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ON THE COMPLETE RELATIVE HOMOLOGY AND COHOMOLOGY OF FROBENIUS EXTENSIONS

By

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Introduction.

Let G be a finite group, K a subgroup of G and M a left G-module. Then for $r\in\boldsymbol{Z}$ the complete relative homology group $H_r(G, K, M)$ and cohomology group $H^r(G, K, M)$ are defined in [6]. Let 1 be the unit element of G. For the case of $K=\{1\}$ $H_r(G, K, M)\cong H^{-r-1}(G, K, M)$ holds. But it is not true that for any G, K, M and r there exists an isomorphism from $H_r(G, K, M)$ into $H^{-r-1}(G, K, M)$. In fact, in [6, p. 262] there are G, K and M such that $H_r(G, K, M)\cong\boldsymbol{Z}/2\boldsymbol{Z}$ and $H^r(G, K, M)=0$ for all $r\in\boldsymbol{Z}$. And if we set $M=\boldsymbol{Q}/\boldsymbol{Z}$ in [6, p. 262], $H_r(G, K, M)=0$ and $H^r(G, K, M)\cong\boldsymbol{Z}/2\boldsymbol{Z}$ hold for all $r\in\boldsymbol{Z}$.

Let Λ be an algebra over a commutative ring K and Γ a subalgebra such that the ring extension Λ/Γ is a Frobenius extension. In section 1 we shall introduce the complete relative cohomology group $H^r(\Lambda, \Gamma, -)$ and homology group $H_r(\Lambda, \Gamma, -)$ for $r\in\boldsymbol{Z}$. When the ring extension Γ/K is also a Frobenius extension, We can define a K-homomorphism $\Psi^r_{\Lambda/\Gamma}: H_r(\Lambda, \Gamma, (-)^{\Delta})\rightarrow H^{-r-1}(\Lambda, \Gamma, -)$ for $r\in\boldsymbol{Z}$, where Δ is the Nakayama automorphism. The main purpose of this paper is to show necessary and sufficient conditions on which $\Psi^r_{\Lambda/\Gamma}$ is an isomorphism. Theorems 6.3, 7.1 and 7.2 provide the necessary and sufficient conditions. In section 8 we apply our results to extensions defined by a finite group G and a subgroup K. In generalization of the well-known duality for Tate cohomology we show that $H_r(G, K, -)\cong H^{-r-1}(G, K, -)$ if and only if K is a Hall subgroup of G.

1. Complete relative homology.

Throughout this paper, let Λ be an algebra over a commutative ring K and Γ a subalgebra such that the ring extension Λ/Γ is a (projective) Frobenius extension in the sense of [9]. Since Λ/Γ is a Frobenius extension, there exist elements $R_1, \cdots, R_n, L_1, \cdots, L_n$ in Λ and a $\Gamma-\Gamma$-homomorphism $H\in\mathrm{Hom}({}_{\Gamma}\Lambda_{\Gamma},$

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${}_\Gamma\Gamma_\Gamma$) such that $x=\sum_{i=1}^{n} H(xR_i)L_i=\sum_{i=1}^{n} R_iH(L_ix)$ for all $x\in\Lambda$. The pair (R_i, L_i) $(1\leqq i\leqq n)$ and H are called the dual projective pair as in [1] and Frobenius homomorphism of Λ/Γ, respectively. Let Λ^o and Γ^o be the opposite rings of Λ and Γ, respectively. Put $P=\Lambda\otimes_K\Lambda^o$, and let S be a subring of P which is the image of the natural homomorphism $\Gamma\otimes_K\Gamma^o\to P$. Note that the ring extension P/S is a Frobenius extension with the dual projective pair $(R_i\otimes_K L_j,\ L_i\otimes_K R_j))$ $1\leqq i,\ j\leqq n)$ and Frobenius homomorphism $H\otimes_K H$.

Regard Λ as a left P-module with the usual way. Then a complete (P, S)-resolution of Λ

$$(1)\qquad \cdots\to X_r\xrightarrow{d_r}X_{r-1}\to\cdots\to X_1\xrightarrow{d_1}X_0\xrightarrow{d_0}X_{-1}\xrightarrow{d_{-1}}\cdots\to X_{-r}\xrightarrow{d_{-r}}X_{-(r+1)}\to\cdots$$

$$X_0\overset{\varepsilon}{\searrow}\Lambda\overset{\eta}{\nearrow}X_{-1}$$

is a (P, S)-exact sequence with a P-epimorphism ε and a P-monomorphism η such that X_r is (P, S)-projective for all $r\in\boldsymbol{Z}$ and $d_0=\eta\circ\varepsilon$ holds. Note that X_r is also (P, S)-injective since for (projective) Frobenius extensions of rings, the relative projectivity and injectivity are equivalent by [9, Theorem 7]. Let M be a left P-module. Then we have the following sequence from (1):

$$\cdots\longleftarrow \mathrm{Hom}\,({}_PX_1,\ {}_PM)\xleftarrow{d_1{}^*}\mathrm{Hom}\,({}_PX_0,\ {}_PM)\xleftarrow{d_0{}^*}\mathrm{Hom}\,({}_PX_{-1},\ {}_PM)\xleftarrow{d_{-1}{}^*}\cdots,$$

where we set $d_r{}^*(f)=f\circ d_r$ for $f\in\mathrm{Hom}\,({}_PX_{r-1},\ {}_PM)$. In [8] the r-th complete cohomology group $H^r(\Lambda, \Gamma, M)$ is given by $H^r(\Lambda, \Gamma, M)=\mathrm{Ker}\ d_{r+1}{}^*/\mathrm{Im}\ d_r{}^*$. We regard $H^r(\Lambda, \Gamma, M)$ as a K-module with the usual way. Since left P-modules can be regarded as right P-modules, (1) gives the following sequence:

$$\cdots\longrightarrow X_1\otimes_PM\xrightarrow{d_1\otimes_P1_M}X_0\otimes_PM\xrightarrow{d_0\otimes_P1_M}X_{-1}\otimes_PM\xrightarrow{d_{-1}\otimes_P1_M}\cdots.$$

We define the r-th complete relative homology group $H_r(\Lambda, \Gamma, M)$ as $\mathrm{Ker}\,(d_r\otimes_P1_M)/\mathrm{Im}\,(d_{r+1}\otimes_P1_M)$. Since $X_r\otimes_PM$ is a K-module, we can regard $H_r(\Lambda, \Gamma, M)$ as a K-module.

We now give a complete (P, S)-resolution of Λ, i.e., (2) in [8]:

$$(2)\qquad \cdots\to X_r\xrightarrow{d_r}X_{r-1}\to\cdots\to X_1\xrightarrow{d_1}X_0\xrightarrow{d_0}X_{-1}\xrightarrow{d_{-1}}\cdots\to X_{-r}\xrightarrow{d_{-r}}X_{-(r+1)}\to\cdots,$$

$$X_0\overset{\varepsilon}{\searrow}\Lambda\overset{\eta}{\nearrow}X_{-1}$$

where $X_r=\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$ ($r+2$ copies) for $r\geqq0$, $X_{-r}=X_{r-1}$ for $r\geqq1$, $d_r(x_0\otimes_\Gamma\cdots$

$\otimes_\Gamma x_{r+1})=\sum_{t=0}^{r}(-1)^t x_0\otimes_\Gamma\cdots\otimes_\Gamma x_t x_{t+1}\otimes_\Gamma\cdots\otimes_\Gamma x_{r+1}$ for $r\geqq 1$, $\varepsilon(x_0\otimes_\Gamma x_1)=x_0x_1$, $\eta(x)=\sum_i R_i\otimes_\Gamma L_i x$, $d_0=\eta\circ\varepsilon$, $d_{-r}(x_0\otimes_\Gamma\cdots\otimes_\Gamma x_r)=\sum_{t=0}^{r}\sum_i(-1)^t x_0\otimes_\Gamma\cdots\otimes_\Gamma x_{t-1}\otimes_\Gamma R_i\otimes_\Gamma L_i x_t\otimes_\Gamma\cdots\otimes_\Gamma x_r$ for $r\geqq 1$.

For the relative complete resolution X of (2) and a left P-module M, we denote the r-cycle module of $X\otimes_P M$ by $C_r^{\Lambda/\Gamma}(M)$, and the r-boundary module of $X\otimes_P M$ by $B_r^{\Lambda/\Gamma}(M)$. Then we have $H_r(\Lambda, \Gamma, M)=C_r^{\Lambda/\Gamma}(M)/B_r^{\Lambda/\Gamma}(M)$. According to the definition of d_r, we have

PROPOSITION 1.1. *Let M be a left P-module. Then $C_0^{\Lambda/\Gamma}(M)$, $C_{-1}^{\Lambda/\Gamma}(M)$, $B_0^{\Lambda/\Gamma}(M)$ and $B_{-1}^{\Lambda/\Gamma}(M)$ are K-submodules of $(\Lambda\otimes_\Gamma\Lambda)\otimes_P M$ such that $C_0^{\Lambda/\Gamma}(M)=\{(1\otimes_\Gamma 1)\otimes_P m\mid\sum_i(R_i\otimes_\Gamma L_i)\otimes_P m=0$ in $(\Lambda\otimes_\Gamma\Lambda)\otimes_P M\}$, $C_{-1}^{\Lambda/\Gamma}(M)=\{(1\otimes_\Gamma 1)\otimes_P m\mid\sum_i(R_i\otimes_\Gamma L_i\otimes_\Gamma 1)\otimes_P m-\sum_i(1\otimes_\Gamma R_i\otimes_\Gamma L_i)\otimes_P m=0$ in $(\Lambda\otimes_\Gamma\Lambda\otimes_\Gamma\Lambda)\otimes_P M\}$, $B_0^{\Lambda/\Gamma}(M)=\{(1\otimes_\Gamma 1)\otimes_P\sum_t(m_t x_t-x_t m_t)$ (finite sum) $\mid x_t\in\Lambda$ and $m_t\in M\}$ and $B_{-1}^{\Lambda/\Gamma}(M)=\{\sum_i(R_i\otimes_\Gamma L_i)\otimes_P m\mid m\in M\}$.*

In the sequel let the ring extension Γ/K be also a Frobenius extension with the dual projective pair (r_j, l_j) and Frobenius homomorphism h. Then since ring extensions Λ/Γ and Γ/K are Frobenius extensions, the ring extension Λ/K is also a Frobenius extension with the dual projective pair $(R_i r_j, l_j L_i)$ and Frobenius homomorphism $h\circ H$. Since the natural homomorphisms $K\otimes_K K^o\to\Gamma\otimes_K\Gamma^o$ and $\Gamma\otimes_K\Gamma^o\to P$ are monomorphisms and the image of the natural homomorphism $\Gamma\otimes_K\Gamma^o\to P$ is the subring S of P, therefore from complete (S, K)-resolutions of Γ we have the complete relative cohomology group $H^r(\Gamma, K, M)$ and homology group $H_r(\Gamma, K, M)$ for any $r\in\boldsymbol{Z}$ and any left S-module M. Moreover we have the complete (S, K)-resolution of Γ of type (2), and have Proposition 1.1 for it. Similarly we have the complete relative cohomology group $H^r(\Lambda, K, M)$ and homology group $H_r(\Lambda, K, M)$ from complete (P, K)-resolutions of Λ for any $r\in\boldsymbol{Z}$ and any left P-module M, and have the complete (P, K)-resolution of Λ of type (2).

Since Λ is a Frobenius K-algebra, as in [10] we have the Nakayama automorphism $\Delta:\Lambda\to\Lambda$ such that $\Delta(x)=\sum_{i,j}h\circ H(R_i r_j x)l_j L_i$ for all $x\in\Lambda$. We denote Δ^{-1} by ∇. Then $\nabla(x)=\sum_{i,j}R_i r_j h\circ H(x l_j L_i)$ holds for all $x\in\Lambda$. Throughout this paper Δ is the Nakayama automorphism of the Frobenius K-algebra Λ and ∇ is Δ^{-1}.

