \in p_1(D_{i'j'})$. For each $x \in X$ there is an $e$ such that $(f(x), g(x)) \in A'_e \times A'_e$, and there is a $B'_{j_0}$ such that $p_2(C_{ij}) \times p_2(D_{i'j'}) \cap B'_{j_0} \times B'_{j_0} \neq \emptyset$. Also, $p_2(C_{ij}) \subset B'_j$, $p_2(D_{i'j'}) \subset B'_{j'}$, and hence $B'_j \times B'_{j'} \cap B'_{j_0} \times B'_{j_0} \neq \emptyset$.

And becaus of $f \in p_1(C_{ij})$, there is an $x \in B'_j$ such that $(f, x) \in C_{ij}$. Similarly $g \in p_1(D_{i'j'})$, there is an $x' \in B'_{j'}$ such that $(g, x) \in D_{i'j'}$, and hence $x, x' \in (B'_{j_0})^* \subset B_j$. Thus $(f(x), f(x')) \in A'_i \times A'_i$, $(g(x), g(x')) \in A'_k \times A'_k$ and $(f(x), g(x)) \in A'_e \times A'_e$, and hence $(f(x), g(x')) \in (A'_e)^* \times (A'_e)^*$. Note that $(A'_e)^* \subset A_i$. Which means both $f(x)$ and $g(x')$ belong to $A_i$. We have known that $(f, x) \in C$, $(g, x') \in D$, so $P(C) \cap A_i \neq \emptyset$ and $P(D) \cap A_i \neq \emptyset$, which implies that $P(C) \mathcal{J} P(D)$.

THEOREM 3.3. *Let $(X, \mathcal{P})$ be a proximity space, $(Y, \mathcal{V})$ be a uniform space, $Y^X$ possess u. c.-proximity $\mathcal{R}$ induced by $\mathcal{V}$ and $\mathcal{J}_{\mathcal{V}}$ be a proximity on $Y$ generated by $\mathcal{V}$. If a net of $(\mathcal{P}, \mathcal{J}_{\mathcal{V}})$-proximally continuous maps $\{f_\alpha : \alpha \in D\}$ $\mathcal{R}$-converges to $f$ in $Y^X$, then $f$ is a $(\mathcal{T}, \mathcal{J}_{\mathcal{V}})$-proximally continuous map.*

PROOF. If $f$ is not a $(\mathcal{P}, \mathcal{J}_{\mathcal{V}})$-proximally continuous map, there is $A \subset X$, $B \subset X$, $A \mathcal{P} B$, but $f(A) \bar{\mathcal{J}}_{\mathcal{V}} f(B)$. Thus there is a symmetric element $V \in \mathcal{V}$ such that

$$(f(x), f(y)) \notin V \qquad \text{for each} \quad x \in A,\ y \in B \tag{3.4}$$

Choose a symmetric element $U \in \mathcal{V}$ such that $U \circ U \circ U \subset V$, since the net $\{f_\alpha : \alpha \in D\}$ $\mathcal{R}$-converges to $f$ there is an $\alpha_0 \in D$ such that

$$(f(x), f_\alpha(x)) \in U \qquad \text{for each} \quad \alpha \geqq \alpha_0,\ x \in X. \tag{3.5}$$

From the $(\mathcal{P}, \mathcal{S}_{\mathcal{V}})$-proximal continuity of $f_\alpha$, we know $f_\alpha(A) \mathcal{S}_{\mathcal{V}} f_\alpha(B)$, implying there are $a \in A$ and $b \in B$ such that

$$(f_\alpha(a), f_\alpha(b)) \in U. \tag{3.6}$$

By (3.5), we obtain

$$(f(a), f_\alpha(a)) \in U. \tag{3.7}$$

$$(f(b), f_\alpha(b)) \in U. \tag{3.8}$$

By (3.6), (3.7) and (3.8), we have

$$(f(a), f(b)) \in U \circ U \circ U \subset V. \tag{3.9}$$

where $a \in A$, $b \in B$, which contradicts (3.4).

