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SOBOLEV SPACES IN THE GENERALIZED DISTRIBUTION SPACES OF BEURLING TYPE

By

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1. Introduction

In this paper we extend the concept of Sobolev spaces to the generalized distribution spaces of Beurling type and investigate the Sobolev imbedding theorem, the Rellich's compactness theorem and etc on these generalized Sobolev spaces.

For this purpose we briefly introduce the basic spaces and theories which we need in this paper. The reader can find the details in [3]. Let $\mathcal{M}_c$ the set of all continuous real valued functions ω on R^n which satisfies the following conditions:

(α) $0=\omega(0)\leqq\omega(\xi+\eta)\leqq\omega(\xi)+\omega(\eta)$, ξ, $\eta\in R^n$.

(β) $\int_{R^n}\frac{\omega(\xi)}{(1+|\xi|)^{n+1}}d\xi<\infty$

(γ) $\omega(\xi)\geqq a+b\log(1+|\xi|)$ for some constants a and $b>0$.

(δ) $\omega(\xi)$ is radial.

With the weight functions ω in $\mathcal{M}_c$ and open set Ω in R^n Björck defines $\mathcal{D}_\omega(\Omega)$ the set of all ϕ in $L^1(R^n)$ such that ϕ has compact support in Ω and

$$\|\varphi\|_\lambda=\int_{R^n}|\hat{\phi}(\xi)|e^{\lambda\omega(\xi)}d\xi<\infty$$

for all $\lambda>0$. The space $\mathcal{D}_\omega(\Omega)$ equipped with the inductive limit topology, as $\mathcal{D}(\Omega)$, is Fréchet and we call $\mathcal{D}'_\omega(\Omega)$, the dual of $\mathcal{D}_\omega(\Omega)$, the Beurling's generalized distribution space. They denote by $\mathcal{E}_\omega(\Omega)$ the set of all complex valued functions ψ in Ω such that $\psi\phi\in\mathcal{D}_\omega(\Omega)$ for all $\phi\in\mathcal{D}_\omega(\Omega)$ and the topology is given by the semi-norms $\|\phi\psi\|_\lambda$ for every $\lambda>0$ and every ϕ in $\mathcal{D}_\omega(\Omega)$. The dual space $\mathcal{E}'_\omega(\Omega)$ of the space $\mathcal{E}_\omega(\Omega)$ can be identified with the set of all elements of $\mathcal{D}'_\omega(\Omega)$ which have compact support contained in Ω. They also extend the Schwartz space denoting by $\mathcal{S}_\omega$ the space of all C^∞-function ϕ in $L^1(R^n)$ with the property that for each multi-index α and each non-negative number λ we have

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$$P_{\alpha,\lambda}(\phi)=\sup_{x\in R^n} e^{\lambda\omega(x)}|D^\alpha\phi(x)|<\infty$$

and

$$\Pi_{\alpha,\lambda}(\hat{\phi})=\sup_{\xi\in R^n} e^{\lambda\omega(\xi)}|D^\alpha\hat{\phi}(\xi)|<\infty$$

and $\mathcal{S}'_\omega$ the dual space of the space $\mathcal{S}_\omega$. Most theorems in the distribution space can be extended in this generalized distribution spaces. For example, we have the following Paley-Wiener type theorem which can be founded in [3];

THEOREM. *Let K be a compact convex set in R^n with support function H. If F is an entire function of n-complex variables $\zeta=\xi+i\eta=(\zeta_1,\cdots,\zeta_n)$, the following three conditions are equivalent:*

(i) *For each $\lambda>0$ and each $\varepsilon>0$ there exists a constant $C_{\lambda,\varepsilon}$ such that for every $\eta\in R^n$.*

$$\int_{R^n}|F(\xi+i\eta)|e^{\lambda\omega(\xi)}d\xi\leqq C_{\lambda,\varepsilon}e^{H(\eta)+\varepsilon|\eta|}.$$

(iii) *For each $\lambda>0$ and each $\varepsilon>0$ there exists a constant $C_{\lambda,\varepsilon}$ such that for $\zeta=\xi+i\eta\in C^n$.*

$$|F(\xi+i\eta)|\leqq C_{\lambda,\varepsilon}e^{H(\eta)+\varepsilon|\eta|-\lambda\omega(\xi)}.$$

(iii) $F(\zeta)=\int_{R^n}e^{-i\langle x,\zeta\rangle}\phi(x)dx$ *for some* $\phi\in\mathcal{D}_\omega(K)$.

