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ON THE COMPLETE RELATIVE COHOMOLOGY OF FROBENIUS EXTENSIONS

By

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Introduction.

Let Λ be an algebra over a commutative ring K and Γ a subalgebra. Suppose that the extension Λ/Γ is a Frobenius extension. Then in [3, section 3], the complete relative cohomology group $H^{r}_{(\Lambda,\Gamma)}(M, -)$ is introduced for an arbitrary left Λ-module M and $r\in \boldsymbol{Z}$. We denote the opposite rings of Λ and Γ by Λ^{0} and Γ^{0} respectively. Put $P=\Lambda\otimes_{K}\Lambda^{0}$ and let S denote the natural image of $\Gamma\otimes_{K}\Gamma^{0}$ in P. Then the extension P/S is also a Frobenius extension. Since Λ is a left P-module with the natural way, we have $H^{r}_{(P,S)}(\Lambda, -)$. We will denote this $H^{r}_{(P,S)}(\Lambda, -)$ by $H^{r}(\Lambda, \Gamma, -)$ for [6, section 3]. In this paper, we will study this complete relative cohomology $H(\Lambda, \Gamma, -)$. In section 1, we will study relative complete resolutions of Λ and in section 2, we will introduce the dual of the fundamental exact sequence of [4, Proposition 1 and Theorem 1] for complete relative cohomology groups. In section 3, we will study an internal product like as in [9, section 2] which we will call the cup product. If the basic ring of the Frobenius extension is commutative, the cup product in this paper coincides with the product $\vee$ in [2, Exercise 2 of Chapter XI] for dimension>0.

1. Relative complete resolutions.

Let P be a ring and S a subring such that the extension P/S is a Frobenius extension. In [3], the complete (P, S)-resolution of a left P-module M is introduced. It is a (P, S)-exact sequence $\cdots\rightarrow X_{1}\xrightarrow{d_1} X_{0}\xrightarrow{d_0} X_{-1}\xrightarrow{d_{-1}}\cdots$ such that X_{n} is (P, S)-projective for all $n\in \boldsymbol{Z}$ and there exist a P-epimorphism $\varepsilon: X_{0}\rightarrow M$ and a P-monomorphism $\eta: M\rightarrow X_{-1}$ which satisfy $\eta\circ\varepsilon=d_{0}$, that is, the complete (P, S)-resolution of M is an exact sequence which consists of a (P, S)-projective resolution and a (P, S)-injective resolution of M since (P, S)-projectivity is equivalent to (P, S)-injectivity. Note that any two complete (P, S)-resolutions

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of M denoted by $\mathcal{U}$ and $\mathcal{U}'$ have the same homotopy type, i. e., for chain maps $F: \mathcal{U}\to\mathcal{U}'$ and $G: \mathcal{U}'\to\mathcal{U}$ over the identity map 1_M, $F\circ G$ and $G\circ F$ are homotopic to $1_{\mathcal{U}'}$ and $1_{\mathcal{U}}$ respectively. Therefore for any subring Q of P, if there exists a complete (P, S)-resolution of M which has a contracting Q-homotopy in addition to the contracting S-homotopy, any complete (P, S)-resolution of M also has a contracting Q-homotopy. Especially if P/Q is also a Frobenius extension such that $Q\supseteq S$ holds and there exists a complete (P, S)-resolution with a contracting Q-homotopy, all complete (P, S)-resolutions of M are complete (P, Q)-resolutions of M since (P, S)-projective modules are (P, Q)-projective modules.

Let Λ be an algebra over a commutative ring K and Γ be a subalgebra of Λ. We suppose that the extension Λ/Γ is a Frobenius extension, that is to say, there exist elements of Λ denoted by $\{r_1, \cdots, r_n\}$, $\{l_1, \cdots, l_n\}$ and a Γ-Γ-homomorphism $h\in \mathrm{Hom}({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ such that $x=\sum_{i=1}^n h(xr_i)l_i=\sum_{i=1}^n r_i h(l_i x)$ for all $x\in\Lambda$. Let Λ^0 and Γ^0 be the opposite rings of Λ and Γ respectively. Put $P=\Lambda\otimes_K\Lambda^0$ and let Q, R and S be the images of natural homomorphisms $\Gamma\otimes_K\Lambda^0\to P$, $\Lambda\otimes_K\Gamma^0\to P$ and $\Gamma\otimes_K\Gamma^0\to P$ respectively. Then the extensions P/Q, P/R and P/S are Frobenius extensions. We regard Λ as a left P-module with the natural way.

PROPOSITION 1.1. *Any complete (P, S)-resolution of Λ has a contracting Q-homotopy and a contracting R-homotopy in addition to the contracting S-homotopy.*

PROOF. We can prove this proposition by constructing such a complete (P, S)-resolution of Λ. Let

$$(1) \qquad \cdots \longrightarrow X_r \xrightarrow{b_r} X_{r-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{b_1} X_0 \xrightarrow{\varepsilon} \Lambda \longrightarrow 0$$

be a (P, S)-projective resolution of Λ such that $X_r=\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$ ($r+2$ copies), $b_r(x_0\otimes\cdots\otimes x_{r+1})=\sum_{i=0}^r(-1)^{r-i}x_0\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_{r+1}$ and $\varepsilon(x_0\otimes x_1)=x_0x_1$. Note that (1) has two types of contracting S-homotopy. The one is a contracting Q-homotopy such that $x_0\otimes\cdots\otimes x_{r+1}\to(-1)^{r+1}1\otimes x_0\otimes\cdots\otimes x_{r+1}$. The other is a contracting R-homotopy such that $x_0\otimes\cdots\otimes x_{r+1}\to x_0\otimes\cdots\otimes x_{r+1}\otimes 1$. $\mathrm{Hom}({}_\Lambda X_r, {}_\Lambda\Lambda)$ and $\mathrm{Hom}(X_{r\Lambda}, \Lambda_\Lambda)$ are regarded as left P-modules by setting $((x\otimes y)\cdot f)(\ )=f((\ )x)y$ and $((x\otimes y)\cdot g)(\ )=xg(y(\ ))$ for $x\otimes y\in P$, $f\in\mathrm{Hom}({}_\Lambda X_r, {}_\Lambda\Lambda)$ and $g\in\mathrm{Hom}(X_{r\Lambda}, \Lambda_\Lambda)$. Applying the functors $\mathrm{Hom}({}_\Lambda -, {}_\Lambda\Lambda)$ and $\mathrm{Hom}(-_\Lambda, \Lambda_\Lambda)$ to (1), we have a (P, Q)-exact sequence and a (P, R)-exact sequence respectively. Let φ_r and ϕ_r denote P-isomorphisms $\mathrm{Hom}({}_\Lambda X_r, {}_\Lambda\Lambda)\simeq\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$ ($r+2$ copies) and $\mathrm{Hom}(X_{r\Lambda}, \Lambda_\Lambda)\simeq\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$($r+2$ copies) respectively such that

$$\varphi_r(f)=\sum_{1\le i_0\le n,\cdots,1\le i_r\le n} r_{i_0}\otimes\cdots\otimes r_{i_r}\otimes f(1\otimes l_{i_r}\otimes\cdots\otimes l_{i_0}),$$

$$\varphi_r^{-1}(\lambda_0\otimes\cdots\otimes\lambda_{r+1})=[x_0\otimes\cdots\otimes x_{r+1}\to x_0h(x_1h(\cdots h(x_rh(x_{r+1}\lambda_0)\lambda_1)\cdots)\lambda_r)\lambda_{r+1}],$$

$$\phi_r(g)=\sum_{1\leq i_0\leq n,\dots,1\leq i_r\leq n}g(r_{i_0}\otimes\cdots\otimes r_{i_r}\otimes 1)\otimes l_{i_r}\otimes\cdots\otimes l_{i_0},$$

$$\phi_r^{-1}(\lambda_0\otimes\cdots\otimes\lambda_{r+1})=[x_0\otimes\cdots\otimes x_{r+1}\to\lambda_0h(\lambda_1h(\cdots h(\lambda_rh(\lambda_{r+1}x_0)x_1)\cdots)x_r)x_{r+1}].$$