2. The homomorphism $\Psi_{\Lambda/\Gamma}$.

Let L and M be left P-modules. Then since P/K is a Frobenius extension with the dual projective pair $(R_i r_j\otimes_K l_{j'}L_{i'}, l_j L_i\otimes_K R_{i'}r_{j'})$, we have the trace map from $\mathrm{Hom}({}_K L, {}_K M)$ into $\mathrm{Hom}({}_P L, {}_P M)$ which is defined by trace $f(x)=\sum_{i,j,i',j'}(R_i r_j \otimes_K l_{j'} L_{i'}) f((l_j L_i \otimes_K R_{i'} r_{j'})x)$ for $x \in L$, where we denote the image of $f \in \mathrm{Hom}({}_K L, {}_K M)$ by trace f as in [9, section 3]. Let M^0 be the module M with a new scalar multiplication $*$ as a left P-module such that $(x \otimes_K y)*m=(\nabla(x)\otimes_K \Delta(y))m$ for $x \otimes_K y \in P$ and $m \in M$. Then in [7, section 4] and [9, section 4], the mapping $\varphi: \mathrm{Hom}({}_K L, {}_K K)\otimes_P M^0 \to \mathrm{Hom}({}_P L, {}_P M)$ is defined by $\varphi(f \otimes_P m)=[x \to \text{trace}\, f(x)m]$, where $f \in \mathrm{Hom}({}_K L, {}_K K)$ is regarded as an element of $\mathrm{Hom}({}_K L, {}_K P)$.

The left P-module M is regarded as a two-sided (Λ, K)-module. Modifying the structure of the right Λ-module as $m \cdot x = m\Delta(x)$ where $m \in M$ and $x \in \Lambda$, we obtain a left P-module M^Δ from M. We shall denote $m \in M^\Delta$ by m^Δ. For the left P-module L, when we regard $\mathrm{Hom}({}_\Lambda L, {}_\Lambda \Lambda)$ as a left P-module with the usual way, there is a K-isomorphism $\kappa: \mathrm{Hom}({}_\Lambda L, {}_\Lambda \Lambda)\otimes_P M^\Delta \xrightarrow{\sim} \mathrm{Hom}({}_K L, {}_K K)\otimes_P M^0$ given by $\kappa(f \otimes_P m^\Delta)=[x \to h \circ H(f(x))]\otimes_P m$ and $\kappa^{-1}(g \otimes_P m)=[x \to \sum_{i,j} R_i r_j g(l_j L_i x)]\otimes_P m^\Delta$. Then putting $\phi=\varphi\circ\kappa$, we have a K-homomorphism

(3) $$\phi: \mathrm{Hom}({}_\Lambda L, {}_\Lambda \Lambda)\otimes_P M^\Delta \longrightarrow \mathrm{Hom}({}_P L, {}_P M)$$

such that $\phi(f \otimes_P m^\Delta)(x)=\sum_{i,j} f(xR_i r_j) m l_j L_i$. When $\mathrm{Hom}({}_\Lambda L, {}_\Lambda \Lambda)\otimes_P M^\Delta$ and $\mathrm{Hom}({}_P L, {}_P M)$ are regarded as functors in P-modules L and M, it is shown by the conventional argument that ϕ is natural in each of L and M. When L is (P, S)-projective, that is, there is an S-module T such that ${}_P L < \oplus_P P \otimes_S T$, $\mathrm{Hom}({}_\Lambda L, {}_\Lambda \Lambda)$ is also (P, S)-projective since we have ${}_P\mathrm{Hom}({}_\Lambda L, {}_\Lambda \Lambda) < \oplus_P \mathrm{Hom}({}_\Lambda P \otimes_S T, {}_\Lambda \Lambda) \cong {}_P\mathrm{Hom}({}_\Lambda \Lambda \otimes_\Gamma T \otimes_\Gamma \Lambda, {}_\Lambda \Lambda) \cong {}_P\mathrm{Hom}({}_\Gamma \Lambda, {}_\Gamma\mathrm{Hom}({}_\Gamma T, {}_\Gamma \Lambda)) \cong {}_P\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma), {}_\Gamma\mathrm{Hom}({}_\Gamma T, {}_\Gamma \Gamma \otimes_\Gamma \Lambda)) \cong {}_P \Lambda \otimes_\Gamma \mathrm{Hom}({}_\Gamma T, {}_\Gamma \Gamma)\otimes_\Gamma \Lambda \cong {}_P P \otimes_S \mathrm{Hom}({}_\Gamma T, {}_\Gamma \Gamma)$. Therefore for any complete (P, S)-resolution X of Λ, since the complex $\mathrm{Hom}({}_\Lambda X, {}_\Lambda \Lambda)$ is a (P, S)-exact sequence by [8, Proposition 1.1], it is also a complete (P, S)-resolution of Λ such that the r-th component is $\mathrm{Hom}({}_\Lambda X_{-r-1}, {}_\Lambda \Lambda)$. Hence ϕ induces K-homomorphisms

$$\Psi^r_{\Lambda/\Gamma}: H_r(\Lambda, \Gamma, M^\Delta) \longrightarrow H^{-r-1}(\Lambda, \Gamma, M)$$

for $r \in \boldsymbol{Z}$.

Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a (P, S)-exact sequence. Then the sequence $0 \to L^\Delta \xrightarrow{f} M^\Delta \xrightarrow{g} N^\Delta \to 0$ is also (P, S)-exact. Then the connecting homomorphisms $\partial^r: H^r(\Lambda, \Gamma, N) \to H^{r+1}(\Lambda, \Gamma, L)$ and $\partial^\Delta_r: H_r(\Lambda, \Gamma, N^\Delta) \to H_{r-1}(\Lambda, \Gamma, L^\Delta)$ are induced for all $r \in \boldsymbol{Z}$ with the usual way. Since ϕ of (3) is natural in each of L and M, we obtain

PROPOSITION 2.1. *Let $0 \to L \to M \to N \to 0$ be a (P, S)-exact sequence. Then* *for the connecting homomorphisms* $\partial^r : H^r(\Lambda, \Gamma, N) \to H^{r+1}(\Lambda, \Gamma, L)$ *and* $\partial_r^\Delta : H_r(\Lambda, \Gamma, N^\Delta) \to H_{r-1}(\Lambda, \Gamma, L^\Delta)$, $\Psi_{\Lambda/\Gamma}^{r-1} \circ \partial_r^\Delta = \partial^{-r-1} \circ \Psi_{\Lambda/\Gamma}^r$ *holds for* $r \in \boldsymbol{Z}$.

For the module X_r of (2), as in [8], we have a P-isomorphism $\varphi_r : X_r \xrightarrow{\sim} \mathrm{Hom}({}_\Lambda X_{-r-1}, {}_\Lambda\Lambda)$ for $r \in \boldsymbol{Z}$ such that $\varphi_r(x_0 \otimes_\Gamma \cdots \otimes_\Gamma x_{\tau(r)}) = [\lambda_0 \otimes_\Gamma \cdots \otimes_\Gamma \lambda_{\tau(r)} \to \lambda_0 H(\cdots H(\lambda_{\tau(r)-1} H(\lambda_{\tau(r)} x_0) x_1) \cdots) x_{\tau(r)}]$, $\varphi_r^{-1}(f) = \sum_{i_0, \dots, i_{\tau(r)-1}} R_{i_0} \otimes_\Gamma \cdots \otimes_\Gamma R_{i_{\tau(r)-1}} \otimes_\Gamma f(1 \otimes_\Gamma L_{i_{\tau(r)-1}} \otimes_\Gamma \cdots \otimes_\Gamma L_{i_0})$, where we put

$$\tau(r) = \begin{cases} r+1 & \text{for } r \geqq 0, \\ -r & \text{for } r < 0. \end{cases}$$

Then for the P-homomorphism d_r of (2) we have $\varphi_{r-1} \circ d_r = (-1)^r d_{-r}^* \circ \varphi_r$ where $d_{-r}^* : \mathrm{Hom}({}_\Lambda X_{-r-1}, {}_\Lambda\Lambda) \to \mathrm{Hom}({}_\Lambda X_{-r}, {}_\Lambda\Lambda)$. Therefore when we put the plus and minus sign $\sigma(\ )$ such that

$$\sigma(r) = \begin{cases} + & \text{for the case of } r \equiv 0 \text{ or } 3 \text{ (modulo 4)}, \\ - & \text{for the case of } r \equiv 1 \text{ or } 2 \text{ (modulo 4)}, \end{cases}$$

$\{\sigma(r)\varphi_r\}_{r \in Z}$ is a chain map from X into $\mathrm{Hom}({}_\Lambda X, {}_\Lambda\Lambda)$. Hence composing ψ of (3) with $\sigma(r)\varphi_r \otimes_P 1 M^\Delta$, we can consider that $\Psi_{\Lambda/\Gamma}^r$ is induced by the K-homomorphism

(4) $$\psi_{\Lambda/\Gamma}^r : X_r \otimes_P M^\Delta \longrightarrow \mathrm{Hom}({}_P X_{-r-1}, {}_P M)$$

such that $\psi_{\Lambda/\Gamma}^r((x_0 \otimes_\Gamma \cdots \otimes_\Gamma x_{\tau(r)}) \otimes_P m^\Delta)(\lambda_0 \otimes_\Gamma \cdots \otimes_\Gamma \lambda_{\tau(r)}) = \sum_{i,j} \sigma(r) \lambda_0 H(\cdots H(\lambda_{\tau(r)-1} \cdot H(\lambda_{\tau(r)} R_i r_j x_0) x_1) \cdots) x_{\tau(r)} m l_j L_i$ for $r \in \boldsymbol{Z}$, $x_i, \lambda_i \in \Lambda$ and $m \in M$.

3. Homomorphisms of complete relative homology.

Let Y be a complete (P, K)-resolution of Λ and M a left P-module. Then we have the K-homomorphisms of change of rings, that is,

(5) $$H_r(Y \otimes_S M) \longrightarrow H_r(Y \otimes_P M) = H_r(\Lambda, K, M)$$

for $r \in \boldsymbol{Z}$. Since P is S-projective, Y is also a complete (S, K)-resolution of Λ. So the natural inclusion $\Gamma \to \Lambda$ induces K-homomorphisms

(6) $$H_r(\Gamma, K, M) \longrightarrow H_r(Y \otimes_S M)$$

for $r \in \boldsymbol{Z}$. Then composing (5) with (6), we have K-homomorphisms

$$\mathrm{Cor}_r : H_r(\Gamma, K, M) \longrightarrow H_r(\Lambda, K, M)$$

for $r \in \boldsymbol{Z}$. Since Y is also a complete (S, K)-resolution of Λ, the Frobenius homomorphism of Λ/Γ $H : \Lambda \to \Gamma$ induces K-homomorphisms

$$H_r(Y\otimes_S M)\longrightarrow H_r(\Gamma,\ K,\ M) \tag{7}$$

for $r\in \boldsymbol{Z}$. Since the dual projective pair of the Frobenius extension P/S is $(R_i\otimes_K L_j,\ L_i\otimes_K R_j)$, we can define a chain map $Y\otimes_P M\to Y\otimes_S M$ such as $y\otimes_P m \to\sum_{i,j} y\cdot(R_i\otimes_K L_j)\otimes_S(L_i\otimes_K R_j)\cdot m=\sum_{i,j} L_j y R_i\otimes_S L_i m R_j$ for $y\in Y_r$ and $m\in M$, and this chain map induces the K-homomorphisms

$$H_r(\Lambda,\ K,\ M)\longrightarrow H_r(Y\otimes_S M) \tag{8}$$

for $r\in \boldsymbol{Z}$. Composing (7) with (8), for $r\in \boldsymbol{Z}$ we have K-homomorphisms

$$\mathrm{Res}_r : H_r(\Lambda,\ K,\ M)\longrightarrow H_r(\Gamma,\ K,\ M)\,.$$

Let X (resp. Y) be a complete $(P,\ S)$- (resp. $(P,\ K)$-) resolution of Λ with the differentiation $d=\{d_r\}$ (resp. $c=\{c_r\}$) and the P-epimorphism $\varepsilon : X_0\to\Lambda$ (resp. $\delta : Y_0\to\Lambda$). Then the identity homomorphism of Λ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & Y_1 & \xrightarrow{c_1} & Y_0 & \xrightarrow{c_0} & Y_{-1} & \xrightarrow{c_{-1}} & Y_{-2} & \longrightarrow\cdots \\
 & \sigma_1\big\downarrow & & \sigma_0\big\downarrow\ \ \delta\searrow & \Lambda & \nearrow\ \ \big\uparrow\sigma_{-1} & & \big\uparrow\sigma_{-2} & \\
 & & & \varepsilon\nearrow & & \searrow & & & \\
\cdots\longrightarrow & X_1 & \xrightarrow[d_1]{} & X_0 & \xrightarrow[d_0]{} & X_{-1} & \xrightarrow[d_{-1}]{} & X_{-2} & \longrightarrow\cdots
\end{array} \tag{9}$$