Moreover, the Fourier transform is isomorphic on the space $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$.

According to the definitions of the above spaces we may assume $b=1$ in the condition (γ) of $\omega\in\mathcal{M}_c$, since it does not effect the size of the spaces. Throughout this paper we assume that the weight function ω satisfies the condition (γ) with $b=1$.

2. Generalized Sobolev Spaces

Since the Beurling's generalized distribution space with weight function $\omega(\xi)=\log(1+|\xi|)$ is exactly the distribution space, we naturally define the generalized Sobolev space as follows:

DEFINITION 2.1. For $s\in R$, we denote by H^s_ω the set of all generalized distributions $u\in\mathcal{S}'_\omega$ such that

$$\|u\|^\omega_s=\left[\int_{R^n}e^{2s\omega(\xi)}|\hat{u}(\xi)|^2d\xi\right]^{1/2}<\infty$$

We call H^s_ω the generalized ω-Sobolev space of order s or simply ω-Sobolev space of order s.

From the definition we can easily see that $\mathcal{S}_\omega$ is contained in H_ω^s for all $s\in R$; $H_\omega^0=L^2(R^n)$, $H^s\subset H_\omega^s$ for $s\leqq 0$ and $H^s\supset H_\omega^s$ for $s\geqq 0$. $H_\omega^s=H^s$ when $\omega(\xi)=\log(1+|\xi|)$.

THEOREM 2.2. *H_ω^s is a Hilbert space with inner product given by* $(u, v)_s^\omega=\int e^{2s\omega(\xi)}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$.

PROOF. It is clearly an inner product on H_ω^s and the completeness follows from the Proposition 2.2.2 and Theorem 2.2.3 in [3].

As we mentioned before, $H_\omega^s=H^s$ when the weight function $\omega(\xi)=\log(1+|\xi|)$.

THEOREM 2.2. *$\mathcal{S}_\omega$ is dense in H_ω^s for all $s\in R$.*

PROOF. Since $\mathcal{D}_\omega$ is dense in C_0^∞, C_0^∞ is dense in $L^2(e^{2s\omega(\xi)}d\xi)$ and $\mathcal{D}_\omega\subset\mathcal{S}_\omega\subset L^2(e^{2s\omega(\xi)}d\xi)$, $\mathcal{S}_\omega$ is dense in $L^2(e^{2s\omega(\xi)}d\xi)$. From the fact that Fourier transform is isomorphism on $\mathcal{S}_\omega$ the theorem follows.

COROLLARY 2.3. *$H_\omega^t\subset H_\omega^s$ for $t>s$, the inclusion is continuous and has dense image.*

From the fact that for multi-index α

$$\|D^\alpha u\|_{s-|\alpha|}^\omega\leqq e^{a|\alpha|}\|u\|_s^\omega,\qquad u\in H_\omega^s$$

we have

COROLLARY 2.4. *The differential D^α is a continuous linear operator from H_ω^s to $H_\omega^{s-|\alpha|}$.*

To find the relation between H_ω^s and H_ω^{-s}. We define the pairing

$$\langle u, \phi\rangle=u(\bar{\phi})\quad \textit{for}\quad u\in\mathcal{S}_\omega'\quad \textit{and}\quad \phi\in\mathcal{S}_\omega.$$

Then we can easily get the following.