Since P/S is a Frobenius extension, (P, S)-projective module $\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$ is (P, S)-injective. Therefore we have two (P, S)-injective resolutions of Λ such that the one has a contracting Q-homotopy and the other has a contracting R-homotopy. But since $\varphi_{r+1}(\varphi_r^{-1}(\lambda_0\otimes\cdots\otimes\lambda_{r+1})\circ b_{r+1})=\phi_{r+1}(\phi_r^{-1}(\lambda_0\otimes\cdots\otimes\lambda_{r+1})\circ b_{r+1})$ holds for all $\lambda_0\otimes\cdots\otimes\lambda_{r+1}\in\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$ ($r+2$ copies), two (P, S)-injective resolutions are same. Connecting this resolution with the standard (P, S)-projective resolution of Λ that is (1) which has $(-1)^rb_r$ instead of b_r as the differentiation, we have a complete (P, S)-resolution of Λ which we want:

$$(2)\quad \cdots\to X_r\xrightarrow{d_r}X_{r-1}\to\cdots\to X_1\xrightarrow{d_1}X_0\xrightarrow{d_0}X_{-1}\xrightarrow{d_{-1}}\cdots\to X_{-r}\xrightarrow{d_{-r}}X_{-(r+1)}\to\cdots.$$
$$X_0\overset{\varepsilon}{\searrow}\Lambda\overset{\eta}{\nearrow}X_{-1}$$

Here we set $d_r=(-1)^rb_r$ and $X_{-r}=\Lambda\otimes_\Gamma\cdots\otimes_\Gamma\Lambda$ ($r+1$ copies) for $r\geqq 1$, and η, d_0 and d_{-r} are given by $\eta(x)=\sum_i r_i\otimes l_ix$, $d_0(x_0\otimes x_1)=\eta\circ\varepsilon(x_0\otimes x_1)=\sum_i x_0r_i\otimes l_ix_1$ and $d_{-r}(x_0\otimes\cdots\otimes x_r)=\sum_{i=0}^r\sum_j(-1)^ix_0\otimes\cdots\otimes x_{i-1}\otimes r_j\otimes l_jx_i\otimes\cdots\otimes x_r$. Let denote the contracting Q-homotopy of (2) by D^Q. D_r^Q: $X_r\to X_{r+1}$ is given by $D_r^Q(x_0\otimes\cdots\otimes x_{r+1})=1\otimes x_0\otimes\cdots\otimes x_{r+1}$ for $r\geqq 0$, $D_{-1}^Q(x_0\otimes x_1)=h(x_0)\otimes x_1$ and $D_{-r}^Q(x_0\otimes\cdots\otimes x_r)=h(x_0)x_1\otimes\cdots\otimes x_r$ for $r\geqq 2$. Let denote the contracting R-homotopy of (2) by D^R. D_r^R: $X_r\to X_{r+1}$ is given by $D_r^R(x_0\otimes\cdots\otimes x_{r+1})=(-1)^{r+1}x_0\otimes\cdots\otimes x_{r+1}\otimes 1$ for $r\geqq 0$, $D_{-1}^R(x_0\otimes x_1)=x_0\otimes h(x_1)$ and $D_{-r}^R(x_0\otimes\cdots\otimes x_r)=(-1)^{r+1}x_0\otimes\cdots\otimes x_{r-1}h(x_r)$ for $r\geqq 2$.

We can see other complete (P, S)-resolutions of Λ in [3], [5] and [8].

Let M be a left P-module and $(X, d, \varepsilon, \eta)$ be any complete (P, S)-resolution of Λ. Then we have the following sequnce:

$$\cdots\longleftarrow \mathrm{Hom}({}_PX_1, {}_PM)\xleftarrow{d_1^*}\mathrm{Hom}({}_PX_0, {}_PM)\xleftarrow{d_0^*}\mathrm{Hom}({}_PX_{-1}, {}_PM)\xleftarrow{d_{-1}^*}\cdots$$

where we set $d_r^*(f)=f\circ d_r$ for $f\in\mathrm{Hom}({}_PX_r, {}_PM)$. The r-th complete relative cohomology group $H^r(\Lambda, \Gamma, M)$ with coefficients in M is given by $H^r(\Lambda, \Gamma, M)=\mathrm{Ker}\, d_{r+1}^*/\mathrm{Im}\, d_r^*$. We put $H^*(\Lambda, \Gamma, M)=\oplus_{r\in Z}H^r(\Lambda, \Gamma, M)$. Let $Z(\Lambda)$ be the center of Λ. Then $\mathrm{Hom}({}_PX_r, {}_PM)$ becomes a $Z(\Lambda)$-module by setting $(c\cdot f)(\ )=cf(\ )$ for $c\in Z(\Lambda)$. Therefore $H^r(\Lambda, \Gamma, M)$ is a $Z(\Lambda)$-module. It is obvious that $H^r(\Lambda, \Gamma, M)$ is independent of the choice of complete (P, S)-resolutions of Λ.

PROPOSITION 1.2. *Put* $M^{\Lambda}=\{m\in M \mid xm=mx \text{ for all } x\in\Lambda\}$, $M^{\Gamma}=\{m\in M \mid xm=mx \text{ for all } x\in\Gamma\}$ *and* $N_{\Lambda/\Gamma}(M)=\{\sum_i r_i m l_i \mid m\in M^{\Gamma}\}$. *Then* $H^0(\Lambda, \Gamma, M)\cong M^{\Lambda}/N_{\Lambda/\Gamma}(M)$ *holds as* $Z(\Lambda)$*-modules.*

PROOF. Take (2) as a complete (P, S)-resolution of Λ and let f be the representative of an elemant $\alpha\in H^0(\Lambda, \Gamma, M)$. Then the isomorphism $H^0(\Lambda, \Gamma, M)\simeq M^{\Lambda}/N_{\Lambda/\Gamma}(M)$ is given by $\alpha\rightarrow f(1\otimes 1)+N_{\Lambda/\Gamma}(M)$.

2. The dual of the fundamental exact sequence.

Let Λ/Γ be a Frobenius extension of K-algebras and P, Q, R, S, $\{r_i\}$, $\{l_i\}$ and h be the same as in section 1. Suppose that Γ/K is also a Frobenius extension in section 2. Note that Λ/K is a Frobenius extension and Q, R and S are isomorphic to $\Gamma\otimes_K\Lambda^o$, $\Lambda\otimes_K\Gamma^o$ and $\Gamma\otimes_K\Gamma^o$ respectively. We have a complete (P, K)-resolution of Λ and a complete (S, K)-resolution of Γ. We denote them by Y and Z respectively.

Now we treat the restriction homomorphism and the corestriction homomorphism introduced in [10] briefly. Let M be a left P-module. Since Y and $Z\otimes_\Gamma\Lambda$ are regarded as complete (Q, K)-resolutions of Λ, $H^r(\mathrm{Hom}({}_QY, {}_QM))\cong H^r(\mathrm{Hom}({}_QZ\otimes_\Gamma\Lambda, {}_QM))$ holds. Since $H^r(\mathrm{Hom}({}_QZ\otimes_\Gamma\Lambda, {}_QM))\cong H^r(\mathrm{Hom}({}_SZ, {}_SM))=H^r(\Gamma, K, M)$ holds, we have an isomorphism

$$(3)\qquad s_r:\ H^r(\mathrm{Hom}({}_QY, {}_QM)) \xrightarrow{\sim} H^r(\Gamma, K, M).$$

Composing s_r with the homomorphism induced by the natural map $\mathrm{Hom}({}_PY_r, {}_PM)\rightarrow\mathrm{Hom}({}_QY_r, {}_QM)$, we obtain the restriction homomorphism $\mathrm{Res}^r: H^r(\Lambda, K, M)\rightarrow H^r(\Gamma, K, M)$. Composing s_r^{-1} with the homomorphism induced by the homomorphism $N_{\Lambda/\Gamma}: \mathrm{Hom}({}_QY_r, {}_QM)\rightarrow\mathrm{Hom}({}_PY_r, {}_PM)$ defined by $N_{\Lambda/\Gamma}(f)(\ )=\sum_i r_i f(l_i(\ ))$, we obtain the corestriction homomorphism $\mathrm{Cor}^r: H^r(\Gamma, K, M)\rightarrow H^r(\Lambda, K, M)$.

