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PROPER ISOPARAMETRIC SEMI-RIEMANNIAN SUBMANIFOLDS IN A SEMI-RIEMANNIAN SPACE FORM

By

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§0. Introduction.

In a sphere, Erbacher [2] and Yano-Ishihara [14] characterized Riemannian submanifolds with non-negative sectional curvature, flat normal connection and parallel mean curvature vector under the additional assumptions. It is a natural question to consider this problem in the semi-Riemannian case. Recently, we characterized proper isoparametric semi-Riemannian hypersurfaces in a semi-Riemannian space form under certain assumptions [1]. The main purpose of this paper is to characterize, in a semi-Riemannian space form, proper isoparametric semi-Riemannian submanifolds with non-negative (or non-positive) sectional curvature and parallel mean curvature vector under certain additional assumptions.

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§1. Preliminaries.

Throughout this paper, all manifolds are smooth and connected and geometrical objects are assumed to be smooth unless mentioned otherwise. In this section, we prepare basic facts about semi-Riemannian submanifolds in a semi-Riemannian manifold. We call a non-degenerate symmetric (0, 2)-tensor field on an n-dimensional manifold M^n a *semi-Riemannian metric* of M^n and a manifold M^n equipped with such a metric a *semi-Riemannian manifold*. Especially, an n-dimensional real vector space equipped with a non-degenerate symmetric bilinear form of signature $(\nu, n-\nu)$ given by

$$\langle x, x\rangle=-\sum_{i=1}^{\nu} x_i^2+\sum_{j=\nu+1}^{n} x_j^2$$

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is called an *n-dimensional semi-Euclidean space* and is denoted by R_ν^n, where $x=(x_1, \cdots, x_n)$ is the natural coordinate. A frame $(e_1, \cdots, e_n)$ is said to be *orthonormal* if $|\langle e_i, e_j\rangle|=\delta_{ij}$. Semi-Riemannian manifolds $S_\nu^n(c)$ and $H_\nu^n(c)$ given by

$$S_\nu^n(c)=\{(x_1, \cdots, x_{n+1})\in R_\nu^{n+1} \mid -\sum_{i=1}^{\nu} x_i^2+\sum_{i=\nu+1}^{n+1} x_i^2=1/c\} \quad (c>0),$$

$$H_\nu^n(c)=\{(x_1, \cdots, x_{n+1})\in R_{\nu+1}^{n+1} \mid -\sum_{i=1}^{\nu+1} x_i^2+\sum_{i=\nu+2}^{n+1} x_i^2=1/c\} \quad (c<0)$$

are called a *semi-sphere* and a *semi-hyperbolic space*, respectively. These spaces are complete and of constant curvature c, that is,

$$R(X, Y)Z=c(X\wedge Y)Z \ (=c(\langle Y, Z\rangle X-\langle X, Z\rangle Y)),$$

where R is the curvature tensor ($n\geqq 2$). It is clear that $S_\nu^n(c)$ is diffeomorphic to $R^\nu\times S^{n-\nu}$ and $H_\nu^n(c)$ is diffeomorphic to $S^\nu\times R^{n-\nu}$, where $S^\mu=S_0^\mu$ and $R^\mu=R_0^\mu$. We note that $S_n^n(c)$ and $H_0^n(c)$ are not connected and $S_{n-1}^n(c)$ and $H_1^n(c)$ are not simply connected. We call these three spaces R_ν^n, $S_\nu^n(c)$ and $H_\nu^n(c)$ *semi-Riemannian space forms.*

A semi-Riemannian manifold M^n isometrically immersed into a semi-Riemannian manifold $\tilde{M}^m$ by an immersion f is called a *semi-Riemannian submanifold* of $\tilde{M}$. Since f can be treated locally as an imbedding, p $(\in M)$ will often be identified with $f(p)$ and the mention of f will be supressed. Especially if $n=m-1$, then M is called a *semi-Riemannian hypersurface* of $\tilde{M}$. Let T_pM (resp. $T_p^\perp M$) be the tangent space (resp. the normal space) of M at $p\in M$, TM (resp. $T^\perp M$) the tangent bundle (resp. the normal bundle) of M and $\Gamma(TM)$ resp. $\Gamma(T^\perp M)$) the space of all cross sections of TM (resp. $T^\perp M$). We denote the semi-Riemannian metrics of $\tilde{M}$ and M by $\langle\,,\rangle$ and the Levi-Civita connections on $\tilde{M}$ (resp. M) by $\tilde{\nabla}$ (resp. ∇). For any $X\in TM$ and any $Y\in\Gamma(TM)$, we have the Gauss formula:

$$\tilde{\nabla}_XY=\nabla_XY+h(X, Y), \tag{1.1}$$

where ∇_XY and $h(X, Y)$ are the tangential and the normal components of $\tilde{\nabla}_XY$ respectively. It is easy to show that h is symmetric. We call h the *second fundamental form* of the semi-Riemannian submanifold M.