LEMMA 2.5. *For $u\in H_\omega^s$, The conjugate linear functional $\langle u, \cdot\rangle$ on $\mathcal{S}_\omega$ extends uniquely to a conjugate linear functional on H_ω^{-s} satisfying*

(i) $\langle u, v\rangle=(2\pi)^{-n}\int\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$

(ii) $|\langle u, v\rangle|\leqq\|u\|_s^\omega\|v\|_{-s}^\omega,\qquad u\in H_\omega^s,\qquad v\in H_\omega^{-s}$

(iii) $\langle u, v\rangle=\overline{\langle v, u\rangle}$

PROOF. For $u\in H_\omega^s$, the map $v\mapsto(2\pi)^{-n}\int\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi$ is a continuous conjugate linear functional, since

$$|(2\pi)^{-n}\int\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi|\leqq(2\pi)^{-n}\Big(\int e^{2s\omega(\xi)}|\hat{u}(\xi)|^2d\xi\Big)^{1/2}\Big(\int e^{-2s\omega(\xi)}|\hat{v}(\xi)|^2d\xi\Big)^{1/2}$$
$$=(2\pi)^{-n}\|u\|_s^\omega\|v\|_{-s}^\omega.$$

The remaining results follow from the density of $\mathcal{S}_\omega$ in H_ω^{-s}.

THEOREM 2.6. *The pairing $\langle\cdot,\cdot\rangle$ identifies H_ω^{-s} isometrically with the antidual of H_ω^s. If $u\in\mathcal{D}'_\omega$, then $u\in H_\omega^s$ if and only if there is a constant c such that $|u(\phi)|\leqq c\|\phi\|_{-s}^\omega$ for $\phi\in\mathcal{D}_\omega$. Moreover, the best value of c is $\|u\|_s^\omega$.*

PROOF. Let $*(H_\omega^s)$ be the antidual of H_ω^s. Define $G: H_\omega^{-s}\to*(H_\omega^s)$ by

$$G(v)(u)=\langle v, u\rangle=(2\pi)^{-n}\int\hat{v}(\xi)\bar{\hat{u}}(\xi)d\xi.$$

From lemma 2.5 (ii) we have $\|G(v)\|\leqq\|v\|_{-s}^\omega$ for all $v\in H_\omega^{-s}$, which implies $\|G\|\leqq1$. On the other hand, if $G(v)=0$, $G(v)(u)=\langle v, u\rangle=(2\pi)^{-n}\int\hat{v}(\xi)\bar{\hat{u}}(\xi)d\xi=0$ for all $u\in S_\omega$. Therefore $v=0$ as a point of S'_ω, and hence $v=0$ as a point of H_ω^{-s}. To show the surjectivity of G, for $w\in*(H_\omega^s)$, there is a $w_1\in H_s^\omega$ due to the Riesz representation theorem such that $w(u)=(w_1, u)_s^\omega$ for all u in H_ω^s. Since $\phi\to\int e^{2s\omega(\xi)}\hat{w}_1(\xi)\hat{\phi}(\xi)d\xi$ is a continuous linear functional on S_ω, there is a distribution $w_2\in H_\omega^{-s}$ such that $\hat{w}_2(\xi)=e^{2s\omega(\xi)}\hat{w}_1(\xi)$ a.e. Then we have

$$w(u)=(w_1, u)_s^\omega=(2\pi)^{-n}\int e^{2s\omega(\xi)}\hat{w}_1(\xi)\bar{\hat{u}}(\xi)d\xi$$
$$=(2\pi)^{-n}\int e^{2s\omega(\xi)}e^{-2s\omega(\xi)}\hat{w}_2(\xi)\bar{\hat{u}}(\xi)d\xi$$
$$=\langle w_2, u\rangle=G(w_2)(u)\qquad\text{for all } u \text{ ih } H_\omega^s,$$

that is, $w=G(w_2)$. To show the isometry, let $v\in H_\omega^{-s}$ and put $u\in H_\omega^s$ such that $\hat{u}(\xi)=e^{-2s\omega(\xi)}\hat{v}(\xi)$ a.e. Then

$$G(v)(u)=(2\pi)^{-n}\int e^{-2s\omega(\xi)}|\hat{v}(\xi)|^2d\xi$$
$$=[\|v\|_{-s}^\omega]^2=\|u\|_s^\omega\|v\|_{-s}^\omega.$$