Next let X be a complete (P, S)-resolution of Λ. Dividing X and Y into the non-negative parts and the negative parts, that is, the relative projective resolutions of Λ and the relative injective resolutions of Λ, then the identity homomorphism of Λ derives a commutative diagram

$$(4)\qquad \begin{array}{ccccccccc}
\cdots\longrightarrow & Y_1 & \xrightarrow{c_1} & Y_0 & \xrightarrow{c_0} & Y_{-1} & \xrightarrow{c_{-1}} & Y_{-2} & \longrightarrow\cdots \\
& \sigma_1\downarrow & & \sigma_0\downarrow & \searrow\ \Lambda\ \nearrow & \uparrow\sigma_{-1} & & \uparrow\sigma_{-2} & \\
& & & & \nearrow\ \ \ \searrow & & & & \\
\cdots\longrightarrow & X_1 & \xrightarrow[d_1]{} & X_0 & \xrightarrow[d_0]{} & X_{-1} & \xrightarrow[d_{-1}]{} & X_{-2} & \longrightarrow\cdots
\end{array}$$

and applying the functor $\mathrm{Hom}({}_P-, {}_PM)$ to (4), σ_r induces homomorphisms $\mathrm{Inf}^r: H^r(\Lambda, \Gamma, M)\to H^r(\Lambda, K, M)$ for $r\geqq 1$ and $\mathrm{Def}^r: H^r(\Lambda, K, M)\to H^r(\Lambda, \Gamma, M)$ for $r\leqq -1$. We will call them the inflation homomorphism and the deflation homomorphism respectively. We can define $\mathrm{Def}^0: H^0(\Lambda, K, M)\to H^0(\Lambda, \Gamma, M)$, that is, $\mathrm{Def}^0: \mathrm{Ker}\, c_1^*/\mathrm{Im}\, c_0^*\to \mathrm{Ker}\, d_1^*/\mathrm{Im}\, d_0^*$ since $\mathrm{Ker}\, c_1^*\cong\mathrm{Hom}({}_P\Lambda, {}_PM)\cong\mathrm{Ker}\, d_1^*$ holds and $\mathrm{Im}\, d_0^*$ contains the image of $\mathrm{Im}\, c_0^*$. If we identify $H^0(\Lambda, K, M)$ and $H^0(\Lambda, \Gamma, M)$ with $M^\Lambda/N_{\Lambda/K}(M)$ and $M^\Lambda/N_{\Lambda/\Gamma}(M)$ respectively by Proposition 1.2, $\mathrm{Def}^0(m+N_{\Lambda/K}(M))=m+N_{\Lambda/\Gamma}(M)$ holds.

Note that Res, Cor, Inf and Def are independent of the choice of relative complete resolutions.

Now we treat on the fundamental exact sequeuce introduced in [4]. Let A be an arbitrary ring and B a subring. By U, V and W we denote a B-projective, an A-projective and an (A, B)-projective resolution of a left A-module M respectively. Then the identity homomorphism of M induces the chain maps $U\to V$ and $V\to W$. They induce $\mathrm{res}^r: \mathrm{Ext}_A^r(M, N)\to \mathrm{Ext}_B^r(M, N)$ and $\mathrm{inf}^r: \mathrm{Ext}_{(A,B)}^r(M, N)\to\mathrm{Ext}_A^r(M, N)$ for $r\geqq 0$ by the natural way where N is any left A-module. Consider $\mathrm{Hom}({}_BA, {}_BN)$ as a left A-module by $(a\cdot f)(\ )=f((\ )a)$ for $a\in A$, $f\in\mathrm{Hom}({}_BA, {}_BN)$. Define left A-modules $N^i\,(i\geqq 0)$ inductively as $N^0=N$ and $N^i=\mathrm{Hom}({}_BA, {}_BN^{i-1})$ for $i\geqq 1$. Then in [4], it is proved that the sequence

$$0\longrightarrow \mathrm{Ext}_{(A,B)}^r(M, N)\xrightarrow{\mathrm{inf}^r}\mathrm{Ext}_A^r(M, N)\xrightarrow{\mathrm{res}^r}\mathrm{Ext}_B^r(M, N)$$

is exact for $r\geqq 1$ if A is left B-projective and $\mathrm{Ext}_B^n(M, N^{r-n})=0$ $(0<n<r)$.

Let A, B and M be P, Q and Λ respectively. Then the P-projective resolution V is a Q-projective resolution of Λ since P is Q-projective. Therefore we may choose V as U. So res is the homomorphism induced by the natural map $\mathrm{Hom}({}_PV, {}_P-)\to\mathrm{Hom}({}_QV, {}_Q-)$. V is also a (P, K)-projective resolution of Λ since Λ and P are K-projective. Therefore we may consider that V is the non-negative part of a complete (P, K)-resolution of Λ. Hence $\mathrm{Ext}_P^r(\Lambda, -)=H^r(\Lambda, K, -)$ and $s_r\circ\mathrm{res}^r=\mathrm{Res}^r$ hold for $r\geqq 1$ where s_r is the same isomorphism of (3). We know by Proposition 1.1 that the complete (P, S)-resolution of Λ is also a complete (P, Q)-resolution of Λ. Therefore as W we may choose the non-negative part of a complete (P, S)-resolution of Λ. So $\mathrm{Ext}_{(P,Q)}^r(\Lambda, -)=H^r(\Lambda, \Gamma, -)$ and $\mathrm{inf}^r=\mathrm{Inf}^r$ hold for $r\geqq 1$. Thus the following theorem holds:

THEOREM 2.1. *Let N be any left P-module and define P-modules $N^i\,(i\geqq 0)$ inductively as $N^0=N$ and $N^i=\mathrm{Hom}({}_QP, {}_QN^{i-1})$ for $i\geqq 1$. Then the sequence*

$$0\longrightarrow H^r(\Lambda, \Gamma, N)\xrightarrow{\mathrm{Inf}^r}H^r(\Lambda, K, N)\xrightarrow{\mathrm{Res}^r}H^r(\Gamma, K, N)$$

is exact for $r\geqq 1$ *if* $H^n(\Gamma, K, N^{r-n})=0$ $(0<n<r)$.

PROOF. $\mathrm{Ext}^n_Q(\Lambda, N^{r-n})=H^n(\mathrm{Hom}({}_QV, {}_QN^{r-n}))\cong H^n(\Gamma, K, N^{r-n})=0$ holds by (3). Therefore the sequence is exact.

We show the dual of Theorem 2.1 till the end of section 2:

PROPOSITION 2.2. *The following sequence is exact for any left P-module M*:

$$0\longleftarrow H^0(\Lambda, \Gamma, M)\xleftarrow{\mathrm{Def}^0} H^0(\Lambda, K, M)\xleftarrow{\mathrm{Cor}^0} H^0(\Gamma, K, M). \tag{5}$$

PROOF. By Proposition 1.2 the exactness of (5) is equivalent to the exactness of $0\leftarrow M^\Lambda/N_{\Lambda/\Gamma}(M)\xleftarrow{\mathrm{Def}^0} M^\Lambda/N_{\Lambda/K}(M)\xleftarrow{\overline{N_{\Lambda/\Gamma}}} M^\Gamma/N_{\Gamma/K}(M)$ where $\mathrm{Def}^0(m+N_{\Lambda/K}(M))=m+N_{\Lambda/\Gamma}(M)$ and $\overline{N_{\Lambda/\Gamma}}(m+N_{\Gamma/K}(M))=\sum_i r_i m l_i+N_{\Lambda/K}(M)$. This sequence is exact. Therefore (5) is also exact.