as (4) in [8]. Let M be a left P-module. Then from the positive part of the diagram (9), we have K-homomorphisms

$$\mathrm{Def}_r : H_r(\Lambda,\ K,\ M)\longrightarrow H_r(\Lambda,\ \Gamma,\ M)$$

for $r\geqq 1$. Since $\delta\otimes_P 1_M$ is an epimorphism, for any element $\alpha\in X_0\otimes_P M$ there is an element $\beta\in Y_0\otimes_P M$ such that $(\varepsilon\otimes_P 1_M)(\alpha)=(\delta\otimes_P 1_M)(\beta)$ holds. Therefore we can define a K-homomorphism $\tau : X_0\otimes_P M/\mathrm{Im}\,(d_1\otimes_P 1_M)\to Y_0\otimes_P M/\mathrm{Im}\,(c_1\otimes_P 1_M)$ such that $\tau(\bar{\alpha})=\bar{\beta}$, where $\bar{\ }$ stands for the residue classes. Then we have the following commutative diagram from the diagram (9):

$$\begin{array}{ccccccc}
0\longrightarrow Y_0\otimes_P M/\mathrm{Im}\,(c_1\otimes_P 1_M) & \xrightarrow{\overline{c_0\otimes_P 1_M}} & Y_{-1}\otimes_P M & \xrightarrow{c_{-1}\otimes_P 1_M} & Y_{-2}\otimes_P M & \xrightarrow{c_{-2}\otimes_P 1_M} & \cdots \\
\big\uparrow\tau & & \big\uparrow\sigma_{-1}\otimes_P 1_M & & \big\uparrow\sigma_{-2}\otimes_P 1_M & & \\
0\longrightarrow X_0\otimes_P M/\mathrm{Im}\,(d_1\otimes_P 1_M) & \xrightarrow[\overline{d_0\otimes_P 1_M}]{} & X_{-1}\otimes_P M & \xrightarrow[d_{-1}\otimes_P 1_M]{} & X_{-2}\otimes_P M & \xrightarrow[d_{-2}\otimes_P 1_M]{} & \cdots
\end{array} \tag{10}$$

where $\overline{c_0\otimes_P 1_M}$ and $\overline{d_0\otimes_P 1_M}$ are homomorphisms induced by $c_0\otimes_P 1_M$ and $d_0\otimes_P 1_M$ respectively with the usual way. Taking the homology of this diagram, for $r\leqq 0$ we have K-homomorphisms

$$\mathrm{Inf}_r : H_r(\Lambda,\ \Gamma,\ M)\longrightarrow H_r(\Lambda,\ K,\ M)\,.$$

PROPOSITION 3.1. *Let $0\to L\to M\to N\to 0$ be a (P, K)-exact sequence. Then for the connecting homomorphisms $\partial_r^\Lambda: H_r(\Lambda, K, N)\to H_{r-1}(\Lambda, K, L)$ and $\partial_r^\Gamma: H_r(\Gamma, K, N)\to H_{r-1}(\Gamma, K, L)$, we have*

(i) $\partial_r^\Lambda\circ\mathrm{Cor}_r=\mathrm{Cor}_{r-1}\circ\partial_r^\Gamma$,

(ii) $\partial_r^\Gamma\circ\mathrm{Res}_r=\mathrm{Res}_{r-1}\circ\partial_r^\Lambda$

for all $r\in\boldsymbol{Z}$. Let $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$ be a (P, S)-exact sequence. Then for the connecting homomorphisms $\partial_r: H_r(\Lambda, \Gamma, N)\to H_{r-1}(\Lambda, \Gamma, L)$ and $\partial_r^\Lambda: H_r(\Lambda, K, N)\to H_{r-1}(\Lambda, K, L)$, we have

(iii) $\partial_r\circ\mathrm{Def}_r=\mathrm{Def}_{r-1}\circ\partial_r^\Lambda$ *for* $r\geqq 2$,

(iv) $\partial_r^\Lambda\circ\mathrm{Inf}_r=\mathrm{Inf}_{r-1}\circ\partial_r$ *for* $r\leqq 0$,

(v) $\mathrm{Inf}_0\circ\partial_1\circ\mathrm{Def}_1=\partial_1^\Lambda$.

PROOF. Since K-homomorphisms (5), (6), (7) and (8) commute with connecting homomorphisms, the composite homomorphisms Cor and Res also commute with them. Hence (i) and (ii) hold. (iii) follows from the definition. When $Y_0\otimes_P M/\mathrm{Im}\,(c_1\otimes_P 1_M)$ and $X_0\otimes_P M/\mathrm{Im}\,(d_1\otimes_P 1_M)$ in the diagram (10) are regarded as functors covariant in M, they are right exact functors, and τ in (10) is a natural transformation. Therefore by the conventional argument (iv) holds. Put $\partial_1\circ\mathrm{Def}_1(\bar{\alpha})=\bar{\beta}$ and $\partial_1^\Lambda(\bar{\alpha})=\bar{\gamma}$ for (v), where $\bar{}$ stands for the residue classes. Then we can choose β and γ such that $\beta=(\sigma_0\otimes_P 1_L)(\gamma)$ for σ_0 in (9). Hence $\mathrm{Inf}_0\circ\partial_1\circ\mathrm{Def}_1(\bar{\alpha})=\bar{\gamma}=\partial_1^\Lambda(\bar{\alpha})$ holds.

It is easy to see that Cor, Res, Inf and Def are independent of the choice of relative complete resolutions. Therefore they are computable from the relative complete resolutions of type (2). Then we have the following proposition.

PROPOSITION 3.2. *Let M be a left P-module, and take the relative complete resolutions of type* (2). *Then*

$$\mathrm{Inf}_0: C_0^{\Lambda/\Gamma}(M)/B_0^{\Lambda/\Gamma}(M)\longrightarrow C_0^{\Lambda/K}(M)/B_0^{\Lambda/K}(M),$$

$$\mathrm{Inf}_{-1}: C_{-1}^{\Lambda/\Gamma}(M)/B_{-1}^{\Lambda/\Gamma}(M)\longrightarrow C_{-1}^{\Lambda/K}(M)/B_{-1}^{\Lambda/K}(M),$$

$$\mathrm{Cor}_{-1}: C_{-1}^{\Gamma/K}(M)/B_{-1}^{\Gamma/K}(M)\longrightarrow C_{-1}^{\Lambda/K}(M)/B_{-1}^{\Lambda/K}(M),$$

$$\mathrm{Res}_0: C_0^{\Lambda/K}(M)/B_0^{\Lambda/K}(M)\longrightarrow C_0^{\Gamma/K}(M)/B_0^{\Gamma/K}(M)$$

satisfy $\mathrm{Inf}_0(\overline{(1\otimes_\Gamma 1)\otimes_P m})=\overline{(1\otimes_K 1)\otimes_P m}$, $\mathrm{Inf}_{-1}(\overline{(1\otimes_\Gamma 1)\otimes_P m})=\overline{\sum_j (r_j\otimes_K l_j)\otimes_P m}$, $\mathrm{Cor}_{-1}(\overline{(1\otimes_K 1)\otimes_S m})=\overline{\sum_i (\nabla(L_i)\otimes_K R_i)\otimes_P m}$ *and* $\mathrm{Res}_0(\overline{(1\otimes_K 1)\otimes_P m})=\overline{\sum_i (1\otimes_K 1)\otimes_S L_i m R_i}$, *where* $\bar{}$ *stands for the residue classes and ∇ is Δ^{-1} as in section* 1.

PROOF. Since we took the relative complete resolutions of type (2), for ε

and δ in (9), $\varepsilon(1\otimes_{\Gamma}1)=1$ and $\delta(1\otimes_K 1)=1$ hold. So we have $\mathrm{Inf}_0(\overline{(1\otimes_{\Gamma}1)\otimes_P m})=\overline{(1\otimes_K 1)\otimes_P m}$. We can take σ_{-r} in (9) for $r\geqq 1$ such that $\sigma_{-r}(x_0\otimes_{\Gamma}\cdots\otimes_{\Gamma}x_r)=\sum_{j_0,\dots,j_{r-1}}x_0 r_{j_0}\otimes_K l_{j_0}x_1 r_{j_1}\otimes_K\cdots\otimes_K l_{j_{r-1}}x_r$. So we have $\mathrm{Inf}_{-1}(\overline{(1\otimes_{\Gamma}1)\otimes_P m})=\overline{\sum_j(r_j\otimes_K l_j)\otimes_P m}$. Let Y (resp. Z) be the complete (P, K)- (resp. (S, K)-) resolution of Λ (resp. Γ) of type (2). Then for the K-homomorphism (6), we need the chain map $F: Z\rightarrow Y$ over the natural inclusion $\Gamma\rightarrow\Lambda$. Put $F=\{F_r\}_{r\in Z}$ where F_r is the right S-homomorphism of Z_r to Y_r. Then we can take F_r such that $F_{-r}(z_0\otimes_K\cdots\otimes_K z_r)=\sum_{i_0,\dots,i_{r-1}}z_0\nabla(L_{i_0})\otimes_K R_{i_0}z_1\nabla(L_{i_1})\otimes_K\cdots\otimes_K R_{i_{r-1}}z_r$ for $r\geqq 1$. Therefore $\mathrm{Cor}_{-1}(\overline{(1\otimes_K 1)\otimes_P m})=\overline{\sum_i(\nabla(L_i)\otimes_K R_i)\otimes_P m}$ holds. For the K-homomorphism (7), we need the chain map $G: Y\rightarrow Z$ over the Frobenius homomorphism $H: \Lambda\rightarrow\Gamma$. Put $G=\{G_r\}_{r\in Z}$ where G_r is the right S-homomorphism of Y_r to Z_r. We can take G_r such that $G_r(y_0\otimes_K\cdots\otimes_K y_{r+1})=\sum_{i_0,\dots,i_r}H(y_0 R_{i_0})\otimes_K H(L_{i_0}y_1 R_{i_1})\otimes_K\cdots\otimes_K H(L_{i_r}y_{r+1})$ for $r\geqq 0$. Therefore $\mathrm{Res}_0(\overline{(1\otimes_K 1)\otimes_P m})=\overline{\sum_i(1\otimes_K 1)\otimes_S L_i m R_i}$ holds.

4. Ψ and homomorphisms Res, Cor, Inf and Def.

Let A and C be rings and B a subring of A. We consider a family of covariant functors $T=\{T_i\}_{i\in Z}$ from the category of A-modules to the category of C-modules with connecting homomorphisms $\partial: T_i(M_3)\rightarrow T_{i-1}(M_1)$ defined for each (A, B)-exact sequence $0\rightarrow M_1\rightarrow M_2\rightarrow M_3\rightarrow 0$, and satisfying the following conditions (11) and (12):

(11) The sequence

$$\cdots\longrightarrow T_i(M_1)\longrightarrow T_i(M_2)\longrightarrow T_i(M_3)\overset{\partial}{\longrightarrow}T_{i-1}(M_1)\longrightarrow\cdots$$

is exact.

(12) If

$$\begin{array}{ccccccccc}0&\longrightarrow&M_1&\longrightarrow&M_2&\longrightarrow&M_3&\longrightarrow&0\\&&\downarrow&&\downarrow&&\downarrow&&\\0&\longrightarrow&N_1&\longrightarrow&N_2&\longrightarrow&N_3&\longrightarrow&0\end{array}$$

is a commutative diagram of (A, B)-exact rows, then, for $i\in\boldsymbol{Z}$, the following diagram is commutative

$$\begin{array}{ccc}T_i(M_3)&\overset{\partial}{\longrightarrow}&T_{i-1}(M_1)\\\downarrow&&\downarrow\\T_i(N_3)&\underset{\partial}{\longrightarrow}&T_{i-1}(N_1)\end{array}$$

This family of functors is a relativized version of "connected sequence of functors" in [2] or "∂-foncteurs" in [4]. Let U be also a family of functors which satisfies the conditions (11) and (12). When a sequence of natural transformations $f_i: T_i \to U_i$ satisfies a condition, that is, for any (A, B)-exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, the diagram

$$\begin{array}{ccc} T_i(M_3) & \xrightarrow{\partial} & T_{i-1}(M_1) \\ f_i \downarrow & & \downarrow f_{i-1} \\ U_i(M_3) & \xrightarrow{\partial} & U_{i-1}(M_1) \end{array}$$

is commutative, we call $\{f_i\}$ a map from T to U. Then the following proposition holds by the same way as [2, Proposition 5.2 in Chapter III] and [4, Proposition 2.2.1].