For any $X\in TM$ and any $E\in\Gamma(T^\perp M)$, we have the Weingarten formula:

$$\tilde{\nabla}_XE=-A_EX+\nabla_X^\perp E, \tag{1.2}$$

where $-A_EX$ and $\nabla_X^\perp E$ are the tangential and the normal components of $\tilde{\nabla}_XE$ respectively. It is easy to verify that $\nabla^\perp$ is a connection of the normal bundle of M. We call A the *shape operator* of the semi-Riemannian submanifold M.

It follows that

$$\langle h(X, Y), E\rangle=\langle A_E X, Y\rangle \tag{1.3}$$

for any $X, Y\in T_pM$ and any $E\in T_p^{\perp}M$ $(p\in M)$.

Let $\tilde{R}$ and R be the curvature tensors of $\tilde{M}$ and M, respectively. The equation of Gauss is given by

$$R(X, Y)Z=(\tilde{R}(X, Y)Z)^T+\sum_{a=1}^{m-n}\varepsilon_a^{\perp}(A_{E_a}X\wedge A_{E_a}Y)Z \quad (\varepsilon_a^{\perp}=\langle E_a, E_a\rangle)$$

for any X, Y and $Z\in T_pM$ $(p\in M)$, where $(\tilde{R}(X, Y)Z)^T$ is the tangential component and $(E_1, \cdots, E_{m-n})$ is an orthonormal frame of $T_p^{\perp}M$. The equation of Codazzi is given by

$$(\tilde{R}(X, Y)E)^T=(\nabla'_Y A)_E X-(\nabla'_X A)_E Y$$

for any $X, Y\in T_pM$ and any $E\in T_p^{\perp}M$ $(p\in M)$, where $(\nabla'_X A)_E Y=\nabla_X(A_E Y)$ $A_{\nabla_X^{\perp}E}Y-A_E(\nabla_X Y)$. In particular, if $\tilde{M}$ is of constant curvature $\tilde{c}$, then these equations can be rewritten as follows:

$$R(X, Y)=\tilde{c}X\wedge Y+\sum_{a=1}^{m-n}\varepsilon_a^{\perp}A_{E_a}X\wedge A_{E_a}Y \tag{1.4}$$

$$(\nabla'_X A)_E Y=(\nabla'_Y A)_E X. \tag{1.5}$$

§2. Shape operators of proper isoparametric semi-Riemannian submanifolds.

Let Q be a (1, 1)-tensor of a real vector space V equipped with a non-degenerate symmetric bilinear form. If Q can be expressed by a real diagonal matrix with respect to an orthonormal frame of V, then Q is said to be *proper*.

LEMMA 2.1. *Let $Q_1, \cdots, Q_k$ be proper (1, 1)-tensors of V such that $[Q_a, Q_b]=0$ $(1\leqq a, b\leqq k)$. Then $Q_1, \cdots, Q_k$ are simultaneously diagonalizable with respect to an orthonormal frame of V.*

PROOF. It is sufficient to show the case where $k=2$. Let $\{\lambda_1, \cdots, \lambda_t\}$ (resp. $\{\mu_1, \cdots, \mu_u\}$) be the set of all distinct eigenvalues of Q_1 (resp. Q_2). Set $V_{\lambda_a}=Ker(Q_1-\lambda_a I)$ $(1\leqq a\leqq t)$, $W_{\mu_b}=Ker(Q_2-\mu_b I)$ $(1\leqq b\leqq u)$. Let v be a vector of V_{λ_a}. There exists a unique $v_b\in W_{\mu_b}$ $(1\leqq b\leqq u)$ such that $v=v_1+\cdots+v_u$ because of $V=\bigoplus_{1\leqq b\leqq u}W_{\mu_b}$, where $\oplus$ means the orthogonal direct sum. By operating Q_1 to both sides of $v=v_1+\cdots+v_u$, we have $\lambda_a v_1+\cdots+\lambda_a v_u=Q_1v_1+\cdots+Q_1v_u$. On the other hand, from $[Q_1, Q_2]=0$, it follows that $Q_1v_b\in W_{\mu_b}$ $(1\leqq b\leqq u)$. Hence, we have $Q_1v_b=\lambda_a v_b$, which means that $v_b\in V_{\lambda_a}\cap W_{\mu_b}$. Therefore, we can obtain

$V_{\lambda_a}=\bigoplus_{b\in G_a}(V_{\lambda_a}\cap W_{\mu_b})$ and hence $V=\bigoplus_{(a,b)\in G}(V_{\lambda_a}\cap W_{\mu_b})$ because of $V=\bigoplus_{1\leq a\leq t}V_{\lambda_a}$, where $G=\{(a,b)\mid 1\leqq a\leqq t,\ 1\leqq b\leqq u,\ (V_{\lambda_a}\cap W_{\mu_b}\neq\{0\})\}$ and $G_a=\{b\mid(a,b)\in G\}$ $(1\leqq a\leqq t)$. Moreover, since $V_{\lambda_a}\cap W_{\mu_b}$ $((a,b)\in G)$ are orthogonal to one another, they are non-degenerate, respectively. So we can take orthonormal frames of $V_{\lambda_a}\cap W_{\mu_b}$ $((a,b)\in G)$ and, by using them, we can construct an orthonormal frame of V. It is clear that Q_1 and Q_2 are simultaneously diagonalizable with respect to this orthonormal frame. This completes the proof. Q.E.D.