Therefore, $\|G(v)\|\geqq\|v\|_{-s}^\omega$, which means $\|G\|\geqq1$. Hence G is an isometry and so G is an isometric isomorphism from H_ω^{-s} onto $*(H_\omega^s)$.

In order to show the last statement, let $u\in H_\omega^s$. Then, by the above identification, we have

$$\|u\|_s^\omega=\sup\left\{\frac{|u(\phi)|}{\|\phi\|_{-s}^\omega} : \phi\in H_\omega^{-s}\right\}\geqq\sup\left\{\frac{|u(\phi)|}{\|\phi\|_{-s}^\omega} : \phi\in\mathcal{D}_\omega\right\},$$

which is required. Conversely, if $u\in\mathcal{D}'_\omega$ and $|u(\phi)|\leqq c\|\phi\|_{-s}^\omega$ for all $\phi\in\mathcal{D}_\omega$, then the map $\phi|\rightarrow u(\bar{\phi})$ extends to an element $\langle u, \cdot\rangle$ of $^*(H_\omega^{-s})$ with norm $\leqq c$. Thus there is a unique $w\in H_\omega^s$ such that $G(w)(\phi)=\langle w, \phi\rangle=u(\bar{\phi})=\langle u, \phi\rangle$. This shows $u=w$.

COROLLARY 2.7. *Let $A: \mathcal{D}_\omega\rightarrow\mathcal{D}'_\omega$ be a linear map. Then A extends uniquely to a continuous linear map $A: H_\omega^s\rightarrow H_\omega^t$ if and only if $|(Au)(v)|\leqq c\|u\|_s^\omega\|v\|_{-t}^\omega$ for $u, v\in\mathcal{D}_\omega$, for some constant c. Moreover, the best value of c is the norm $\|A\|$ of the linear operator $A: H_\omega^s\rightarrow H_\omega^t$.*

PROOF. If $u\in\mathcal{D}_\omega$ and $Au\in H_\omega^t$, we have, from theorem 2.6, $|(Au)(v)|=|\langle Au, v\rangle|\leqq\|Au\|_t^\omega\|v\|_{-t}^\omega=\|A\|\|u\|_s^\omega\|v\|_{-t}^\omega$ for $u, v\in\mathcal{D}_\omega$. The converse is obvious from the density of $\mathcal{D}_\omega$.

DEFINITION 2.2. If Ω is an open subset of R^n and $s\in R$, then we define $H_{\omega\,\mathrm{loc}}^s(\Omega)=\}u\in\mathcal{D}'_\omega(\Omega): \phi u\in H_\omega^s$ for all $\phi\in\mathcal{D}_\omega(\Omega)\}$.

LEMMA 2.8. *If $\phi\in D_\omega$ and $u\in H_\omega^s$, then $\phi u\in H_\omega^s$.*