PROPOSITION 4.1. *Let T and U be families of covariant functors which satisfy the conditions* (11) *and* (12). *When a natural transformation $f_{i_0}: T_{i_0} \to U_{i_0}$ is given for some $i_0 \in \boldsymbol{Z}$,*

(i) *f_{i_0} extends uniquely to a map $\{f_i\}: T \to U$ defined for all $i \leqq i_0$ if $T_i(N) = 0$ holds for all $i < i_0$ and any (A, B)-injective module N,*

(ii) *f_{i_0} extends uniquely to a map $\{f_i\}: T \to U$ defined for all $i \geqq i_0$ if $U_i(N) = 0$ holds for all $i > i_0$ and any (A, B)-projective module N.*

Let Q be a subring of P which is the image of the natural homomorphism $\Gamma \otimes_K \Lambda^o \to P$. Note that Q is isomorphic to $\Gamma \otimes_K \Lambda^o$ and the ring extension P/Q is a Frobenius extension.

LEMMA 4.2. *Let M be a (P, Q)-projective module. Then for any $r \in \boldsymbol{Z}$ $H_r(\Lambda, \Gamma, M) = 0$ and $H^r(\Lambda, \Gamma, M) = 0$ hold. These equations also hold for M^Δ.*

PROOF. Let X be a complete (P, S)-resolution of Λ. Then [8, Proposition 1.1] shows that X is (P, Q)-exact. Since P/Q is a Frobenius extension, M is (P, Q)-injective. Hence $\mathrm{Hom}\,({}_PX, {}_PM)$ is an exact complex, and thus $H^r(\Lambda, \Gamma, M) = 0$ holds. Regard the differentiations of X as right P-homomorphisms. Then [8, Proposition 1.1] also shows that X is (P, Q)-exact. Hence $X \otimes_P M$ is an exact complex, and thus $H_r(\Lambda, \Gamma, M) = 0$ holds. M is (P, Q)-projective, i.e., there is a Q-module N such that ${}_PM < \oplus_P(P \otimes_Q N)$ holds. Then we have ${}_PM^\Delta < \oplus_P(P \otimes_Q N)^\Delta \cong {}_P(\Lambda \otimes_\Gamma N)^\Delta = {}_P\Lambda \otimes_\Gamma N^\Delta \cong {}_PP \otimes_Q N^\Delta$. So M^Δ is (P, Q)-projective. Hence this lemma also holds for M^Δ.

The natural homomorphisms $K\otimes_K\Lambda^o\to P$ and $K\otimes_K\Gamma^o\to P$ are monomorphisms since Λ^o and Γ^o are K-projective. So we can regard $K\otimes_K\Lambda^o$ and $K\otimes_K\Gamma^o$ as subrings of P and S, respectively. Moreover the ring extensions $P/K\otimes_K\Lambda^o$ and $S/K\otimes_K\Gamma^o$ are Frobenius extensions. Therefore by the same way as Lemma 4.2, the following corollary follows from [8, Proposition 1.1].

COROLLARY 4.3. *Let M be a $(P, K\otimes_K\Lambda^o)$-projective module. Then for any $r\in\boldsymbol{Z}$, $H^r(\Lambda, K, M)=0$, $H_r(\Lambda, K, M^\Delta)=0$, $H^r(\Gamma, K, M)=0$ and $H_r(\Gamma, K, M^\Delta)=0$ hold.*

PROOF. When M is regarded as a left S-module, M is $(S, K\otimes_K\Gamma^o)$-projective since Λ is Γ-projective. Therefore $H^r(\Gamma, K, M)=0$ and $H_r(\Gamma, K, M^\Delta)=0$ also hold.

In [8], for a left P-module M, we have defined K-homomorphisms $\mathrm{Res}^r: H^r(\Lambda, K, M)\to H^r(\Gamma, K, M)$ for $r\in\boldsymbol{Z}$, $\mathrm{Cor}^r: H^r(\Gamma, K, M)\to H^r(\Lambda, K, M)$ for $r\in\boldsymbol{Z}$, $\mathrm{Inf}^r: H^r(\Lambda, \Gamma, M)\to H^r(\Lambda, K, M)$ for $r\geqq 1$ and $\mathrm{Def}^r: H^r(\Lambda, K, M)\to H^r(\Lambda, \Gamma, M)$ for $r\leqq 0$. For these homomorphisms, the following holds.

LEMMA 4.4. *Let M be a left P-module, and take the relative complete resolutions of type* (2). *Then we have*

$$\mathrm{Cor}^0(\bar{f})=\overline{[y_0\otimes_K y_1 \longrightarrow \textstyle\sum_i y_0R_if(1\otimes_K 1)L_iy_1]}\,,$$

$$\mathrm{Def}^0(\bar{g})=\overline{[x_0\otimes_\Gamma x_1 \longrightarrow g(x_0\otimes_K x_1)]}\,,$$

$$\mathrm{Res}^{-1}(\bar{k})=\overline{[z_0\otimes_K z_1 \longrightarrow \textstyle\sum_i k(z_0L_i\otimes_K\Delta(R_i)z_1)]}\,,$$

where $\bar{\ }$ *stands for residue classes.*

PROOF. The proofs about Cor^0 and Def^0 are given in [8, Proposition 2.2]. Let Y (resp. Z) be the complete (P, K)- (resp. (S, K)-) resolution of Λ (resp. Γ) of type (2). Then the identity homomorphism of Λ induces a chain map $G: Z\otimes_\Gamma\Lambda\to Y$ which consists of left Q-homomorphisms. Put $G=\{G_r\}_{r\in Z}$ where $G_r: Z_r\otimes_\Gamma\Lambda\to Y_r$. Take G_r such that $G_{-r}((z_0\otimes_K\cdots\otimes_K z_r)\otimes_\Gamma\lambda)=\sum_{i_0,\dots,i_{r-1}} z_0L_{i_0}\otimes_K\Delta(R_{i_0})z_1L_{i_1}\otimes_K\cdots\otimes_K\Delta(R_{i_{r-1}})z_r\lambda$ for $r\geqq 1$. Then $\mathrm{Res}^{-1}(\bar{k})=\overline{[z_0\otimes_K z_1\to\sum_i k(z_0L_i\otimes_K\Delta(R_i)z_1)]}$ holds by the definition of Res in [8].

By the same argument as in section 2 we have the K-homomorphisms $\Psi^r_{\Lambda/K}: H_r(\Lambda, K, M^\Delta)\to H^{-r-1}(\Lambda, K, M)$ and $\Psi^r_{\Gamma/K}: H_r(\Gamma, K, N^\Delta)\to H^{-r-1}(\Gamma, K, N)$ for $r\in\boldsymbol{Z}$, any left P-module M and S-module N. Note that the restriction of Δ to Γ is the Nakayama automorphism of the Frobenius K-algebra Γ. Then the following holds.

PROPOSITION 4.5. *Let M be a left P-module. Then for the K-homomorphisms* Cor_r, Res_r, Def_r *and* Inf_r *of* M^{Δ}, *we have the equations*

(i) $\Psi^r_{\Lambda/K}\circ\mathrm{Cor}_r=\mathrm{Cor}^{-r-1}\circ\Psi^r_{\Gamma/K}$ *for any* $r\in\boldsymbol{Z}$,

(ii) $\Psi^r_{\Gamma/K}\circ\mathrm{Res}_r=\mathrm{Res}^{-r-1}\circ\Psi^r_{\Lambda/K}$ *for any* $r\in\boldsymbol{Z}$,

(iii) $\Psi^r_{\Lambda/\Gamma}\circ\mathrm{Def}_r=\mathrm{Def}^{-r-1}\circ\Psi^r_{\Lambda/K}$ *for* $r\geqq 1$,

(iv) $\Psi^{-r}_{\Lambda/\Gamma}=\mathrm{Def}^{r-1}\circ\Psi^{-r}_{\Lambda/K}\circ\mathrm{Inf}_{-r}$ *for* $r=0, 1$,

(v) $\Psi^{-r}_{\Lambda/K}\circ\mathrm{Inf}_{-r}=\mathrm{Inf}^{r-1}\circ\Psi^{-r}_{\Lambda/\Gamma}$ *for* $r\geqq 2$.

PROOF. By (4), Proposition 3.2 and Lemma 4.4, (i), (ii) and (iv) hold for the cases of $r=-1$, $r=0$ and $r=1$, respectively. All the composite K-homomorphisms in the equations above commute with connecting homomorphisms by Propositions 2.1, 3.1 and [8, Lemmas 2.5 and 3.8]. Therefore the uniqueness of Proposition 4.1 and Corollary 4.3 shows that (i) and (ii) hold. Similarly the uniqueness of Proposition 4.1 and Lemma 4.2 shows that (iv) holds. The case of $r=1$ of (iii) also holds. In fact, for the (P, Q)-exact sequence

$$0\longrightarrow \mathrm{Ker}\,\xi\longrightarrow P\otimes_Q M\overset{\xi}{\longrightarrow} M\longrightarrow 0\,, \tag{13}$$

where ξ is a P-homomorphism such that $\xi(p\otimes_Q m)=p\cdot m$, we have the connecting homomorphisms $\partial^{\Delta}_r: H_r(\Lambda, \Gamma, M^{\Delta})\to H_{r-1}(\Lambda, \Gamma, (\mathrm{Ker}\,\xi)^{\Delta})$, $\partial^r: H^r(\Lambda, \Gamma, M)\to H^{r+1}(\Lambda, \Gamma, \mathrm{Ker}\,\xi)$, $\partial^{\Delta,\Lambda}_r: H_r(\Lambda, K, M^{\Delta})\to H_{r-1}(\Lambda, K, (\mathrm{Ker}\,\xi)^{\Delta})$ and $\partial^r_\Lambda: H^r(\Lambda, K, M)\to H^{r+1}(\Lambda, K, \mathrm{Ker}\,\xi)$ for all $r\in\boldsymbol{Z}$. Then by (iv) of this proposition and Proposition 3.1 (v) we have $\partial^{-2}\circ\Psi^1_{\Lambda/\Gamma}\circ\mathrm{Def}_1=\Psi^0_{\Lambda/\Gamma}\circ\partial^{\Delta}_1\circ\mathrm{Def}_1=\mathrm{Def}^{-1}\circ\Psi^0_{\Lambda/K}\circ\mathrm{Inf}_0\circ\partial^{\Delta}_1\circ\mathrm{Def}_1=\mathrm{Def}^{-1}\circ\Psi^0_{\Lambda/K}\circ\partial^{\Delta,\Lambda}_1=\mathrm{Def}^{-1}\circ\partial^{-2}_\Lambda\circ\Psi^1_{\Lambda/K}=\partial^{-2}\circ\mathrm{Def}^{-2}\circ\Psi^1_{\Lambda/K}$. Since ∂^{-2} is an isomorphism by Lemma 4.2, the case of $r=1$ also holds. Similarly by using the connecting homomorphisms of the (P, Q)-exact sequence

$$0\longrightarrow M\overset{\tau}{\longrightarrow}\mathrm{Hom}\,({}_QP, {}_QM)\longrightarrow \mathrm{Coker}\,\tau\longrightarrow 0\,, \tag{14}$$

where $\tau(m)=[p\to p\cdot m]$, the case of $r=2$ of (v) also holds by (iv) of this proposition and [8, Lemma 3.8 (ii)]. Therefore (iii) and (v) also hold by the uniqueness of Proposition 4.1 and Lemma 4.2.