LEMMA 2.3. $H^r(\Gamma, K, M)\cong H^r(\Lambda, K, \mathrm{Hom}({}_QP, {}_QM))\cong H^r(\Lambda, K, P\otimes_Q M)$ *holds for any left P-module M and all* $r\in \boldsymbol{Z}$.

PROOF. For a complete (P, K)-resolution Y of Λ, $H^r(\Gamma, K, M)\cong H^r(\mathrm{Hom}({}_QY, {}_QM))$ holds by (3) and $H^r(\mathrm{Hom}({}_QY, {}_QM))\cong H^r(\Lambda, K, \mathrm{Hom}({}_QP, {}_QM))\cong H^r(\Lambda, K, P\otimes_Q M)$ holds.

LEMMA 2.4. *Let* $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$ *be a* (P, S)*-exact sequence. Then we have the following long exact sequence*

$$\cdots\to H^r(\Lambda, \Gamma, L)\longrightarrow H^r(\Lambda, \Gamma, M)\longrightarrow H^r(\Lambda, \Gamma, N)\xrightarrow{\partial} H^{r+1}(\Lambda, \Gamma, L)\to\cdots$$

where ∂ *is the connecting homomorphism. We have similar long exact sequences for* $H^*(\Lambda, K, -)$ *and* $H^*(\Gamma, K, -)$.

PROOF. This can be proved by the usual way for short exact sequences.

LEMMA 2.5. *Let* $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$ *be a* (P, S)*-exact sequence. Then for the connecting homomorphisms* ∂: $H^r(\Lambda, \Gamma, N)\to H^{r+1}(\Lambda, \Gamma, L)$ *and* ∂^Λ: $H^r(\Lambda, K, N)\to H^{r+1}(\Lambda, K, L)$, (i) $\partial\circ\mathrm{Def}^r=\mathrm{Der}^{r+1}\circ\partial^\Lambda$ *holds for* $r\leqq -1$. *Let* $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$ *be a* (P, K)*-exact sequence. Then for the connecting homomorphisms* ∂^Λ: $H^r(\Lambda, K, N)\to H^{r+1}(\Lambda, K, L)$ *and* ∂^Γ: $H^r(\Gamma, K, N)\to H^{r+1}(\Gamma, K, L)$, (ii) $\partial^\Lambda\circ\mathrm{Cor}^r=\mathrm{Cor}^{r+1}\circ\partial^\Gamma$ *holds for all* $r\in\boldsymbol{Z}$.

PROOF. We use (4) for the proof. (i) holds for $r\leqq -2$ by the commutativity of (4). Let φ denote the isomorphism $\mathrm{Ker}\,c_1^*\to\mathrm{Ker}\,d_1^*$ by which we defined Def^0. Then $\varphi\circ(f_*|_{\mathrm{Ker}\,c_1^*})=f_*\circ\varphi$, $\varphi\circ c_0^*=d_0^*\circ\sigma_{-1}^*$ and $\sigma_{-1}^*\circ g_*=g_*\circ\sigma_{-1}^*$ hold where f_* and g_* are homomorphisms induced by f and g respectively with the natural way. Therefore (i) holds for $r=-1$. Let Z be a complete (S, K)-resolution of Λ with a differentiation e. Then Cor is induced by a chain map ϕ: $\mathrm{Hom}({}_SZ, {}_S-)\to\mathrm{Hom}({}_PY, {}_P-)$. $\phi\circ f_*=f_*\circ\phi$, $\phi\circ e^*=c^*\circ\phi$ and $\phi\circ g_*=g_*\circ\phi$ hold. Therefore (ii) also holds.

THEOREM 2.6. *Let M be any left P-module and define P-modules M_i $(i\geqq 0)$ inductively as $M_0=M$ and $M_i=P\otimes_Q M_{i-1}$ for $i\geqq 1$. Then the sequence*

$$0\longleftarrow H^{-r}(\Lambda, \Gamma, M)\xleftarrow{\ \mathrm{Def}^{-r}\ } H^{-r}(\Lambda, K, M)\xleftarrow{\ \mathrm{Cor}^{-r}\ } H^{-r}(\Gamma, K, M)$$

is exact for $r\geqq 0$ if $H^{-n}(\Gamma, K, M_{r-n})=0$ $(0\leqq n\leqq r-1)$.

PROOF. By induction on r. The case of $r=0$ is proved by Proposition 2.2. Assume that the case of $r=t$ holds. Consider the case of $r=t+1$. By M' we denote the kernel of a P-homomorphism d: $M_1\to M$ such that $d(p\otimes m)=pm$. Put $M'_0=M'$ and $M'_i=P\otimes_Q M'_{i-1}$ for all $i\geqq 1$. Then there holds ${}_SM'_i <_{\oplus} {}_SM_{i+1}$ for all $i\geqq 0$. Therefore $H^{-n}(\Gamma, K, M'_{t-n})=0$ holds for $0\leqq n\leqq t$. Hence the following sequence

$$0\longleftarrow H^{-t}(\Lambda, \Gamma, M')\xleftarrow{\ \mathrm{Def}^{-t}\ } H^{-t}(\Lambda, K, M')\xleftarrow{\ \mathrm{Cor}^{-t}\ } H^{-t}(\Gamma, K, M')$$

is exact by the assumption of induction. Note that $H^{-t}(\Gamma, K, M')=0$ holds. The (P, S)-, (P, K)- and (S, K)-exact sequence

$$(6)\qquad 0\longrightarrow M'\longrightarrow M_1\xrightarrow{\ d\ } M\longrightarrow 0$$

induces the following commutative diagram by Lemma 2.5

$$\begin{array}{ccccccc}
0\longleftarrow H^{-t}(\Lambda, \Gamma, M') & \xleftarrow{\ \mathrm{Def}^{-t}\ } & H^{-t}(\Lambda, K, M') & \xleftarrow{\ \mathrm{Cor}^{-t}\ } & H^{-t}(\Gamma, K, M') \\
\uparrow \partial & & \uparrow \partial^{\Lambda} & & \uparrow \partial^{\Gamma} \\
H^{-t-1}(\Lambda, \Gamma, M) & \xleftarrow{\ \mathrm{Def}^{-t-1}\ } & H^{-t-1}(\Lambda, K, M) & \xleftarrow{\ \mathrm{Cor}^{-t-1}\ } & H^{-t-1}(\Gamma, K, M) \\
& & \uparrow \bar{d} & \nearrow \tau & \\
& & H^{-t-1}(\Lambda, K, M_1) & &
\end{array}$$

where ∂, ∂^{Λ} and ∂^{Γ} are connecting homomorphisms for (6), $\bar{d}$ is a homomorphism induced by d and τ is the isomorphism of Lemma 2.3. The isomorphism $H^r(\Lambda, K, M_1)\to H^r(\mathrm{Hom}({}_QY, {}_QM))$ in the proof of Lemma 2.3 is induced by an isomorphism $u: \mathrm{Hom}({}_PY_r, {}_PM_1) \to \mathrm{Hom}({}_QY_r, {}_QM)$ such that $u(f)=\mu\circ f$ where the Q-homomorhism $\mu: M_1\to M$ is defined by $\mu((x\otimes y)\otimes m)=h(x)my$ for $x\otimes y\in P$ and $m\in M$. Therefore $\mathrm{Cor}^{-t-1}\circ\tau=\bar{d}$ holds. M_1 is (P, Q)-injective since P/Q is a Frobenius extension. So by Proposition 1.1, $H^i(\Lambda, \Gamma, M_1)=0$ holds for all $i\in\mathbf{Z}$. Therefore ∂ is an isomorphism. And ∂^Λ is an epimorphism because $H^{-t}(\Lambda, K, M_1)\cong H^{-t}(\Gamma, K, M)$ holds by Lemma 2.3 and $H^{-t}(\Gamma, K, M)=0$ holds by $H^{-t}(\Gamma, K, M)\oplus H^{-t}(\Gamma, K, M')\cong H^{-t}(\Gamma, K, M_1)=0$. Hence for the middle sequence of the above commutative diagram, Theorem 2.6 holds.