PROOF. By the Minkowski inequality, we have

$$\begin{aligned}\|\phi u\|_s^\omega&=\left(\int e^{2s\omega(\xi)}|\widehat{\phi u}(\xi)|^2d\xi\right)^{1/2}\\&=\left(\int e^{2s\omega(\xi)}\left[(2\pi)^{-n}\int\hat{\phi}(\eta)\hat{u}(\xi-\eta)d\eta\right]^2d\xi\right)^{1/2}\\&\leqq(2\pi)^{-n/2}\int|\hat{\phi}(\eta)|\left(\int|\hat{u}(\xi-\eta)|^2e^{2s\omega(\xi)}d\xi\right)^{1/2}d\eta\\&\underset{(\alpha)}{\leqq}\begin{cases}(2\pi)^{-n/2}\left(\int|\hat{\phi}(\eta)|e^{s\omega(\eta)}d\eta\right)\left(\int|\hat{u}(\xi-\eta)|^2e^{2s\omega(\xi-\eta)}d\xi\right)^{1/2} & \text{if } s\geqq0\\(2\pi)^{-n/2}\left(\int|\hat{\phi}(\eta)|e^{-s\omega(\eta)}u\eta\right)\left(\int|\hat{u}(\xi-\eta)|^2e^{2s\omega(\xi-\eta)}d\xi\right)^{1/2} & \text{if } s<0\end{cases}\\&=(2\pi)^{-n/2}\|\phi\|_{|s|}\|u\|_s^\omega<\infty,\quad\text{which shows } \phi u\in H_\omega^s\end{aligned}$$

THEOREM 2.9. *If $P(x, D)=\sum_{|\alpha|\leqq m}a_\alpha(x)D^\alpha$ with $a_\alpha\in\mathcal{E}_\omega(\Omega)$, then $P(x, D)$ is continuous linear map from $H_{\omega\,\mathrm{loc}}^s(\Omega)$ into $H_{\omega\,\mathrm{loc}}^{s-m}(\Omega)$.*

PROOF. Let $u\in H_{\omega\,\mathrm{loc}}^s$ and $\phi\in\mathcal{D}_\omega$. Choose χ in $\mathcal{D}_\omega(\Omega)$ such that $\chi\equiv1$ on a neighborhood of supp(ϕ). Then we get the result from corollary 2.4 and lemma 2.8.

Next we give some examples.

EXAMPLE.

(1) δ_x is an element of H_ω^s for any $s<-(n/2)$.

(2) $D^\alpha\delta_x$ is an element of $H_\omega^{s-|\alpha|}$ for $s<-(n/2)$.

(3) On R, the characteristic function $u(x)=\chi_{(-1,1)}$ is an element of H_ω^s when $s\leqq 0$.

3. Imbedding Theorem and Compactness

In this section we extend the Sobolev imbedding theorem and Rellich's compactness theorem to the generalized Sobolev space. For this we need several definitions.

DEFINITION 3.1. We denote by $H_\omega^\infty=\bigcap_{s\in R} H_\omega^s$ and $H_\omega^{-\infty}=\bigcup_{s\in R} H_\omega^s$. We provide H_ω^∞(resp. $H_\omega^{-\infty}$) with the weakest (resp. the strongest) topology such that the cannonical injection $H_\omega^\infty\to H_\omega^s$(resp. $H_\omega^s\to H_\omega^{-\infty}$) is continuous for all $s\in R$.

DEFINITION 3.2. Let k be a non-negative integer. We denote by $\mathcal{E}_\omega^k(\Omega)$ the vector space of all locally integrable functions u on Ω such that

$$\int e^{k\omega(\xi)}|\widehat{\phi u}(\xi)d\xi<\infty \qquad \text{for all} \quad \phi\in\mathcal{D}_\omega(\Omega).$$

We notice that the intersection of such spaces $\mathcal{E}_\omega^k(\Omega)$ is $\mathcal{E}_\omega(\Omega)$. Moreover we get

PROPOSITION 3.1. *For any nonnegative integer* k, $\mathcal{E}_\omega^k(\Omega)\subset C^k(\Omega)$.

PROOF. Let $u\in\mathcal{E}_\omega^k(\Omega)$ and $x\in\Omega$. Choose $\phi\in\mathcal{D}_\omega(\Omega)$ with $\phi=1$ in a neighborhood of x. Then

$$(\phi u)(x)=\int e^{i\langle x,\xi\rangle}\widehat{\phi u}(\xi)d\xi,$$

since $\phi u\in L^1(R^n)$. Using the inequality for $|\alpha|\leqq k$,

$$|\xi|^{|\alpha|}\leqq(1+|\xi|)^{|\alpha|}\leqq e^{-a|\alpha|}e^{|\alpha|\omega(\xi)}\leqq e^{-a|\alpha|}e^{k\omega(\xi)}$$

we know that $D^\alpha(\phi u)(x)=\int e^{i\langle x,\xi\rangle}\xi^\alpha(\widehat{\phi u})(\xi)d\xi$ has finite value for $|\alpha|\leqq k$. This means $u\in C^k(\Omega)$.