5. Fundamental exact sequences.

We now show that we can define Cor_r by another way. Let Y and Z be a complete (P, K)-resolution of Λ and a complete (S, K)-resolution of Γ, respectively. For the subring Q of P in section 4, $\Lambda\otimes_\Gamma Z$ is a complete (Q, K)-resolution of the right Q-module Λ. Since Y is also a complete (Q, K)-resolution of Λ, the identity homomorphism of Λ induces

(15) $$H_r((\Lambda\otimes_\Gamma Z)\otimes_Q M)\cong H_r(Y\otimes_Q M)$$

for any left Q-module M and $r\in \mathbf{Z}$. And we have $H_r((\Lambda\otimes_\Gamma Z)\otimes_Q M)\cong H_r((Z\otimes_S Q)\otimes_Q M)\cong H_r(Z\otimes_S M)=H_r(\Gamma, K, M)$ with the usual way. Hence for $r\in \mathbf{Z}$ there is an isomorphism

(16) $$s_r: H_r(\Gamma, K, M)\xrightarrow{\simeq} H_r(Y\otimes_Q M)$$

LEMMA 5.1. *Let M be a left P-module. Then* Cor_r *and* Res_r *coincide with the following composite homomorphisms* (17) *and* (18), *respectively*:

(17) $$H_r(\Gamma, K, M)\underset{s_r}{\xrightarrow{\simeq}} H_r(Y\otimes_Q M)\longrightarrow H_r(\Lambda, K, M),$$

(18) $$H_r(\Lambda, K, M)\longrightarrow H_r(Y\otimes_Q M)\underset{s_r^{-1}}{\xrightarrow{\simeq}} H_r(\Gamma, K, M),$$

where $H_r(Y\otimes_Q M)\to H_r(\Lambda, K, M)$ is induced by the K-homomorphisms of change of rings, that is, $Y_r\otimes_Q M\to Y_r\otimes_P M$, and $H_r(\Lambda, K, M)\to H_r(Y\otimes_Q M)$ is induced by a chain map $\kappa: Y\otimes_P M\to Y\otimes_Q M$ such that $\kappa(y\otimes_P m)=\sum_i yR_i\otimes_Q L_i m$ for $y\in Y_r$ and $m\in M$.

PROOF. Let $F: Z\to Y$ be the chain map over the natural inclusion $\Gamma\to\Lambda$ which induces the K-homomorphism (6), and $G: Y\to Z$ the chain map over $H: \Lambda\to\Gamma$ which induces the K-homomorphism (7). Then the isomorphism (15) is induced by a chain map $F': \Lambda\otimes_\Gamma Z\to Y$ such that $F'(\lambda\otimes_\Gamma z)=\lambda F(z)$, and the inverse isomorphism of (15) is induced by a chain map $G': Y\to\Lambda\otimes_\Gamma Z$ such that $G'(y)=\sum_i R_i\otimes_\Gamma G(L_i y)$. Using these chain maps F' and G', we can see that the K-homomorphism (17) is induced by the chain map $F'': Z\otimes_S M\to Y\otimes_P M$ such that $F''(z\otimes_S m)=F(z)\otimes_P m$, and the K-homomorphism (18) is induced by the chain map $G'': Y\otimes_P M\to Z\otimes_S M$ such that $G''(y\otimes_P m)=\sum_{i,j} G(L_j y R_i)\otimes_S L_i m R_j$. By the definitions of Cor_r and Res_r, these mean that (17) and (18) coincide with Cor_r and Res_r, respectively.

In the introduction of [5], it is said that fundamental exact sequences of Tor can be obtained. Therefore by the same way as [8, section 2] we have the following theorem from them and Lemma 5.1:

THEOREM 5.2. *Let N be a left P-module and define left P-modules N_i ($i\geqq 0$) inductively as $N_0=N$ and $N_i=P\otimes_Q N_{i-1}$ for $i\geqq 1$. Then the sequence*

$$0\longleftarrow H_r(\Lambda, \Gamma, N)\xleftarrow{\mathrm{Def}_r} H_r(\Lambda, K, N)\xleftarrow{\mathrm{Cor}_r} H_r(\Gamma, K, N)$$

is exact for $r\geqq 1$ *if* $H_n(\Gamma, K, N_{r-n})=0$ $(0<n<r)$.

PROPOSITION 5.3. *The following sequence is exact for any left P-module*:

$$0 \longrightarrow H_0(\Lambda, \Gamma, M) \xrightarrow{\text{Inf}_0} H_0(\Lambda, K, M) \xrightarrow{\text{Res}_0} H_0(\Gamma, K, M).$$

PPOOF. Take relative complete resolutions of type (2). Then by Propositions 1.1 and 3.2 Inf_0 is a monomorphism and $\text{Ker}\,\text{Res}_0 \subset \text{Im}\,\text{Inf}_0$ holds. In fact, if $\text{Res}_0(\overline{(1\otimes_K 1)\otimes_P m})=\bar{0}$, $\sum_i L_i m R_i=\sum_t (m_t z_t - z_t m_t)$ (finite sum) for some $z_t\in\Gamma$ and $m_t\in M$ by Proposition 1.1, and so $\sum_i (R_i\otimes_\Gamma L_i)\otimes_P m=\sum_i (1\otimes_\Gamma 1)\otimes_P L_i m R_i =0$ holds in $(\Lambda\otimes_\Gamma\Lambda)\otimes_P M$, that is, $(1\otimes_\Gamma 1)\otimes_P m\in C_0^{\Lambda/\Gamma}(M)$. So $\text{Ker}\,\text{Res}_0\subset\text{Im}\,\text{Inf}_0$ holds. Define a K-homomorphism $\varphi: (\Lambda\otimes_\Gamma\Lambda)\otimes_P M\to(\Gamma\otimes_K\Gamma)\otimes_S M/B_0^{\Gamma/K}(M)$ such that $\varphi((\lambda_0\otimes_\Gamma\lambda_1)\otimes_P m)=\overline{(1\otimes_K 1)\otimes_S \lambda_1 m\lambda_0}$. Then for $(1\otimes_\Gamma 1)\otimes_P m\in C_0^{\Lambda/\Gamma}(M)$, $\bar{0}=\varphi(0)=\varphi(\sum_i (R_i\otimes_\Gamma L_i)\otimes_P m)=\overline{\sum_i (1\otimes_K 1)\otimes_S L_i m R_i}$ holds. So $\sum_i (1\otimes_K 1)\otimes_S L_i m R_i \in B_0^{\Gamma/K}(M)$ holds. Therefore $\text{Res}_0\circ\text{Inf}_0(\overline{(1\otimes_\Gamma 1)\otimes_P m})=\overline{\sum_i (1\otimes_K 1)\otimes_S L_i m R_i}=\bar{0}$ holds. Hence the proof is complete.

LEMMA 5.4. $H_r(\Gamma, K, M)\cong H_r(\Lambda, K, P\otimes_Q M)\cong H_r(\Lambda, K, \text{Hom}\,({}_QP, {}_QM))$ *holds for any left P-module M and all* $r\in \boldsymbol{Z}$.

PROOF. For a complete (P, K)-resolution Y of Λ, we have $H_r(\Gamma, K, M)\cong H_r(Y\otimes_Q M)$ by (16). Since P/Q is a Frobenius extension, $P\otimes_Q M\cong\text{Hom}\,({}_QP, {}_QM)$ holds as left P-modules. Therefore $H_r(Y\otimes_Q M)\cong H_r(Y\otimes_P(P\otimes_Q M))= H_r(\Lambda, K, P\otimes_Q M)\cong H_r(\Lambda, K, \text{Hom}\,({}_QP, {}_QM))$ holds.

THEOREM 5.5. *Let M be any left P-module and define left P-modules* M^i $(i\geqq 0)$ *inductively as* $M^0=M$ *and* $M^i=\text{Hom}\,({}_QP, {}_QM^{i-1})$ *for* $i\geqq 1$. *Then the sequence*

$$0 \longrightarrow H_{-r}(\Lambda, \Gamma, M) \xrightarrow{\text{Inf}_{-r}} H_{-r}(\Lambda, K, M) \xrightarrow{\text{Res}_{-r}} H_{-r}(\Gamma, K, M)$$

is exact for $r\geqq 0$ *if* $H_{-n}(\Gamma, K, M^{r-n})=0$ $(0\leqq n\leqq r-1)$.

PROOF. By induction on r. The case of $r=0$ is proved by Proposition 5.3. Assume that the case of $r=t$ holds. Consider the case of $r=t+1$. We use the exact sequence (14) in section 4. Put $N=\text{Coker}\,\tau$. Then since (14) is (P, Q)-exact, we have ${}_QN<\oplus_Q M^1$. Therefore ${}_SN^i<\oplus_S M^{i+1}$ holds for all $i\geqq 0$, where we put $N^0=N$ and $N^i=\text{Hom}\,({}_QP, {}_QN^{i-1})$ for $i\geqq 1$ inductively. So we have $H_{-n}(\Gamma, K, N^{t-n})=0$ for $0\leqq n\leqq t$. Hence the theorem holds for N and the case of $r=t$. Moreover $H_{-t}(\Gamma, K, N)=0$ holds. Then by Proposition 3.1, (14) induces the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & H_{-t}(\Lambda, \Gamma, N) & \xrightarrow{\text{Inf}_{-t}} & H_{-t}(\Lambda, K, N) & \xrightarrow{\text{Res}_{-t}} & H_{-t}(\Gamma, K, N) \\
& \downarrow \partial & & \downarrow \partial^{\Lambda} & & \downarrow \partial^{\Gamma} \\
& H_{-t-1}(\Lambda, \Gamma, M) & \xrightarrow{\text{Inf}_{-t-1}} & H_{-t-1}(\Lambda, K, M) & \xrightarrow{\text{Res}_{-t-1}} & H_{-t-1}(\Gamma, K, M) \\
& & & \downarrow \bar{\tau} & \swarrow \varphi & \\
& & & H_{-t-1}(\Lambda, K, M^1) & &
\end{array}$$

where ∂, ∂^{Λ} and ∂^{Γ} are connecting homomorphisms for (14), $\bar{\tau}$ is the homomorphism induced by τ, and φ is the isomorphism of Lemma 5.4. The isomorphism $H_r(Y \otimes_Q M) \to H_r(\Lambda, K, M^1)$ in the proof of Lemma 5.4 is induced by an isomorphism $u: Y_r \otimes_Q M \to Y_r \otimes_P M^1$ such that $u(y \otimes_Q m) = y \otimes_P [x_0 \otimes x_1 \to H(x_0) m x_1]$ for $y \in Y_r$, $m \in M$ and $x_0 \otimes x_1 \in P$. Therefore $\varphi \circ \text{Res}_{-t-1} = \bar{\tau}$ holds by Lemma 5.1. M^1 is (P, Q)-projective since P/Q is a Frobenius extension. Therefore ∂ is an isomorphism by Lemma 4.2. And ∂^{Λ} is a monomorphism because by Lemma 5.4 $H_{-t}(\Lambda, K, M^1) \cong H_{-t}(\Gamma, K, M)$ and $H_{-t}(\Gamma, K, M) = 0$ holds by $H_{-t}(\Gamma, K, M) \oplus H_{-t}(\Gamma, K, N) \cong H_{-t}(\Gamma, K, M^1) = 0$. Hence for the middle sequence of the commutative diagram above, Theorem 5.5 holds.

6. $\Psi_{\Lambda/\Gamma}$ and fundamental exact sequences.

The complete (P, K)-resolution of Λ of type (2) is the complete projective resolution defined in [7] and [9] since the modules of the resolution are P-projective. Therefore the absolute homology and cohomology groups in [7] and [9] are $H_r(\Lambda, K, -)$ and $H^r(\Lambda, K, -)$. Similarly for the complete (S, K)-resolution of Γ of type (2), we have the same argument. Hence [7, Satz 2] and [9, Theorem 10] show that the following holds.

THEOREM 6.1. *$\Psi^r_{\Lambda/K}$ (resp. $\Psi^r_{\Gamma/K}$) is an isomorphism for any left P- (resp. S-) module and any $r \in \mathbf{Z}$.*

[10] also shows the result above by using a complete resolution.