COROLLARY. *Under the hypothesis of the theorem, the family* $C(X, Y)$ *of all* $(\mathcal{P}, \mathcal{S}_{\mathcal{V}})$*-proximally continuous functions is a* $\mathcal{T}_{\mathcal{R}}$*-closed set in* $Y^X$.

## 4. Uniformly convergent proximity on a family of subsets

Let $X$ be a set, $(Y, \mathcal{V})$ be a uniform space, $F \subset Y^X$ and $\mathcal{K}$ be a family of subsets of $X$ closed for finite union, in other words, if $K_1 \in \mathcal{K}$, $K_2 \in \mathcal{K}$, then $K_1 \cup K_2 \in \mathcal{K}$.

THEOREM 4.1. *Suppose that* $\mathcal{S}$ *is a binary relation on the family of all subsets of* $F$. *If for each* $A \subset F$, $B \subset F$, $A \mathcal{S} B$ *iff for each* $V \in \mathcal{V}$, $K \in \mathcal{K}$, *there are* $f \in A$ *and* $g \in B$ *such that* $(f(x), g(x)) \in V$ *for each* $x \in K$, *then* $\mathcal{S}$ *is a proximity on* $F$.

PROOF. It is obvious that $\mathcal{S}$ satisfies $(P_1)$-$(P_4)$ (cf. [2]). If $A \bar{\mathcal{S}} C$, and $B \bar{\mathcal{S}} C$, then there are $V_1 \in \mathcal{V}$ and $K_1 \in \mathcal{K}$ such that for each $f \in A$, $g \in C$, there is an $x_1 \in K_1$ such that $(f(x_1), g(x_1)) \notin V$, and there are $V_2 \in \mathcal{V}$ and $K_2 \in \mathcal{K}$ such that for each $\varphi \in B$, $g \in C$, there is an $x_2 \in K_2$ such that $(\varphi(x_2), g(x_2)) \notin V_2$.

Thus there are $V = V_1 \cap V_2 \in \mathcal{V}$ and $K = K_1 \cup K_2 \in \mathcal{K}$ such that there is an $x \in K_1 \cup K_2$ satisfying $(\psi(x), g(x)) \notin V$ for each $\psi \in A \cup B$, $g \in C$. So it satisfies $(P_5)$.

If $A \bar{\mathcal{S}} B$, then there are $V \in \mathcal{V}$ and $K \in \mathcal{K}$, for each $f \in A$, $g \in B$, there is an $x \in K$ such that

$$(f(x),\ g(x)) \notin V. \tag{4.1}$$

Take a symmetric element $U \in \mathcal{V}$, such that $U \circ U \subset V$. Let

$P = \{f \in F$: there is a $g \in A$, $(f(x), g(x)) \in U$ for each $x \in K\}$,

$Q = \{f \in F$: there is a $g \in B$, $(f(x), g(x)) \in U$ for each $x \in K\}$,

If $P \cap Q \neq \emptyset$, then there is $f \in P \cap Q$, $g \in A$, $g' \in B$ such that $(f(x), g(x)) \in U$ and $(f(x), g'(x)) \in U$ for each $x \in K$, thus $(g(x), g'(x)) \in V$ for each $x \in K$. It is contrary to (4.1). So $P \cap Q = \emptyset$.

In addition, there are $U \in \mathcal{V}$ and $K \in \mathcal{K}$ such that there must be an $x \in X$ such that $(g(x), f(x)) \notin U$ for each $g \in A, f \in F \backslash P$. Hence $A \bar{\mathcal{S}} F \backslash P$. Similarly, $B \bar{\mathcal{S}} F \backslash Q$. Which means that $(P_6)$ is satisfied and $\mathcal{S}$ is a proximity on $F$.

DEFINITION 4.1. The proximity on $F$ defined by Theorem 4.1 is called the *uniformly convergent proximity on $\mathcal{K}$ induced by $\mathcal{V}$*, denoted by $(\mathcal{K})$ u.c.-proximity.