THEOREM 3.2 (Sobolev imbedding theorem). *For* $s>(n/2)+k$, $k\in N$, *we* *have* $H_\omega^s \subset \mathcal{E}_\omega^k(R^n)$. *Moerover,*

$$\sum_{|\alpha|\leqq k} \|D^\alpha u\|_\infty \leqq C_{s,k} \|u\|_s^\omega \quad \textit{for all} \quad u \in H_\omega^s.$$

PROOF. Let $u \in H_\omega^s$ and $\phi \in \mathcal{D}_\omega(R^n)$. From lemma 2.8 we have

$$\int c^{k\omega(\xi)} |\widehat{\phi u}(\xi)| d\xi = \int e^{s\omega(\xi)} |\widehat{\phi u}(\xi)| e^{(k-s)\omega(\xi)} d\xi$$

$$\leqq \left[\int e^{2s\omega(\xi)} |\widehat{\phi u}(\xi)|^2 d\xi\right]^{1/2} \left[\int e^{2(k-s)\omega(\xi)} d\xi\right]^{1/2}$$

$$\leqq C\|\phi u\|_s^\omega \quad \text{for} \quad s > \frac{n}{2} + k$$

Hence $u \in \mathcal{E}_\omega^k \subset C^k$. Furthermore, for $|\alpha| \leqq k$, we have

$$|\xi^\alpha \hat{u}(\xi)| = |\xi^\alpha e^{-s\omega(\xi)} e^{s\omega(\xi)} \hat{u}(\xi)|$$

$$\leqq (1+|\xi|)^{k-s} e^{s\omega(\xi)} |\hat{u}(\xi)|$$

The last term is product of L^2-functions and so $(\widehat{D^\alpha u})(\xi)$ is integrable. By the Riemann-Lebesgue lemma we get $\lim_{|x|\to\infty} D^\alpha u(x) = 0$ and

$$|D^\alpha u(x)| = |(2\pi)^{-n} \int e^{i\langle x, \xi\rangle} \xi^\alpha \hat{u}(\xi) d\xi|$$

$$\leqq \|\xi^\alpha \hat{u}(\xi)\|_{L^1} \leqq C\|u\|_s^\omega \quad \text{for all} \quad x \in R^n,$$

which shows the inequality.

We can easily see that $H_\omega^\infty \subset \mathcal{E}_\omega$ and $\lim_{|x|\to\infty} D^\alpha u(x) = 0$ for every $\alpha \in N^n$ and $u \in H_\omega^\infty$. Moreover we have

PROPOSITION 3.3. $\mathcal{E}'_\omega \subset H_\omega^{-\infty}$.

PROOF. Let $u \in \mathcal{E}'_\omega$. According to the Paley-Wiener type theorem, we can find some constant $\lambda > 0$ and C_λ such that $|\hat{u}(\xi)| \leqq C_\lambda e^{\lambda\omega(\xi)}$, $\xi \in R^n$. Thus we have

$$\int e^{2s\omega(\xi)} |\hat{u}(\xi)|^2 d\xi \leqq C_\lambda^2 \int e^{2(s+\lambda)\omega(\xi)} d\xi$$

$$\leqq C\int (1+|\xi|)^{2(s+\lambda)} d\xi < \infty$$

if $s+\lambda < -(u/2)$. That is, u is in such H_ω^s and $H_\omega^{-\infty}$.

COLLARY 3.4. *If* $\phi \in \mathcal{S}_\omega$, *then the multiplication operator* T_ϕ, *given by* $T_\phi u = \phi u$, *is a bounded linear operator on* H_ω^s *for all* $s \in R$.