We have the following diagrams for any left P-module M from Proposition 4.5 and Theorem 6.1:

$$\begin{array}{ccccccc}
0 \longleftarrow & H_r(\Lambda, \Gamma, M^{\Delta}) & \xleftarrow{\text{Def}_r} & H_r(\Lambda, K, M^{\Delta}) & \xleftarrow{\text{Cor}_r} & H_r(\Gamma, K, M^{\Delta}) \\
& \downarrow \Psi^r_{\Lambda/\Gamma} & & \wr\downarrow \Psi^r_{\Lambda/K} & & \wr\downarrow \Psi^r_{\Gamma/K} \\
0 \longleftarrow & H^{-r-1}(\Lambda, \Gamma, M) & \xleftarrow[\text{Def}^{-r-1}]{} & H^{-r-1}(\Lambda, K, M) & \xleftarrow[\text{Cor}^{-r-1}]{} & H^{-r-1}(\Gamma, K, M),
\end{array} \tag{19}$$

$$(20)\qquad \begin{array}{ccccccc} 0 \longrightarrow & H_{-r}(\Lambda, \Gamma, M^{\Delta}) & \xrightarrow{\mathrm{Inf}_{-r}} & H_{-r}(\Lambda, K, M^{\Delta}) & \xrightarrow{\mathrm{Res}_{-r}} & H_{-r}(\Gamma, K, M^{\Delta}) \\ & \downarrow \Psi^{-r}_{\Lambda/\Gamma} & & \wr\downarrow \Psi^{-r}_{\Lambda/K} & & \wr\downarrow \Psi^{-r}_{\Gamma/K} \\ 0 \longleftarrow & H^{r-1}(\Lambda, \Gamma, M) & \xleftarrow[\mathrm{Def}^{r-1}]{} & H^{r-1}(\Lambda, K, M) & \xleftarrow[\mathrm{Cor}^{r-1}]{} & H^{r-1}(\Gamma, K, M), \end{array}$$

$$(21)\qquad \begin{array}{ccccccc} 0 \longrightarrow & H_{-r}(\Lambda, \Gamma, M^{\Delta}) & \xrightarrow{\mathrm{Inf}_{-r}} & H_{-r}(\Lambda, K, M^{\Delta}) & \xrightarrow{\mathrm{Res}_{-r}} & H_{-r}(\Gamma, K, M^{\Delta}) \\ & \downarrow \Psi^{-r}_{\Lambda/\Gamma} & & \wr\downarrow \Psi^{-r}_{\Lambda/K} & & \wr\downarrow \Psi^{-r}_{\Gamma/K} \\ 0 \longrightarrow & H^{r-1}(\Lambda, \Gamma, M) & \xrightarrow[\mathrm{Inf}^{r-1}]{} & H^{r-1}(\Lambda, K, M) & \xrightarrow[\mathrm{Res}^{r-1}]{} & H^{r-1}(\Gamma, K, M), \end{array}$$

where diagrams (19) and (21) are commutative for $r\geqq 1$ and $r\geqq 2$, respectively, and the left half of (20) is commutative for $r=0, 1$. So if the top and bottom rows of (19) (resp. (21)) are exact, $\Psi^{r}_{\Lambda/\Gamma}$ in (19) (resp. $\Psi^{-r}_{\Lambda/\Gamma}$ in (21)) is an isomorphism, and if the top and bottom rows of (20) are exact and $H_{-r}(\Gamma, K, M^{\Delta})=0$, that is, $H^{r-1}(\Gamma, K, M)=0$, $\Psi^{-r}_{\Lambda/\Gamma}$ in (20) is an isomorphism. Hence by results of scction 5 we have

THEOREM 6.2. *For any left P-module N, put left P-modules N_i ($i\geqq 0$) inductively as $N_0=N$, $N_i=P\otimes_Q N_{i-1}$ for $i\geqq 1$. Then for a left P-module M, the following statements hold.*

(i) $\Psi^{r}_{\Lambda/\Gamma}: H_r(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{-r-1}(\Lambda, \Gamma, M)$ *for $r\geqq 1$ is an isomorphism if M satisfies the condition $H_n(\Gamma, K, (M^{\Delta})_{r-n})=0$ for $-1\leqq n\leqq r-1$.*

(ii) $\Psi^{0}_{\Lambda/\Gamma}: H_0(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{-1}(\Lambda, \Gamma, M)$ *is an isomorphism if M satisfies the conditions $H_0(\Gamma, K, M^{\Delta})=0$ and $H^0(\Gamma, K, M_1)=0$.*

(iii) $\Psi^{-1}_{\Lambda/\Gamma}: H_{-1}(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{0}(\Lambda, \Gamma, M)$ *is an isomorphism if M satisfies the conditions $H_0(\Gamma, K, (M^{\Delta})_1)=0$ and $H^0(\Gamma, K, M)=0$.*

(iv) $\Psi^{-r}_{\Lambda/\Gamma}: H_{-r}(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{r-1}(\Lambda, \Gamma, M)$ *for $r\geqq 2$ is an isomorpnism if M satisfies the condition $H^n(\Gamma, K, M_{r-1-n})=0$ for $-1\leqq n\leqq r-2$.*

PROOF. For any left P-module N and all $i\geqq 0$, note that $N_i\cong N^i$ holds as left P-modules where N^i is the P-module as in Theorem 5.5. Moreover for all $i\geqq 0$ $(N_i)^{\Delta}\cong(N^{\Delta})_i$ holds as left P-modules by induction. In fact, by induction, we have $(N_i)^{\Delta}\cong(\Lambda\otimes_\Gamma N_{i-1}\otimes_\Lambda \Lambda)^{\Delta}\cong \Lambda\otimes_\Gamma(N_{i-1})^{\Delta}\cong\Lambda\otimes_\Gamma(N^{\Delta})_{i-1}\cong(N^{\Delta})_i$. (i) follows from [8, Theorem 2.6], Theorems 5.2 and 6.1. In fact, [8, Theorem 2.6] shows that the bottom row of (19) is exact if $H^{-n}(\Gamma, K, M_{r+1-n})=0$ for $0\leqq n\leqq r$, and Theorem 5.2 shows that the top row of (19) is exact if $H_n(\Gamma, K, (M^{\Delta})_{r-n})=0$ for $0<n<r$. Therefore (i) holds by Theorem 6.1 and the isomorphism $(N_i)^{\Delta}\cong(N^{\Delta})_i$. Similarly by using the diagrm (20), (ii) and (iii) follow from [8, Proposi-

tion 2.2 and Theorem 2.6], Proposition 5.3 and Theorem 5.5. And by using the diagram (21), (iv) follows from [8, Theorem 2.1], Theorems 5.5 and 6.1.

By using Proposition 2.1 for (13) and (14), since the connecting homomorphisms are isomorphisms by Lemma 4.2, $\Psi^{z}_{\Lambda/\Gamma}$ is an isomorphism for any left P-module and any $z\in \boldsymbol{Z}$ if and only if $\Psi^{r}_{\Lambda/\Gamma}$ is an isomorphism for any left P-module and some $r\in \boldsymbol{Z}$. Hence we have

THEOREM 6.3. *The following conditions are equivalent*:

(i) $\Psi^{r}_{\Lambda/\Gamma}: H_r(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{-r-1}(\Lambda, \Gamma, M)$ *is an isomorphism for any left P-module M and any $r\in \boldsymbol{Z}$.*

(ii) $0\leftarrow H^{-2}(\Lambda, \Gamma, M)\xleftarrow{\mathrm{Def}^{-2}} H^{-2}(\Lambda, K, M)\xleftarrow{\mathrm{Cor}^{-2}} H^{-2}(\Gamma, K, M)$ *is exact for any left P-module M.*

(iii) $\mathrm{Inf}_{-1}: H_{-1}(\Lambda, \Gamma, M^{\Delta})\rightarrow H_{-1}(\Lambda, K, M^{\Delta})$ *is a monomorphism and* $H^{0}(\Lambda, K, M) =\mathrm{Im}\,\Psi^{-1}_{\Lambda/K}\circ\mathrm{Inf}_{-1}\oplus\mathrm{Ker}\,\mathrm{Def}^{0}$ *for any left P-module M.*

(iv) $\mathrm{Def}^{-1}: H^{-1}(\Lambda, K, M)\rightarrow H^{-1}(\Lambda, \Gamma, M)$ *is an epimorphism and* $H_0(\Lambda, K, M^{\Delta}) =\mathrm{Ker}\,\mathrm{Def}^{-1}\circ\Psi^{0}_{\Lambda/K}\oplus\mathrm{Im}\,\mathrm{Inf}_0$ *for any left P-module M.*

(v) $0\rightarrow H_{-2}(\Lambda, \Gamma, M^{\Delta})\xrightarrow{\mathrm{Inf}^{-2}} H_{-2}(\Lambda, K, M^{\Delta})\xrightarrow{\mathrm{Res}^{-2}} H_{-2}(\Gamma, K, M^{\Delta})$ *is exact for any left P-module M.*

PROOF. We use (19), (20) and (21) for the proof.

(i)⇔(ii). For any left P-module M the top row of (19) is exact for the case of $r=1$ by Theorem 5.2 without any condition. Therefore $\Psi^{1}_{\Lambda/\Gamma}$ is an isomorphism for any left P-module M if and only if (ii) holds. Hence (i)⇔(ii) holds.

(i)⇔(iii). The bottom row of (20) is exact for the case of $r=1$ by [8, Proposition 2.2]. So if $\Psi^{-1}_{\Lambda/\Gamma}$ is an isomorphism, Inf_{-1} is a monomorphism and the bottom row of (20) is split by $\Psi^{-1}_{\Lambda/K}\circ\mathrm{Inf}_{-1}\circ(\Psi^{-1}_{\Lambda/\Gamma})^{-1}$. Therefore (i)⇒(iii) holds. If (iii) holds, it is easy to see that $\Psi^{-1}_{\Lambda/\Gamma}$ of (20) is an isomorphism. Hence (iii)⇒(i) holds.

(i)⇔(iv). The top row of (20) is exact for the case of $r=0$ by Proposition 5.3. Hence (i)⇔(iv) holds by the similar way to (i)⇔(iii).

(i)⇔(v). The bottom row of (21) is exact for the case of $r=2$ by [8, Theorem 2.1] without any condition. Therefore (i)⇔(v) holds by the similar way to (i)⇔(ii).

7. The necessary and sufficient conditions on which $\Psi_{\Lambda/\Gamma}$ is an isomorphism.

Put $\tilde{M}=\{m\in M|(1\otimes_{\Gamma}1)\otimes_{P}m^{\Delta}\in C_{-1}^{\Lambda/\Gamma}(M^{\Delta})\}$ for any left P-module M where $C_{-1}^{\Lambda/\Gamma}$ is the same as in Proposition 1.1, and put $M^{\Lambda}=\{m\in M|xm=mx$ for all $x\in\Lambda\}$ and $M^{\Gamma}=\{m\in M|zm=mz$ for all $z\in\Gamma\}$. $\tilde{M}$, M^{Λ} and M^{Γ} are K-submodules of M. Then we have the following theorem.

THEOREM 7.1. *The following conditions are equivalent*:

(i) $\Psi_{\Lambda/\Gamma}^{r}: H_{r}(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{-r-1}(\Lambda, \Gamma, M)$ *is an isomorphism for any left P-module M and any* $r\in \boldsymbol{Z}$.

(ii) $\Psi_{\Lambda/\Gamma}^{r}: H_{r}(\Lambda, \Gamma, M^{\Delta})\rightarrow H^{-r-1}(\Lambda, \Gamma, M)$ *is an epimorphism for any left P-module M and any* $r\in \boldsymbol{Z}$.

(iii) *There are elements* $\lambda\in\tilde{\Lambda}$ *and* $\xi\in\Lambda^{\Gamma}$ *such that* $1=\sum_{j}r_{j}\lambda l_{j}+\sum_{i}R_{i}\xi L_{i}$.

PROOF. Induce $\Psi_{\Lambda/\Gamma}$ from (4). By Proposition 1.1 and [8, Proposition 1.2] $\Psi_{\Lambda/\Gamma}^{-1}$ can be regarded as a K-homomorphism from $C_{-1}^{\Lambda/\Gamma}(M^{\Delta})/B_{-1}^{\Lambda/\Gamma}(M^{\Delta})$ into $M^{\Lambda}/N_{\Lambda/\Gamma}(M)$ such that

$$\Psi_{\Lambda/\Gamma}^{-1}(\overline{(1\otimes_{\Gamma}1)\otimes_{P}m^{\Delta}})=\overline{\textstyle\sum_{j}r_{j}ml_{j}} \tag{22}$$

for $(1\otimes_{\Gamma}1)\otimes_{P}m^{\Delta}\in C_{-1}^{\Lambda/\Gamma}(M^{\Delta})$, where $N_{\Lambda/\Gamma}(M)=\{\sum_{i}R_{i}mL_{i}|m\in M^{\Gamma}\}$ and $^{-}$ stands for the residue classes.

(i)⇒(ii). This holds obviously.

(ii)⇒(iii). Consider the case of $M=\Lambda$. $\Psi_{\Lambda/\Gamma}^{-1}$ is an epimorphism. So since $1\in\Lambda^{\Lambda}$, there is $(1\otimes_{\Gamma}1)\otimes_{P}\lambda^{\Delta}\in C_{-1}^{\Lambda/\Gamma}(\Lambda^{\Delta})$ such that $\Psi_{\Lambda/\Gamma}^{-1}(\overline{(1\otimes_{\Gamma}1)\otimes_{P}\lambda^{\Delta}})=\bar{1}$, that is, $\overline{\sum_{j}r_{j}\lambda l_{j}}=\bar{1}$ holds. Therefore $1-\sum_{j}r_{j}\lambda l_{j}\in N_{\Lambda/\Gamma}(\Lambda)$, that is, there is $\xi\in\Lambda^{\Gamma}$ such that $1=\sum_{j}r_{j}\lambda l_{j}+\sum_{i}R_{i}\xi L_{i}$.