3. The cup product on the complete relative cohomology.

The cup product on the complete cohomology of Frobenius algebras is defined in [9]. In this section we will introduce the cup product on the complete relative cohomology of Frobenius extensions. Let Λ/Γ be a Frobenius extension of K-algebras and $P, Q, R, S, \{r_i\}, \{l_i\}, h$ and $Z(\Lambda)$ be the same as in section 1. Γ/K does not need to be a Frobenius extension.

DEFINITION 3.1. Let A and B be any left P-modules and let r and s be any integers. Assume that an element $\alpha\cup\beta\in H^{r+s}(\Lambda, \Gamma, A\otimes_\Lambda B)$ is defined uniquely for every $\alpha\in H^r(\Lambda, \Gamma, A)$ and $\beta\in H^s(\Lambda, \Gamma, B)$. If $\cup$ satisfies the following conditions (i), (ii), (iii) and (iv), we will call $\cup$ the cup product on $H^*(\Lambda, \Gamma, -)$ and call $\alpha\cup\beta$ the cup product of α and β.

(i) $\cup$ induces a $Z(\Lambda)$-homomorphism:

$$H^*(\Lambda, \Gamma, A)\otimes_{Z(\Lambda)}H^*(\Lambda, \Gamma, B) \xrightarrow{\cup} H^*(\Lambda, \Gamma, A\otimes_\Lambda B).$$

(ii) Let $0\to A_1\to A_2\to A_3\to 0$ be a (P, S)-exact sequence and B be a left P-module. If $0\to A_1\otimes_\Lambda B\to A_2\otimes_\Lambda B\to A_3\otimes_\Lambda B\to 0$ is also (P, S)-exact, there holds $\partial(\alpha\cup\beta)=\partial(\alpha)\cup\beta$ for every $\alpha\in H^r(\Lambda, \Gamma, A_3)$ and $\beta\in H^s(\Lambda, \Gamma, B)$, where ∂ denotes the connecting homomorphism.

(iii) Let $0\to B_1\to B_2\to B_3\to 0$ be a (P, S)-exact sequence and A be a left P-module. If $0\to A\otimes_\Lambda B_1\to A\otimes_\Lambda B_2\to A\otimes_\Lambda B_3\to 0$ is also (P, S)-exact, there holds $\partial(\alpha\cup\beta)=(-1)^r\alpha\cup\partial(\beta)$ for every $\alpha\in H^r(\Lambda, \Gamma, A)$ and $\beta\in H^s(\Lambda, \Gamma, B_3)$, where ∂ denotes the conneting homomorphism.

(iv) The diagram

$$\begin{array}{ccc} H^0(\Lambda, \Gamma, A)\otimes_{Z(\Lambda)}H^0(\Lambda, \Gamma, B) & \xrightarrow{\cup} & H^0(\Lambda, \Gamma, A\otimes_\Lambda B) \\ \downarrow & & \downarrow \\ A^\Lambda/N_{\Lambda/\Gamma}(A)\otimes_{Z(\Lambda)}B^\Lambda/N_{\Lambda/\Gamma}(B) & \longrightarrow & (A\otimes_\Lambda B)^\Lambda/N_{\Lambda/\Gamma}(A\otimes_\Lambda B) \end{array}$$

commutes, in which the vertical homomorphisms are isomorphisms by Proposition 1.2 and the homomorphism in the bottom row is defined by

$$(a+N_{\Lambda/\Gamma}(A))\otimes(b+N_{\Lambda/\Gamma}(B)) \longrightarrow a\otimes b+N_{\Lambda/\Gamma}(A\otimes_{\Lambda}B).$$

PROPOSITION 3.2. *If $\cup$ and $\cup'$ satisfy the conditions* (i), (ii), (iii) *and* (iv) *of Definition* 3.1 *respectively, then $\cup=\cup'$ holds.*

PROOF. This proposition is proved by the same method as [1, VI, Lemma 5.8], that is, proved inductively by using the following lemma of dimension-shiftings:

LEMMA 3.3. *Let M be a left P-module. Then we have the following four natural (P, Q)- (or (P, R)-) exact sequences for M:*

$$0 \longrightarrow \mathrm{Ker}\,\phi \longrightarrow P\otimes_{Q}M \xrightarrow{\phi} M \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow \mathrm{Ker}\,\phi' \longrightarrow P\otimes_{R}M \xrightarrow{\phi'} M \longrightarrow 0, \tag{8}$$

$$0 \longrightarrow M \xrightarrow{i} \mathrm{Hom}({}_{Q}P, {}_{Q}M) \longrightarrow \mathrm{Coker}\,i \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow M \xrightarrow{i'} \mathrm{Hom}({}_{R}P, {}_{R}M) \longrightarrow \mathrm{Coker}\,i' \longrightarrow 0 \tag{10}$$

where $\phi(p\otimes m)=pm$, $\phi'(p\otimes m)=pm$, $i(m)=[p\rightarrow pm]$ and $i'(m)=[p\rightarrow pm]$. For any left P-module N, $0\rightarrow\mathrm{Ker}\,\phi\otimes_{\Lambda}N\rightarrow(P\otimes_{Q}M)\otimes_{\Lambda}N\rightarrow M\otimes_{\Lambda}N\rightarrow 0$ is also a (P, Q)-exact sequence. With this sequence and (7) *there hold*

(i) $\partial: H^{r}(\Lambda, \Gamma, M)\cong H^{r+1}(\Lambda, \Gamma, \mathrm{Ker}\,\phi)$,

$\partial: H^{r}(\Lambda, \Gamma, M\otimes_{\Lambda}N)\cong H^{r+1}(\Lambda, \Gamma, \mathrm{Ker}\,\phi\otimes_{\Lambda}N)$

where ∂ is the connecting homomorphism. Similarly there hold

(ii) $\partial: H^{r}(\Lambda, \Gamma, M)\cong H^{r+1}(\Lambda, \Gamma, \mathrm{Ker}\,\phi')$,

$\partial: H^{r}(\Lambda, \Gamma, N\otimes_{\Lambda}M)\cong H^{r+1}(\Lambda, \Gamma, N\otimes_{\Lambda}\mathrm{Ker}\,\phi')$,

(iii) $\partial: H^{r-1}(\Lambda, \Gamma, \mathrm{Coker}\,i)\cong H^{r}(\Lambda, \Gamma, M)$,

$\partial: H^{r-1}(\Lambda, \Gamma, \mathrm{Coker}\,i\otimes_{\Lambda}N)\cong H^{r}(\Lambda, \Gamma, M\otimes_{\Lambda}N)$,

(iv) $\partial: H^{r-1}(\Lambda, \Gamma, \mathrm{Coker}\,i')\cong H^{r}(\Lambda, \Gamma, M)$,

$\partial: H^{r-1}(\Lambda, \Gamma, N\otimes_{\Lambda}\mathrm{Coker}\,i')\cong H^{r}(\Lambda, \Gamma, N\otimes_{\Lambda}M)$

with (8), (9) *and* (10) *respectively.*

PROOF. By Proposition 1.1 any complete (P, S)-resolution or Λ is a (P, Q)-exact sequence. $P\otimes_Q M$ and $(P\otimes_Q M)\otimes_\Lambda N\cong P\otimes_Q(M\otimes_\Lambda N)$ are (P, Q)-injective since P/Q is a Frobenius extension. Therefore $H^*(\Lambda, \Gamma, P\otimes_Q M)=0$ and $H^*(\Lambda, \Gamma, (P\otimes_Q M)\otimes_\Lambda N)=0$ hold. Hence (i) holds. Similar arguments prove (ii), (iii) and (iv).

Note that the cup product is independent of the choice of complete (P, S)-resolutions of Λ.