DEFINITION 3.3. We denote by $\mathring{H}_\omega^s(\Omega)$ the closure of $\mathcal{D}_\omega(\Omega)$ in the H_ω^s-norm for every $s\in R$.

THEOREM 3.5 (Trace Theorem). *If* $u\in\mathring{H}_{\omega}^{s}(\Omega)$ *for some* $s>n/2$, *then* $u\equiv 0$ *on* $\partial\Omega$.

PROOF. If $s>n/2$ and $u\in\mathring{H}_{\omega}^{s}(\Omega)$, then there is a sequence $\{u_n\}$ of functions in $\mathcal{D}_{\omega}(\Omega)$ such that u_n converges to u in H_{ω}^{s}. According to Sobolev imbedding theorm,

$$\sup_{x\in\partial\Omega}|u(x)|=\sup_{x\in\partial\Omega}|(u-u_n)(x)|\leqq\sup_{\Omega}|u-u_n|\leqq C_s\|u_n-u\|_s^{\omega}$$

for all $n=1, 2, \cdots$. Hence $u\equiv 0$ on $\partial\Omega$.

THEOREM 3.6 (Rellich's compactness theorem). *If* Ω *is bounded open in* R^n *and* $t<s$, *the inclusion map from* $\mathring{H}_{\omega}^{s}(\Omega)$ *into* H_{ω}^{t} *is compact*

PROOF. Let $\{u_k\}$ be a bounded sequence in $\mathring{H}_{\omega}^{s}(\Omega)$ and let χ be a local unit for $\bar{\Omega}$. Then

$$\begin{aligned}e^{s\omega(\xi)}|\hat{u}_k(\xi)|&\leqq\int e^{s\omega(\xi)-s\omega(\eta)}|\hat{\chi}(\xi-\eta)|e^{s\omega(\eta)}|\hat{u}_k(\eta)|d\eta\\&\leqq\left[\int e^{2s(\omega(\xi)-\omega(\eta))}|\hat{\chi}(\xi-\eta)|^2d\eta\right]^{1/2}\left[\int e^{2s\omega(\eta)}|\hat{u}_k(\eta)|^2d\eta\right]^{1/2}\\&\leqq\|u_k\|_s^{\omega}\left[\int e^{2|s|\omega(\xi-\eta)}|\hat{\chi}(\xi-\eta)|^2d\eta\right]^{1/2}\\&\leqq C_1<\infty\end{aligned}$$

for some constant C_1, independent of ξ and k. In a similar way we have $e^{s\omega(\xi)}|D_j\hat{u}_k(\xi)|\leqq C_2$ for some constant C_2, independent of ξ and k. We now claim that $e^{s\omega(\xi)}\hat{u}_k(\xi)$ is an equicontinuous sequence of functions on any compact set. Let K be any given compact subset of R^n and $C_3=\sup\{e^{|s|\omega(\xi)}: \xi\in\overline{B(0, R)}\}$ where $R=\sup_{x\in K}\|x\|$. Then $|\hat{u}_k(\xi)|\leqq C_1e^{-s\omega(\xi)}\leqq C_2C_3$ and $|D_j\hat{u}_k(\xi)|\leqq C_2C_3$ for all ξ in K and all positive integer k. From the uniform continuity of $e^{s\omega(\xi)}$ on K, we can find, for each $\varepsilon>0$, a constant δ_1 such that $|\xi-\eta|<\delta_1$ implies $|e^{s\omega(\xi)}-e^{s\omega(\eta)}|<\varepsilon/2C_1C_3$. On the other hand, we have, from the mean value theorem,

$$\begin{aligned}|\hat{u}_k(\xi)-\hat{u}_k(\eta)|&\leqq\sum_{j=1}^{n}|\xi_j-\eta_j||D_j\hat{u}_k(\xi+\theta(\eta-\xi))|\\&\leqq C_2'\|\xi-\eta\|\end{aligned}$$

for some constant C_2' and for all $k=1, 2, \cdots$ and $\xi, \eta\in K$. Hence, for each $\varepsilon>0$ there is a constant $\delta_2>0$ such that $|\xi-\eta|\leqq\delta_2$, $\xi, \eta\in K$ imply the inequality