(iii)⇒(i). By using Proposition 2.1 for (13) and (14), since the connecting homomorphisms are isomorphisms by Lemma 4.2, (iii)⇒(i) holds if $\Psi_{\Lambda/\Gamma}^{-1}$ is an isomorphism for any left P-module M. Let M be any left P-module. For λ of (iii), define a K-homomorphism $\varphi: M^{\Lambda}\rightarrow C_{-1}^{\Lambda/\Gamma}(M^{\Delta})$ such that $\varphi(m)=(1\otimes_{\Gamma}1)\otimes_{P}(\lambda m)^{\Delta}$. This is well-defined. In fact since $m\in M^{\Lambda}$, we can define a K-homomorphism $\kappa_{m}:(\Lambda\otimes_{\Gamma}\Lambda\otimes_{\Gamma}\Lambda)\otimes_{P}\Lambda^{\Delta}\rightarrow(\Lambda\otimes_{\Gamma}\Lambda\otimes_{\Gamma}\Lambda)\otimes_{P}M^{\Delta}$ as $\kappa_{m}((x_{0}\otimes_{\Gamma}x_{1}\otimes_{\Gamma}x_{2})\otimes_{P}x^{\Delta})=(x_{0}\otimes_{\Gamma}x_{1}\otimes_{\Gamma}x_{2})\otimes_{P}(xm)^{\Delta}$. Since $(1\otimes_{\Gamma}1)\otimes_{P}\lambda^{\Delta}\in C_{-1}^{\Lambda/\Gamma}(\Lambda^{\Delta})$, we have

$$\begin{aligned}0=\kappa_{m}(0)&=\kappa_{m}(\textstyle\sum_{i}(R_{i}\otimes_{\Gamma}L_{i}\otimes_{\Gamma}1)\otimes_{P}\lambda^{\Delta}-\sum_{i}(1\otimes_{\Gamma}R_{i}\otimes_{\Gamma}L_{i})\otimes_{P}\lambda^{\Delta})\\&=\textstyle\sum_{i}(R_{i}\otimes_{\Gamma}L_{i}\otimes_{\Gamma}1)\otimes_{P}(\lambda m)^{\Delta}-\sum_{i}(1\otimes_{\Gamma}R_{i}\otimes_{\Gamma}L_{i})\otimes_{P}(\lambda m)^{\Delta}.\end{aligned}$$

Therefore φ is well-defined. $\varphi(N_{\Lambda/\Gamma}(M))\subset B_{-1}^{\Lambda/\Gamma}(M^{\Delta})$ holds. In fact, for any

$\sum_i R_i m L_i \in N_{\Lambda/\Gamma}(M)$, we have $\varphi(\sum_i R_i m L_i)=\sum_i (1\otimes_\Gamma 1)\otimes_P(\lambda R_i m L_i)^\Delta=\sum_i (1\otimes_\Gamma 1)\otimes_P(R_i m L_i \lambda)^\Delta$, and since $m\in M^\Gamma$, we can define a K-homomorphism κ_m: $(\Lambda\otimes_\Gamma\Lambda\otimes_\Gamma\Lambda)\otimes_P\Lambda^\Delta\to(\Lambda\otimes_\Gamma\Lambda)\otimes_P M^\Delta$ as $\kappa_m((x_0\otimes_\Gamma x_1\otimes_\Gamma x_2)\otimes_P x^\Delta)=(x_0\otimes_\Gamma x_1)\otimes_P (m x_2 x)^\Delta$. Then we have

$$0=\kappa_m(0)=\kappa_m(\textstyle\sum_i (R_i\otimes_\Gamma L_i\otimes_\Gamma 1)\otimes_P\lambda^\Delta-\sum_i (1\otimes_\Gamma R_i\otimes_\Gamma L_i)\otimes_P\lambda^\Delta)$$
$$=\textstyle\sum_i (R_i\otimes_\Gamma L_i)\otimes_P(m\lambda)^\Delta-\varphi(\sum_i R_i m L_i)\,.$$

Therefore $\varphi(N_{\Lambda/\Gamma}(M))\subset B_{-1}^{\Lambda/\Gamma}(M^\Delta)$ holds. So φ induces a K-homomorphism $\Phi: M^\Lambda/N_{\Lambda/\Gamma}(M)\to C_{-1}^{\Lambda/\Gamma}(M^\Delta)/B_{-1}^{\Lambda/\Gamma}(M^\Delta)$. And we have $\Phi\circ\Psi_{\Lambda/\Gamma}^{-1}(\overline{(1\otimes_\Gamma 1)\otimes_P m^\Delta})=\overline{\sum_j (1\otimes_\Gamma 1)\otimes_P(\lambda r_j m l_j)^\Delta}$. Further we have

$$\textstyle\sum_j (1\otimes_\Gamma 1)\otimes_P(\lambda r_j m l_j)^\Delta=\sum_j (\nabla(l_j)\otimes_\Gamma \lambda r_j)\otimes_P m^\Delta$$
$$=\textstyle\sum_j (1\otimes_\Gamma r_j\lambda l_j)\otimes_P m^\Delta$$
$$=(1\otimes_\Gamma 1)\otimes_P m^\Delta-\textstyle\sum_i (1\otimes_\Gamma R_i\xi L_i)\otimes_P m^\Delta \quad \text{by (iii).}$$

Since $\xi\in\Lambda^\Gamma$, we can define a K-homomorphism $\kappa_\xi:(\Lambda\otimes_P\Lambda\otimes_\Gamma\Lambda)\otimes_P M^\Delta\to(\Lambda\otimes_\Gamma\Lambda)\otimes_P M^\Delta$ as $\kappa_\xi((x_0\otimes_\Gamma x_1\otimes_\Gamma x_2)\otimes_P n^\Delta)=(x_0\otimes_\Gamma x_1)\otimes_P(\xi x_2 n)^\Delta$ for $x_0\otimes_\Gamma x_1\otimes_\Gamma x_2\in\Lambda\otimes_\Gamma\Lambda\otimes_\Gamma\Lambda$ and $n\in M$. Then since $(1\otimes_\Gamma 1)\otimes_P m^\Delta\in C_{-1}^{\Lambda/\Gamma}(M^\Delta)$, we have

$$0=\kappa_\xi(0)=\kappa_\xi(\textstyle\sum_i (R_i\otimes_\Gamma L_i\otimes_\Gamma 1)\otimes_P m^\Delta-\sum_i (1\otimes_\Gamma R_i\otimes_\Gamma L_i)\otimes_P m^\Delta)$$
$$=\textstyle\sum_i (R_i\otimes_\Gamma L_i)\otimes_P(\xi m)^\Delta-\sum_i (1\otimes_\Gamma R_i\xi L_i)\otimes_P m^\Delta\,,$$

that is, $\sum_i (1\otimes_\Gamma R_i\xi L_i)\otimes_P m^\Delta\in B_{-1}^{\Lambda/\Gamma}(M^\Delta)$ holds. Therefore $\Phi\circ\Psi_{\Lambda/\Gamma}^{-1}=1$ holds. For any $\bar{m}\in M^\Lambda/N_{\Lambda/\Gamma}(M)$, we have

$$\Psi_{\Lambda/\Gamma}^{-1}\circ\Phi(\bar{m})=\overline{\textstyle\sum_j r_j\lambda m l_j}=\overline{\textstyle\sum_j r_j\lambda l_j m}=\bar{m}-\overline{\textstyle\sum_i R_i\xi L_i m}=\bar{m}-\overline{\textstyle\sum_i R_i(\xi m)L_i}=\bar{m},$$

that is, $\Psi_{\Lambda/\Gamma}^{-1}\circ\Phi=1$ holds. Thus $\Psi_{\Lambda/\Gamma}^{-1}$ is an isomorphism. Hence (i) holds.

Let M be a left P-module. Put $N_{\Lambda/K}(M)=\{\sum_{i,j} R_i r_j m l_j L_i\,|\,m\in M\}$ and $N_{\Gamma/K}(\tilde{M})=\{\sum_j r_j m l_j\,|\,m\in\tilde{M}\}$. Let $N_{\Lambda/\Gamma}(M)$ be the same as in the proof of Theorem 7.1. Then by Theorems 6.3 and 7.1 we have

THEOREM 7.2. *The following conditions are equivalent*:

(i) $\Psi_{\Lambda/\Gamma}^r: H_r(\Lambda,\Gamma,M^\Delta)\leftarrow H^{-r-1}(\Lambda,\Gamma,M)$ *is an isomorphism for any left* P-*module* M *and any* $r\in\boldsymbol{Z}$.

(ii) $M^\Lambda/N_{\Lambda/K}(M)=N_{\Gamma/K}(\tilde{M})/N_{\Lambda/K}(M)\oplus N_{\Lambda/\Gamma}(M)/N_{\Lambda/K}(M)$ *holds for any left* P-*module* M.

(iii) $M^\Lambda=N_{\Gamma/K}(\tilde{M})+N_{\Lambda/\Gamma}(M)$ *holds for any left* P-*module* M.

(iv) $\Lambda^\Lambda/N_{\Lambda/K}(\Lambda)=N_{\Gamma/K}(\tilde{\Lambda})/N_{\Lambda/K}(\Lambda)\oplus N_{\Lambda/\Gamma}(\Lambda)/N_{\Lambda/K}(\Lambda)$ *holds.*

PROOF. (i)$\Rightarrow$(ii). By Theorem 6.3 (iii), $H^0(\Lambda, K, M) = \operatorname{Im} \Psi_{\Lambda/K}^{-1} \circ \operatorname{Inf}_{-1} \oplus \operatorname{Ker} \operatorname{Def}^0$ holds. And we have $H^0(\Lambda, K, M) \cong M^{\Lambda}/N_{\Lambda/K}(M)$, $\operatorname{Im} \Psi_{\Lambda/K}^{-1} \circ \operatorname{Inf}_{-1} \cong N_{\Gamma/K}(\tilde{M})/N_{\Lambda/K}(M)$ and $\operatorname{Ker} \operatorname{Def}^0 \cong \operatorname{Im} \operatorname{Cor}^0 \cong N_{\Lambda/\Gamma}(M)/N_{\Lambda/K}(M)$ by (4), Propositions 3.2, Lemma 4.4 and [8, Propositions 1.2 and 2.2]. Hence (ii) holds.

(ii)$\Rightarrow$(iii). This is trivial.

(iii)$\Rightarrow$(i). Put $M=\Lambda$. Then Theorem 7.1 (iii) holds. So (i) holds.

(ii)$\Rightarrow$(iv). Put $M=\Lambda$. Then (iv) holds.

(iv)$\Rightarrow$(i). If (iv) holds, Theorem 7.1 (iii) holds. So (i) holds.

Appendix of section 7.

Let M and N be left Λ-modules. In [3], by using complete (Λ, Γ)-projective resolutions of M, the complete cohomology group $H^r_{(\Lambda,\Gamma)}(M, N)$ is defined for all $r \in \boldsymbol{Z}$. Similarly, when M is a right Λ-module, the complete homology group $H_r^{(\Lambda,\Gamma)}(M, N)$ is defined for all $r \in \boldsymbol{Z}$. For a left Λ-module M, $\operatorname{Hom}({}_K M, {}_K K)$ is regarded as a right Λ-module with the usual way, which we denote by M^*. Let N^0 be the module N with a different structure as a left Λ-module such that $\lambda \cdot n = \nabla(\lambda) n$ for $\lambda \in \Lambda$ and $n \in N$. Then for the left P-module $M^* \otimes_K N$, we have $H_r(\Lambda, \Gamma, (M^* \otimes_K N)^{\Delta}) \cong H_r^{(\Lambda,\Gamma)}(M^*, N^0)$ by the simple argument. Similarly $H^r(\Lambda, \Gamma, \operatorname{Hom}({}_K M, {}_K N)) \cong H^r_{(\Lambda,\Gamma)}(M, N)$ holds. Hence if $\Psi^r_{\Lambda/\Gamma}$ is an isomorphism and $M^* \otimes_K N \cong \operatorname{Hom}({}_K M, {}_K N)$ holds as left P-modules, we have $H_r^{(\Lambda,\Gamma)}(M^*, N^0) \cong H^{-r-1}_{(\Lambda,\Gamma)}(M, N)$. For the case of $\Gamma = K$, this means that we obtain the same result as [7, Satz 2] and [9, Theorem 10]. When we return to general cases, we have

THEOREM 7.3. *Let M be a left Λ-module. Assume that $\Psi^r_{\Lambda/\Gamma}$ is an isomorphism for any left P-modules and any $r \in \boldsymbol{Z}$. Then if M is finitely generated and projective as a K-module, we have*

$$H_r^{(\Lambda,\Gamma)}(M^*, N^0) \cong H^{-r-1}_{(\Lambda,\Gamma)}(M, N)$$

for any left Λ-module N and any $r \in \boldsymbol{Z}$.