LEMMA 3.4. *Let $(X, d, \varepsilon, \eta)$ be a complete (P, S)-resolution of Λ. Then for any integers r and s there exists a left P-homomorphism $\Delta_{r,s}: X_{r+s}\to X_r\otimes_\Lambda X_s$ which satisfies the following conditions:*

(i) $(\varepsilon\otimes_\Lambda\varepsilon)\circ\Delta_{0,0}=\varepsilon$,

(ii) $\Delta_{r,s}\circ d_{r+s+1}=(d_{r+1}\otimes_\Lambda 1_{X_s})\circ\Delta_{r+1,s}+(-1)^r(1_{X_r}\otimes_\Lambda d_{s+1})\circ\Delta_{r,s+1}$.

PROOF. This lemma is proved by using the same method as [1, p. 140]: For $n\in\mathbf{Z}$ put $(X\hat{\otimes}_\Lambda X)_n=\prod_{p+q=n}X_p\otimes_\Lambda X_q$ and define $\delta_n: (X\hat{\otimes}_\Lambda X)_n\to(X\hat{\otimes}_\Lambda X)_{n-1}$ by $\delta_n=\prod_{p+q=n}d_p\otimes_\Lambda 1_{X_q}+\prod_{p+q=n}(-1)^p 1_{X_p}\otimes_\Lambda d_q$. Then $(X\hat{\otimes}_\Lambda X, \delta)$ is a chain complex and has a contracting S-homotopy $\prod_{p+q=n}D_p^Q\otimes_\Lambda 1_{X_q}: (X\hat{\otimes}_\Lambda X)_n\to(X\hat{\otimes}_\Lambda X)_{n+1}$ where D^Q is a contracting Q-homotopy of X which exists by Proposition 1.1. Therefore $(X\hat{\otimes}_\Lambda X, \delta)$ is (P, S)-exact. The direct product of relative injectives is relative injective and the (P, S)-projective module $X_p\otimes_\Lambda X_q$ is (P, S)-injective since P/S is a Frobenius extension. So $(X\hat{\otimes}_\Lambda X, \delta)$ is dimension-wise (P, S)-injective. Therefore if there exists a P-homomorphism $\alpha: X_0\to(X\hat{\otimes}_\Lambda X)_0$ such that $(\varepsilon\otimes_\Lambda\varepsilon)\circ\alpha=\varepsilon$ and $\delta_0\circ\alpha\circ d_1=0$ holds, α extends to a chain map $\Delta: X\to X\hat{\otimes}_\Lambda X$ which satisfies the conditions (i) and (ii). Put $\alpha=(\alpha_p)$ where $\alpha_p: X_0\to X_p\otimes_\Lambda X_{-p}$. Then since X_0 is (P, S)-projective, we can take α such that the condition $(\varepsilon\otimes_\Lambda\varepsilon)\circ\alpha=(\varepsilon\otimes_\Lambda\varepsilon)\circ\alpha_0=\varepsilon$ holds. Put $\delta'_{pq}=d_p\otimes_\Lambda 1_{X_q}$ and $\delta''_{pq}=(-1)^p 1_{X_p}\otimes_\Lambda d_q$. Then the condition $\delta_0\circ\alpha\circ d_1=0$ is equivalent to a condition (iii) $\delta'_{p,-p}\circ\alpha_p+\delta''_{p-1,1-p}\circ\alpha_{p-1}=0$ on $\mathrm{Im}\, d_1$ for all $p\in\mathbf{Z}$. Consider the sequence $(X\otimes_\Lambda X_q, \delta'_{\cdot,q})$ for any fixed q. X_q is (P, S)-projective, that is, ${}_P X_q\langle\oplus_P(\Lambda\otimes_\Gamma M\otimes_\Gamma\Lambda)$ holds for an S-module M, and X has a contracting R-homotopy by Proposition 1.1. Therefore $(X\otimes_\Lambda X_q, \delta'_{\cdot,q})$ has a contracting P-homotopy H. Now assume that $p>0$ and that α_{p-1} has been defined. Set $\alpha_p=-H\circ\delta''_{p-1,1-p}\circ\alpha_{p-1}$. Then α_p satisfies the condition (iii). In fact,

$$\delta'\circ\alpha_p+\delta''\circ\alpha_{p-1}=-\delta'\circ H\circ\delta''\circ\alpha_{p-1}+\delta''\circ\alpha_{p-1}$$

$$=H\circ\delta'\circ\delta''\circ\alpha_{p-1} \quad \text{by the definition of } H$$

$$=-H\circ\delta''\circ\delta'\circ\alpha_{p-1} \quad \text{because } \delta' \text{ and } \delta'' \text{ anti-commute}$$

where we have ommitted the subscripts on δ' and δ'' to simplify the notations. If $p=1$, then $H\circ\delta''\circ\delta'\circ\alpha_{p-1}=H\circ(d_0\otimes_\Lambda d_0)\circ\alpha_0=H\circ(\eta\otimes_\Lambda\eta)\circ(\varepsilon\otimes_\Lambda\varepsilon)\circ\alpha_0=H\circ(\eta\otimes_\Lambda\eta)\circ\varepsilon$ $=0$ holds on $\operatorname{Im} d_1$. If $p>1$, then by the inductive hypothesis $\delta'\circ\alpha_{p-1}+\delta''\circ\alpha_{p-2}$ $=0$ holds on $\operatorname{Im} d_1$. So $H\circ\delta''\circ\delta'\circ\alpha_{p-1}=-H\circ\delta''\circ\delta''\circ\alpha_{p-2}=0$ holds on $\operatorname{Im} d_1$. A similar argument constructs α_p for $p<0$ by descending induction. Thus the proof of this lemma is complete.

By Lemma 3.4 we have the cup product of $\alpha\in H^r(\Lambda, \Gamma, A)$ and $\beta\in H^s(\Lambda, \Gamma, B)$: Put $\alpha=\bar{f}$ and $\beta=\bar{g}$ where f and g are representatives. Then the cup product is given by $\alpha\cup\beta=\overline{(f\otimes_\Lambda g)\circ\Delta_{r,s}}$. Thus we obtain the following theorem:

THEOREM 3.5. *There is a cup product uniquely on* $H^*(\Lambda, \Gamma, -)$.

The cup product has the following anti-commutativity:

THEOREM 3.6. *Let M be a P-module. Then for arbitrary* $\alpha\in H^r(\Lambda, \Gamma, \Lambda)$ *and* $\beta\in H^s(\Lambda, \Gamma, M)$, $\alpha\cup\beta=(-1)^{rs}\beta\cup\alpha$ *holds.*

PROOF. Let $(X, d, \varepsilon, \eta)$ be (2) in section 1. Put $\varphi_n=(1_{X_n}\otimes_\Lambda\varepsilon)\circ\Delta_{n,0}$ and $\phi_n=(\varepsilon\otimes_\Lambda 1_{X_n})\circ\Delta_{0,n}$ for any $n\in\mathbf{Z}$ where Δ is the same as in Lemma 3.4. φ: $X\to X$ and ϕ: $X\to X$ are chain maps. Since $\varepsilon=\varepsilon\circ\varphi_0=\varepsilon\circ\phi_0$ holds, φ is homotopic to ϕ, that is, there exists a P-homomorphism ν_n: $X_n\to X_{n+1}$ such that $\varphi_n-\phi_n=\nu_{n-1}\circ d_n+d_{n+1}\circ\nu_n$ holds for all n. Let f and g be representatives of $\alpha\in H^r(\Lambda, \Gamma, \Lambda)$ and $\beta\in H^s(\Lambda, \Gamma, M)$ respectively. Consider the case of $s=0$. Since $g(1\otimes 1)\in M^\Lambda$ holds by Proposition 1.2, there holds $(f\otimes_\Lambda g)\circ\Delta_{r,0}=g(1\otimes 1)f\circ\varphi_r=g(1\otimes 1)f\circ\phi_r+g(1\otimes 1)f\circ\nu_{r-1}\circ d_r=(g\otimes_\Lambda f)\circ\Delta_{0,r}+g(1\otimes 1)f\circ\nu_{r-1}\circ d_r$. Therefore $\alpha\cup\beta$ $=(-1)^0\beta\cup\alpha$ holds for any $r\in\mathbf{Z}$. Since Λ is flat as a left Λ-module and as a right Λ-module, we can use (ii) and (iii) of Definition 3.1. Therefore by using Lemma 3.3 for $H^s(\Lambda, \Gamma, M)$, $\alpha\cup\beta=(-1)^{rs}\beta\cup\alpha$ holds for any r and s.