In view of Definition 4.1, we obtain the following proposition immediately:

PROPOSITION 4.1. *If the $(\mathcal{K}_1)$ u. c.-proximity on $F$ induced by $\mathcal{V}$ is $\mathcal{S}_1$ and $(\mathcal{K}_2)$ u. c.-proximity is $\mathcal{S}_2$, then:*

(a) *If $\mathcal{K}_1 \subset \mathcal{K}_2$, then $\mathcal{S}_1 < \mathcal{S}_2$,*

(b) *If there is a $K_0 \in \mathcal{K}_2$ such that $K_0 \supset \bigcup_{K \in \mathcal{K}_1} K$, then $\mathcal{S}_1 < \mathcal{S}_2$,*

(c) *u. c.-proximity is more refined than any other $(\mathcal{K})$ u. c.-proximity,*

(d) *If $\mathcal{K}_0$ consists of all finite subsets of $X$, then $(\mathcal{K}_0)$ u. c.-proximity is the coarsest one of all $(\mathcal{K})$ u. c.-proximity satisfying $\bigcup_{K \in \mathcal{K}} K = X$. ($(\mathcal{K}_0)$ u. c.-proximity is called a point-wise convergent u. c.-proximity).*

THEOREM 4.2. *Suppose that the $(\mathcal{K})$ u. c.-proximity on $F$ induced by $\mathcal{V}$ is $\mathcal{S}$, then a net $\{f_\alpha : \alpha \in D\}$ in $F$ $\mathcal{S}$-converges to $f \in F$ iff the net $\mathcal{V}$-uniformly converges to $f$ on $K$ for each $K \in \mathcal{K}$.*

The concept of joint proximally continuous proximity may be generalized to the family $\mathcal{K}$ of subsets of $X$.

DEFINITION 4.2. Let $(X, \mathcal{P})$ and $(Y, \mathcal{J})$ be proximity spaces, $F \subset Y^X$ and $\mathcal{K}$ be a family of subsets of $X$ which is closed for finite join operation. If there is a proximity $\mathcal{R}$ such that $P$: $F \times K \to Y$ $(P(f, x) = f(x))$ is a $(\mathcal{R} \times \mathcal{P} | K, \mathcal{J})$-proximally continuous map for each $K \in \mathcal{K}$, then $\mathcal{R}$ is called a *joint proximally continuous proximity on $\mathcal{K}$* on $F$, denoted by $(\mathcal{K})$ J. P. C.-proximity for short.

Theorem 3.2 can be generalized as follows:

THEOREM 4.3. *Let* $(X, \mathcal{P})$ *and* $(Y, \mathcal{J})$ *be proximity spaces, F be a family of* $(\mathcal{P}, \mathcal{J})$*-equiproximally continous functions and* $\mathcal{K}$ *be a family of subsets which is closed for finite join operation. Then the* $(\mathcal{K})$ *u. c.-proximity* $\mathcal{R}$ *on F induced by* $\mathcal{V}_{\mathcal{J}}$ *is a* $(\mathcal{K})$ *J. P. C.-proximity.*

The proofs of above two theorems are omitted.

## References

[1] Efremovič, V.A. (Ефремович, В.А.), Инфинитезимальные пространства, ДАН СССР 76 (1951), 341-343.
[2] Császár, A., General topology, Budapest, (1978).
[3] Engelking, R., General topology, PWN. Warsaw, (1977).
[4] Arens, R., A topology for spaces of transformations, Ann. of Math., 47 (1946), 480-495.
[5] Arens, R. and Dugundji, J., Topologies for function spaces, Pacific J. of Math., 1 (1951), 5-31.
[6] Stevenson, F.W., Product proximities, Fund. Math., 76 (1972), 157-166.
[7] Naimpally, S.A. and Warrack, B.D., Proximity spaces, Cambridge, (1970).
[8] Kelley, J.L., General topology, New York, (1955).

Mathematics Department
Heilongjiang University
Harbin
China