PROOF. Let $\kappa_M : M^* \otimes_K N \to \operatorname{Hom}({}_K M, {}_K N)$ be a left P-homomorphism such that $\kappa_M(g \otimes_K n) = [m \to g(m) n]$. If M is finitely generated and projective as a K-module, it is easy to see that κ_M is an isomorphism. Hence $H_r^{(\Lambda,\Gamma)}(M^*, N^0) \cong H^{-r-1}_{(\Lambda,\Gamma)}(M, N)$ holds for any left Λ-module N and any $r \in \boldsymbol{Z}$.

8. The necessary and sufficient conditions for
$H_r(G, K, -)\cong H^{-r-1}(G, K, -)$.

Let G be a group and K a subgroup of finite index in G. Then in [6, section 4], from complete (ZG, ZK)-resolutions of Z the complete relative homology group $H_r(G, K, M)$ and cohomology group $H^r(G, K, M)$ are defined for $-\infty<r<\infty$ where M is a left G-module. In this section we treat the case where G is a finite group.

For a subgroup K, let $G=\bigcup_{i=1}^n g_iK$ be a left coset decomposition and $H: ZG\to ZK$ a two-sided ZK-homomorphism such that for $g\in G$

$$H(g)=\begin{cases}0 & g\notin K,\\ g & g\in K.\end{cases}$$

Then for $x\in ZG$ $x=\sum_{i=1}^n g_iH(g_i^{-1}x)=\sum_{i=1}^n H(xg_i)g_i^{-1}$ holds. So the ring extension ZG/ZK is a Frobenius extension. Let 1 be the unit element of G, and $h: ZK\to Z$ a Z-homomorphism such that for $k\in K$

$$h(k)=\begin{cases}0 & k\neq 1,\\ 1 & k=1.\end{cases}$$

Then for $x\in ZK$ $x=\sum_{k\in K} kh(k^{-1}x)=\sum_{k\in K} h(xk)k^{-1}$ holds. So ZK is a Frobenius Z-algebra. Hence ZG is a Frobenius Z-algebra. Therefore we have the Nakayama automorphism $\Delta: ZG\to ZG$. By the definition in section 1 Δ is the identity homomorphism of ZG. We put $P=ZG\otimes_Z(ZG)^o$, and let Q and S be the images of the natural homomorphisms $ZK\otimes_Z(ZG)^o\to P$ and $ZK\otimes_Z(ZK)^o\to P$ respectively as in the previous sections.

We have the augmentation map $\varepsilon: ZG\to Z$ with $\varepsilon(g)=1$ for $g\in G$. Let M be a left G-module. Then by putting $(x_0\otimes_Z x_1)\cdot m=x_0m\varepsilon(x_1)$ for $x_0\otimes_Z x_1\in P$ and $m\in M$, M can be regarded as a left P-module, and then we shall denote M by M_ε. As in the previous sections, we have the Z-homomorphism $\Psi^r_{ZG/ZK}$: $H_r(ZG, ZK, M_\varepsilon)\to H^{-r-1}(ZG, ZK, M_\varepsilon)$ for $-\infty<r<\infty$. Since $H_r(ZG, ZK, M_\varepsilon)\cong H_r(G, K, M)$ and $H^r(ZG, ZK, M_\varepsilon)\cong H^r(G, K, M)$ hold, we have a Z-homomorphism for all $r\in Z$:

$$\Psi^r_{G/K}: H_r(G, K, M)\longrightarrow H^{-r-1}(G, K, M).$$

Let M be a left G-module. $ZG\otimes_{ZK}M$ and $\mathrm{Hom}({}_{ZK}ZG, {}_{ZK}M)$ are left ZG-modules with the usual way. Then there are (P, Q)-exact sequences

$$0\longrightarrow (\mathrm{Ker}\,\xi)_\varepsilon\longrightarrow (ZG\otimes_{ZK}M)_\varepsilon\xrightarrow{\ \xi\ } M_\varepsilon\longrightarrow 0,$$

$$0 \longrightarrow M_\varepsilon \xrightarrow{\tau} (\mathrm{Hom}\,({}_{ZK}ZG,\ {}_{ZK}M))_\varepsilon \longrightarrow (\mathrm{Coker}\,\tau)_\varepsilon \longrightarrow 0\,,$$

where $\xi(x\otimes_{ZK}m)=xm$ and $\tau(m)=[x\to xm]$ for $x\in ZG$ and $m\in M$. Since $(ZG\otimes_{ZK}M)_\varepsilon=ZG\otimes_{ZK}M_\varepsilon\cong P\otimes_Q M_\varepsilon$ and so $(\mathrm{Hom}\,({}_{ZK}ZG,\ {}_{ZK}M))_\varepsilon\cong(ZG\otimes_{ZK}M)_\varepsilon\cong P\otimes_Q M_\varepsilon$ holds, $(ZG\otimes_{ZK}M)_\varepsilon$ and $(\mathrm{Hom}\,({}_{ZK}ZG,\ {}_{ZK}M))_\varepsilon$ are (P, Q)-projective. Therefore by using Proposition 2.1 for these exact sequences, since the connecting homomorphisms are isomorphisms by Lemma 4.2, $\Psi^r_{ZG/ZK}: H_r(ZG, ZK, M_\varepsilon)\to H^{-r-1}(ZG, ZK, M_\varepsilon)$ is an isomorphism for any $r\in Z$ and any left G-module M if and only if it is an isomorphism for some r and any left G-module M.

Z is regarded as a left G-module by ε with the usual way. Then Z_ε is a left P-module. We defined $\tilde{Z}_\varepsilon$ as $\{z\in Z_\varepsilon\,|\,(1\otimes_{ZK}1)\otimes_P z\in C^{ZG/ZK}_{-1}(Z_\varepsilon)\}$ in section 7. $\tilde{Z}_\varepsilon$ is an ideal of the ring Z, and we have

LEMMA 8.1. *Let $|K|$ be the order of K and $(G:K)$ the index of K in G. Then $\tilde{Z}_\varepsilon$ contains $|K|$ and $(G:K)$.*

PROOF. Let $G=\bigcup_{i=1}^{(G:K)} g_iK$ be a left coset decomposition. Then by Proposition 1.1 we have $B^{ZG/ZK}_{-1}(Z_\varepsilon)=\{\sum_i (g_i\otimes_{ZK}g_i^{-1})\otimes_P z\,|\,z\in Z_\varepsilon\}=\{(1\otimes_{ZK}1)\otimes_P (G:K)z\,|\,z\in Z_\varepsilon\}$. Since $B^{ZG/ZK}_{-1}(Z_\varepsilon)\subset C^{ZG/ZK}_{-1}(Z_\varepsilon)$ holds, $\tilde{Z}_\varepsilon$ contains $(G:K)$. In $(ZG\otimes_{ZK}ZG\otimes_{ZK}ZG)\otimes_P Z_\varepsilon$, we have $\sum_i (g_i\otimes_{ZK}g_i^{-1}\otimes_{ZK}1)\otimes_P|K|-\sum_i (1\otimes_{ZK}g_i\otimes_{ZK}g_i^{-1})\otimes_P|K|=\sum_i (1\otimes_{ZK}g_i^{-1}\otimes_{ZK}1)\otimes_P|K|-\sum_i (1\otimes_{ZK}g_i\otimes_{ZK}1)\otimes_P|K|=\sum_i\sum_{k\in K}(1\otimes_{ZK}k^{-1}g_i^{-1}\otimes_{ZK}1)\otimes_P 1-\sum_i\sum_{k\in K}(1\otimes_{ZK}g_ik\otimes_{ZK}1)\otimes_P 1=\sum_{g\in G}(1\otimes_{ZK}g\otimes_{ZK}1)\otimes_P 1-\sum_{g\in G}(1\otimes_{ZK}g\otimes_{ZK}1)\otimes_P 1=0$. Thus $|K|\in\tilde{Z}_\varepsilon$ holds by Proposition 1.1.

THEOREM 8.2. *The following conditions are equivalent*:

(i) *$\Psi^r_{G/K}: H_r(G, K, M)\to H^{-r-1}(G, K, M)$ is an isomorphism for any left G-module M and any $r\in Z$.*

(ii) *$\Psi^r_{G/K}: H_r(G, K, M)\to H^{-r-1}(G, K, M)$ is an epimorphism for any left G-module M and any $r\in Z$.*

(iii) *K is a Hall subgroup of G, that is, there are $t, z\in Z$ such that $1=|K|t+(G:K)z$.*

PROOF. The proof is similar to the proof of Theorem 7.1.

(i)$\Rightarrow$(ii). This is obvious.

(ii)$\Rightarrow$(iii). Let M be a left G-module. For the P-module M_ε, (22) in the proof of Theorem 7.1 is

$$\Psi^{-1}_{ZG/ZK}(\overline{(1\otimes_{ZK}1)\otimes_P m})=\overline{\textstyle\sum_{k\in K} km\varepsilon(k^{-1})}=\overline{\textstyle\sum_{k\in K} km}$$

for $(1\otimes_{ZK}1)\otimes_P m\in C^{ZG/ZK}_{-1}(M_\varepsilon)$. Regard Z as a left G-module by ε with the usual way and put $M=\boldsymbol{Z}$. Then $\Psi_{ZG/ZK}^{-1}$ is a homomorphism from $C_{-1}^{ZG/ZK}(\boldsymbol{Z}_{\varepsilon})/B_{-1}^{ZG/ZK}(\boldsymbol{Z}_{\varepsilon})$ into $(\boldsymbol{Z}_{\varepsilon})^{ZG}/N_{ZG/ZK}(\boldsymbol{Z}_{\varepsilon})$. $(\boldsymbol{Z}_{\varepsilon})^{ZG}$ is $\boldsymbol{Z}_{\varepsilon}$, and $N_{ZG/ZK}(\boldsymbol{Z}_{\varepsilon})$ is $(G:K)\boldsymbol{Z}_{\varepsilon}$. Therefore since (ii) holds, there is $t\in\tilde{\boldsymbol{Z}}_{\varepsilon}$ such that $\bar{1}=\Psi_{ZG/ZK}^{-1}(\overline{(1\otimes_{ZK}1)\otimes_{P}t})$, that is, $\bar{1}=\overline{\sum_{k\in K}kt}=\overline{|K|t}$ holds. Hence there is $z\in\boldsymbol{Z}$ such that $1=|K|t+(G:K)z$.

(iii)$\Rightarrow$(i). By the argument before Lemma 8.1, (iii)$\Rightarrow$(i) holds if $\Psi_{ZG/ZK}^{-1}$: $C_{-1}^{ZG/ZK}(M_{\varepsilon})/B_{-1}^{ZG/ZK}(M_{\varepsilon})\to(M_{\varepsilon})^{ZG}/N_{ZG/ZK}(M_{\varepsilon})$ is an isomorphism for any left G-module M. If (iii) holds, since $\tilde{\boldsymbol{Z}}_{\varepsilon}$ is an ideal of $\boldsymbol{Z}$, by Lemma 8.1 we have $1\in\tilde{\boldsymbol{Z}}_{\varepsilon}$, and so $t=t\cdot 1\in\tilde{\boldsymbol{Z}}_{\varepsilon}$ holds. Therefore when we define a $\boldsymbol{Z}$-homomorphism $\varphi:(M_{\varepsilon})^{ZG}\to C_{-1}^{ZG/ZK}(M_{\varepsilon})$ such that $\varphi(m)=(1\otimes_{ZK}1)\otimes_{P}tm$, it can be shown by the same procedure as the proof of Theorem 7.1 that φ is well-defined, φ induces the $\boldsymbol{Z}$-homomorphism $\Phi:(M_{\varepsilon})^{ZG}/N_{ZG/ZK}(M_{\varepsilon})\to C_{-1}^{ZG/ZK}(M_{\varepsilon})/B_{-1}^{ZG/ZK}(M_{\varepsilon})$, and Φ is the inverse isomorphism of $\Psi_{ZG/ZK}^{-1}$. Hence (i) holds.

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