The cup product has the following associatitivity:

THEOREM 3.7. *Let A, B and C be P-modules. Then for* $\alpha\in H^r(\Lambda, \Gamma, A)$, $\beta\in H^s(\Lambda, \Gamma, B)$ *and* $\gamma\in H^t(\Lambda, \Gamma, C)$, $(\alpha\cup\beta)\cup\gamma=\alpha\cup(\beta\cup\gamma)$ *holds.*

PROOF. We can prove this theorem by the method like the proof of Theorem 3.6: Let $(X, d, \varepsilon, \eta)$ be (2) in section 1. Put

$$\varphi_n=(\varepsilon\otimes_\Lambda 1_{X_n}\otimes_\Lambda\varepsilon)\circ(\Delta_{0,n}\otimes_\Lambda 1_{X_0})\circ\Delta_{n,0} \quad\text{and}\quad \phi_n=(\varepsilon\otimes_\Lambda 1_{X_n}\otimes_\Lambda\varepsilon)\circ(1_{X_0}\otimes_\Lambda\Delta_{n,0})\circ\Delta_{0,n}$$

for $n\in\mathbf{Z}$ where Δ is the same as in Lemma 3.4. $\varphi: X\to X$ and $\phi: X\to X$ are chain maps. Since $\varepsilon=\varepsilon\circ\varphi_0=\varepsilon\circ\phi_0$ holds, φ is homotopic to ϕ, that is, there exists a P-homomorphism $\nu_n: X_n\to X_{n+1}$ which satisfies $\varphi_n-\phi_n=\nu_{n-1}\circ d_n+d_{n+1}\circ\nu_n$. Let f, g and k be representatives of $\alpha\in H^r(\Lambda, \Gamma, A)$, $\beta\in H^s(\Lambda, \Gamma, B)$ and $\gamma\in H^t(\Lambda, \Gamma, C)$ respectively. Consider the case of $r=t=0$. Since $f(1\otimes 1)\in A^\Lambda$ and $k(1\otimes 1)\in C^\Lambda$ hold, there holds

$$\begin{aligned}((f\otimes_\Lambda g)\otimes_\Lambda k)\circ(\Delta_{0,s}\otimes_\Lambda 1_{X_0})\circ\Delta_{s,0}&=f(1\otimes 1)\otimes_\Lambda g\circ\varphi_s\otimes_\Lambda k(1\otimes 1)\\ &=f(1\otimes 1)\otimes_\Lambda g\circ\phi_s\otimes_\Lambda k(1\otimes 1)+(f(1\otimes 1)\otimes_\Lambda g\circ\nu_{s-1}\otimes_\Lambda k(1\otimes 1))\circ d_s\\ &=(f\otimes_\Lambda(g\otimes_\Lambda k))\circ(1_{X_0}\otimes_\Lambda\Delta_{s,0})\circ\Delta_{0,s}+(f(1\otimes 1)\otimes_\Lambda g\circ\nu_{s-1}\otimes_\Lambda k(1\otimes 1))\circ d_s.\end{aligned}$$

Therefore $(\alpha\cup\beta)\cup\gamma=\alpha\cup(\beta\cup\gamma)$ holds for the case of $r=t=0$. By using Lemma 3.3 for $H^r(\Lambda, \Gamma, A)$ and $H^t(\Lambda, \Gamma, C)$, we have $(\alpha\cup\beta)\cup\gamma=\alpha\cup(\beta\cup\gamma)$ for any $r, s, t\in\mathbf{Z}$.

By Theorem 3.7 $H^*(\Lambda, \Gamma, \Lambda)=\bigoplus_{r\in\mathbf{Z}}H^r(\Lambda, \Gamma, \Lambda)$ is a ring with the identity element which is the image of $\bar{1}\in Z(\Lambda)/N_{\Lambda/\Gamma}(\Lambda)$ on the isomorphism $Z(\Lambda)/N_{\Lambda/\Gamma}(\Lambda)\simeq H^0(\Lambda, \Gamma, \Lambda)$ of Proposition 1.2.

Now assume that Γ/K is also a Frobenius extension. Then since Λ/K is a Frobenius extension, we have the cup product $\cup$ on $H^*(\Lambda, K, -)$.

LEMMA 3.8. *For any (P, S)-exact sequence $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$, we have two connecting homomorphisms $\partial: H^r(\Lambda, \Gamma, N)\to H^{r+1}(\Lambda, \Gamma, L)$ and $\partial^\Lambda: H^r(\Lambda, K, N)\to H^{r+1}(\Lambda, K, L)$ for all $r\in\mathbf{Z}$ by Lemma 2.4. Then we have*

(i) $\partial^\Lambda\circ\mathrm{Inf}^r=\mathrm{Inf}^{r+1}\circ\partial$ *for* $r\geqq 1$,

(ii) $\mathrm{Inf}^1\circ\partial\circ\mathrm{Def}^0=\partial^\Lambda$.

PROOF. We use (4) in section 2 for the proof. (i) holds by the commutativity of (4). Let A be any left P-module. By $K(A)$ and $K'(A)$ we denote the kernels of $c_1^*: \mathrm{Hom}({}_PY_0, {}_PA)\to\mathrm{Hom}({}_PY_1, {}_PA)$ and $d_1^*: \mathrm{Hom}({}_PX_0, {}_PA)\to\mathrm{Hom}({}_PX_1, {}_PA)$ respectively. Then the diagram

$$\begin{array}{ccc} g_*^{-1}(K'(N)) & \xrightarrow{\sigma_0^*} & g_*^{-1}(K(N)) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ K'(N) & \longleftarrow & K(N) \end{array}$$

is commutative where g_* is the homomorphism induced by g with the natural way and $K(\ )\to K'(\ )$ is the same isomorphism by which we defined Def^0 in section 2. $\sigma_1^*\circ f_*=f_*\circ\sigma_1^*$ and $\sigma_1^*\circ d_1^*=c_1^*\circ\sigma_0^*$ hold. Therefore (ii) holds.

PROPOSITION 3.9. *Let A and B be left P-modules and let α, β, α' and β' be elements of $H^r(\Lambda, \Gamma, A)$, $H^s(\Lambda, \Gamma, B)$, $H^r(\Lambda, K, A)$ and $H^s(\Lambda, K, B)$ respectively. Then we have*

(i) $\mathrm{Inf}^{r+s}(\alpha\cup\beta)=\mathrm{Inf}^r(\alpha)\cup\mathrm{Inf}^s(\beta)$ *for* $r\geqq 1$ *and* $s\geqq 1$,

(ii) $\mathrm{Def}^{r+s}(\alpha'\cup\beta')=\mathrm{Def}^r(\alpha')\cup\mathrm{Def}^s(\beta')$ *for* $r\leqq 0$ *and* $s\leqq 0$,

(iii-i) $\mathrm{Def}^{r+s}(\alpha'\cup\mathrm{Inf}^s(\beta))=\mathrm{Def}^r(\alpha')\cup\beta$ *for* $r<0$, $s\geqq 1$ *and* $r+s\leqq 0$,

(iii-ii) $\mathrm{Def}^{r+s}(\mathrm{Inf}^r(\alpha)\cup\beta')=\alpha\cup\mathrm{Def}^s(\beta')$ *for* $r\geqq 1$, $s<0$ *and* $r+s\leqq 0$,

(iv-i) $\mathrm{Inf}^{r+s}(\mathrm{Def}^r(\alpha')\cup\beta)=\alpha'\cup\mathrm{Inf}^s(\beta)$ *for* $r\leqq 0$, $s\geqq 1$ *and* $r+s\geqq 1$,

(iv-ii) $\mathrm{Inf}^{r+s}(\alpha\cup\mathrm{Def}^s(\beta'))=\mathrm{Inf}^r(\alpha)\cup\beta'$ *for* $r\geqq 1$, $s\leqq 0$ *and* $r+s\geqq 1$.