$|\hat{u}_k(\xi)-\hat{u}_k(\eta)| < \varepsilon/2C_2'C_3$. Choosing $\delta=\min\{\delta_1, \delta_2\}$, we have $|e^{s\omega(\xi)}\hat{u}_k(\xi)-e^{s\omega(\eta)}\hat{u}_k(\eta)| \leqq e^{s\omega(\xi)}|\hat{u}_k(\xi)-\hat{u}_k(\eta)|+|e^{s\omega(\xi)}-e^{s\omega(\eta)}||\hat{u}_k(\eta)|<\varepsilon$, provided that $|\xi-\eta|<\delta$ and $\xi, \eta\in K$, which shows our claim. Since $\{e^{s\omega(\xi)}\hat{u}_k(\xi)\}$ is a pointwise bounded sequence on K, the Arzela-Ascoli theorem gives a subsequence $\{e^{s\omega(\xi)}\hat{u}_{k_j}(\xi)\}$ converging uniformly on each compact sets in R^n. Now, for each real number $R>0$, we have

$$
\begin{aligned}
(\|u_{k_i}-u_{k_j}\|_t^{(\omega)})^2 &= \int e^{2t\omega(\xi)}|(\hat{u}_{k_i}-\hat{u}_{k_j})(\xi)|^2 d\xi \\
&= \int e^{2(t-s)\omega(\xi)}e^{2s\omega(\xi)}|(\hat{u}_{k_i}-\hat{u}_{k_j})(\xi)|^2 d\xi \\
&\leqq \int_{|\xi|\leq R} e^{2s\omega(\xi)}|(\hat{u}_{k_i}-\hat{u}_{k_j})(\xi)|^2 d\xi \\
&\quad + e^{2(t-s)a}(1+R)^{2(t-s)}\int_{|\xi|\geq R} e^{2s\omega(\xi)}|(\hat{u}_{k_i}-\hat{u}_{k_j})(\xi)|^2 d\xi \\
&\leqq \int_{|\xi|\leq R} e^{2s\omega(\xi)}|(\hat{u}_{k_i}-\hat{u}_{k_j})(\xi)|^2 d\xi + Ce^{2(s-s)a}(1+R)^{2(t-s)}
\end{aligned}
$$

for some constant C. The last inequality follows from the boundedness of $\{u_k\}$ in $\mathring{H}_\omega^s(\Omega)$. Now, given $\varepsilon>0$, we can take R so large that $Ce^{2(t-s)a}(1+R)^{2(t-s)}<\varepsilon/2$. The uniform convergence on $B(0, R)$ of the subsequence $e^{s\omega(\xi)}\hat{u}_{k_j}(\xi)$ then gives a constant M such that $k_i, k_j \geqq M$ imply

$$\int_{|\xi|\leq R} e^{2s\omega(\xi)}|\hat{u}_{k_i}(\xi)-\hat{u}_{k_j}(\xi)|^2<\varepsilon/2.$$

Therefore, $\{u_{k_j}\}$ is a convergent subsequence of u_k in H_ω^t.

We remark that this theorem is not true when $t=s$.

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References

[1] Beals, R. M.: Lecture note on partial differential equations, Rutgers University (1986).

[2] Beuring, A., Quasi-analyticity and general distributions, Lectures 4 and 5. AMS Summer institute, Stanford 1961.

[3] Björck, G.: Linear partial differential operators and generalized distributions, Ark. Mat. 6 (1965), p. 351-407.

[4] Petersen, B. E., Introduction to the Fourier Transform and Pseudodifferential Operators, Boston, London, Melbourne: Pitman 1983.

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