PROOF. Let X be (2) in section 1. Then we can take Δ of Lemma 3.4 such that $\Delta_{0,0}(x_0\otimes_\Gamma x_1)=(x_0\otimes_\Gamma 1)\otimes_\Lambda(1\otimes_\Gamma x_1)$ and $\Delta_{-1,1}(x_0\otimes_\Gamma x_1)=\sum_i(x_0r_i\otimes_\Gamma 1)\otimes_\Lambda(1\otimes_\Gamma l_i\otimes_\Gamma x_1)$ hold. Since Λ/K is a Frobenius extension, we have a complete (P, K)-resolution Y of Λ whose type is (2) in section 1. Then σ_r of (4) in section 2 is given by

$$\sigma_r(x_0\otimes_K\cdots\otimes_K x_{r+1})=x_0\otimes_\Gamma\cdots\otimes_\Gamma x_{r+1}\qquad\text{for}\quad r\geqq 0,$$

$$\sigma_{-r}(x_0\otimes_\Gamma\cdots\otimes_\Gamma x_r)=\sum_{i_0,\cdots,i_{r-1}}x_0r'_{i_0}\otimes_K l'_{i_0}x_1r'_{i_1}\otimes_K\cdots\otimes_K l'_{i_{r-1}}x_r\quad\text{for } r\geqq 1$$

where $\{r'_i\}$ and $\{l'_i\}$ are elements of Γ with respect to the Frobenius extension Γ/K like as $\{r_i\}$ and $\{l_i\}$ of Λ respectively. Let $\Delta^\Lambda_{r,s}$ be the P-homomorphism of Lemma 3.4 for Y. Then $\Delta^\Lambda_{-1,1}(x_0\otimes_K x_1)=\sum_{i,j}(x_0r_ir'_j\otimes_K 1)\otimes_\Lambda(1\otimes_K l'_jl_i\otimes_K x_1)$ holds. Now we show (iii-i). Put $\alpha'=\bar{f}$ and $\beta=\bar{g}$ where f and g are representatives of α' and β respectively. At first we prove the case of $r+s=0$ by induction on s. Since there holds

$$\begin{aligned}\mathrm{Def}^0(\bar{f}\cup\mathrm{Inf}^1(\bar{g}))&=\mathrm{Def}^0\overline{((f\otimes_\Lambda g\circ\sigma_1)\circ\Delta^\Lambda_{-1,1})}\\&=\overline{[x_0\otimes_\Gamma x_1\longrightarrow\textstyle\sum_{i,j}f(x_0r_ir'_j\otimes_K 1)\otimes_\Lambda g(1\otimes_\Gamma l'_jl_i\otimes_\Gamma x_1)]}\\&=\overline{(f\circ\sigma_{-1}\otimes_\Lambda g)\circ\Delta_{-1,1}}\\&=\mathrm{Def}^{-1}(\bar{f})\cup\bar{g},\end{aligned}$$

the case of $s=1$ holds. Assume that (iii-i) holds for some s and for any left P-modfules A and B. Let α' and β be elements of $H^{-(s+1)}(\Lambda, K, A)$ and $H^{s+1}(\Lambda, \Gamma, B)$ respectively. Then with (7), $\partial^\Lambda(\alpha')\in H^{-s}(\Lambda, K, \mathrm{Ker}\,\phi)$ holds where ∂^Λ is the connecting homomorphism. By (iv) of Lemma 3.3 there exists $\beta''\in H^s(\Lambda, \Gamma, \mathrm{Coker}\,i')$ such that $\partial(\beta'')=\beta$ holds. By the assumption of induction $\mathrm{Def}^0(\partial^\Lambda(\alpha')\cup\mathrm{Inf}^s(\beta''))=\mathrm{Def}^{-s}(\partial^\Lambda(\alpha'))\cup\beta''$ holds. So we have $\partial(\mathrm{Def}^{-1}(\alpha'\cup\mathrm{Inf}^s(\beta'')))=\partial(\mathrm{Def}^{-(s+1)}(\alpha')\cup\beta'')$ by Lemma 2.5. Since this ∂ is an isomorphism, we can cancel ∂. Therefore by Lemmas 2.5 and 3.8 (iii-i) holds for α' and β.

Assume that (iii-i) holds for the case of $r+s=-n$ $(n\geqq 0)$. Consider the case of $r+s=-(n+1)$. By (ii) of Lemma 3.3, $\partial(\beta)\in H^{s+1}(\Lambda, \Gamma, \mathrm{Ker}\,\phi')$ holds. So $\mathrm{Def}^{-n}(\alpha'\cup\mathrm{Inf}^{s+1}(\partial(\beta)))=\mathrm{Def}^{r}(\alpha')\cup\partial(\beta)$ holds. By Lemmas 2.5 and 3.8 $\partial(\mathrm{Def}^{-(n+1)}(\alpha'\cup\mathrm{Inf}^{s}(\beta)))=\partial(\mathrm{Def}^{r}(\alpha')\cup\beta)$ holds. This ∂ is an isomorphism. Hence (iii-i) holds. (iii-ii) is shown by the same method. Next we show (iv-i). At first we show the case of $r+s=1$ by induction on r. For $r=0$ (iv-i) holds by the computation like (iii-i). Assume that (iv-i) holds for some r and for any left P-modules A and B. Let α' and β be elements of $H^{r-1}(\Lambda, K, A)$ and $H^{2-r}(\Lambda, \Gamma, B)$ respectively. By (iv) of Lemma 3.3 there exists $\beta''\in H^{1-r}(\Lambda, \Gamma, \mathrm{Coker}\,i')$ such that $\partial(\beta'')=\beta$ holds. Then $\mathrm{Def}^{0}(\alpha'\cup\mathrm{Inf}^{1-r}(\beta''))=\mathrm{Def}^{r-1}(\alpha')\cup\beta''$ holds by (iii-i). Therefore by Lemma 3.8,

$$\begin{aligned}\mathrm{Inf}^{1}(\mathrm{Def}^{r-1}(\alpha')\cup\beta)&=\mathrm{Inf}^{1}(\mathrm{Def}^{r-1}(\alpha')\cup\partial(\beta''))\\&=(-1)^{r-1}\,\mathrm{Inf}^{1}\circ\partial(\mathrm{Def}^{r-1}(\alpha')\cup\beta'')\\&=(-1)^{r-1}\,\mathrm{Inf}^{1}\circ\partial\circ\mathrm{Def}^{0}(\alpha'\cup\mathrm{Inf}^{1-r}(\beta''))\\&=(-1)^{r-1}\partial^{\Lambda}(\alpha'\cup\mathrm{Inf}^{1-r}(\beta''))\\&=\alpha'\cup\mathrm{Inf}^{2-r}(\beta)\end{aligned}$$

holds. Next assume that (iv-i) holds for the case of $r+s=n$ $(n\geqq 1)$. Consider the case of $r+s=n+1$. By (iv) of Lemma 3.3 there exists $\beta''\in H^{s-1}(\Lambda, \Gamma, \mathrm{Coker}\,i')$ such that $\partial(\beta'')=\beta$ holds. Since $\mathrm{Inf}^{n}(\mathrm{Def}^{r}(\alpha')\cup\beta'')=\alpha'\cup\mathrm{Inf}^{s-1}(\beta'')$ holds, (iv-i) holds for α' and β by using Lemma 3.8. (iv-ii) is shown by the same method. (i) and (ii) are also shown by induction easier than (iii-i) and (iv-i).

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