Let A be the shape operator of a semi-Riemannian submanifold M of a semi-Riemannian manifold $\tilde{M}$. The submanifold M is said to be *proper* if A_E is proper for any $E\in T^{\perp}M$. If the normal connection is flat and the characteristic polynomial of A_E is constant over the domain of E for any local parallel normal vector field E, then M is said to be *isoparametric* [3, 11]. By a similar method to the proof of Lemma 2 in [2], we can show the following.

LEMMA 2.2. *Let M^n be a proper semi-Riemannian submanifold in a semi-Riemannian space form $\tilde{M}^{n+r}$ of constant curvature $\tilde{c}$ with flat normal connection and parallel mean curvature vector. Then we have*

$$\Delta\langle A,A\rangle=2\langle\nabla'A,\nabla'A\rangle+\sum_{i,j=1}^{n}\sum_{a=1}^{r}K_{ij}(\lambda_i^a-\lambda_j^a)^2\langle E_a,E_a\rangle\,,$$

where $(e_1,\cdots,e_n)$ and $(E_1,\cdots,E_r)$ are an orthonormal tangent frame and an orthonormal normal frame of M such that $A_{E_a}e_i=\lambda_i^a e_i$ $(1\leqq i\leqq n,\ 1\leqq a\leqq r)$, K_{ij} is the sectional curvature with respect to the 2-dimensional subspace spanned by e_i and e_j $(i\neq j)$, and Δ is the Laplacian operator of M.

Note that $\langle A,A\rangle$ and $\langle\nabla'A,\nabla'A\rangle$ are defined as follows:

$$\langle A,A\rangle=\sum_{i=1}^{n}\sum_{a=1}^{r}\varepsilon_i\varepsilon_a^{\perp}\langle A_{E_a}e_i,A_{E_a}e_i\rangle \quad\text{and}$$

$$\langle\nabla'A,\nabla'A\rangle=\sum_{i,j=1}^{n}\sum_{a=1}^{r}\varepsilon_i\varepsilon_j\varepsilon_a^{\perp}\langle(\nabla'_{e_i}A)_{E_a}e_j,(\nabla'_{e_i}A)_{E_a}e_j\rangle\,,$$

where $\varepsilon_i=\langle e_i,e_i\rangle$ $(1\leqq i\leqq n)$ and $\varepsilon_a^{\perp}=\langle E_a,E_a\rangle$ $(1\leqq a\leqq r)$.

We denote by $B_1\oplus\cdots\oplus B_l$ the following matrix:

$$\begin{pmatrix} B_1 & & 0\\ & \ddots & \\ 0 & & B_l\end{pmatrix}$$

where B_i $(1\leqq i\leqq l)$ are square matrices, respectively.

By using Lemma 2.1 and 2.2, we can show the following theorem.

THEOREM 2.3. *Let M^n be a proper isoparametric semi-Riemannian submanifold in R_ν^{n+r} with parallel mean curvature vector and $\langle\nabla' A, \nabla' A\rangle \geqq 0$. Furthermore, suppose that all sectional curvatures of M are non-negative (resp. non-positive) and $\langle\ ,\ \rangle|_{T^\perp M}$ is positive definite (resp. negative definite). Then, for any point p of M, there exists a parallel orthonormal normal frame field $(E_1, \cdots, E_r)$ on a neighborhood U of p with the property (#): At each point of U, $A_{E_1}, \cdots, A_{E_r}$ can be expressed with respect to a certain orthonormal tangent frame $(e_1, \cdots, e_n)$ as follows:*

$$A_{E_1}=\lambda_1 I_{l_1}\oplus 0_{k_1},$$
$$A_{E_2}=0_{l_1}\oplus\lambda_2 I_{l_2}\oplus 0_{k_2},$$
$$\cdots\cdots,$$
$$A_{E_s}=\left(\bigoplus_{a=1}^{s-1} 0_{l_a}\right)\oplus\lambda_s I_{l_s}\oplus 0_{k_s},$$
$$A_{E_{s+1}}=\cdots=A_{E_r}=0,$$

where $\lambda_a\neq 0$, $k_a=n-\sum_{b=1}^{a} l_b$, $l_a\geqq 1$ $(1\leqq a\leqq s)$, $k_s\geqq 0$ and 0_l and I_l are the zero matrix of type (l, l) and the identity matrix of type (l, l), respectively.

PROOF. Fix a point p of M. Since the normal connection of M is flat, there exists a parallel orthonormal normal frame field $(E_1, \cdots, E_r)$ on a neighborhood U of p and moreover $[A_{E_a}, A_{E_b}]=0$ holds $(1\leqq a, b\leqq r)$. Hence, by Lemma 2.1, $A_{E_1}, \cdots, A_{E_r}$ are simultaneously diagonalizable with respect to an orthonormal tangent frame at each point of U. Suppose that $A_{E_1}, \cdots, A_{E_r}$ are expressed with respect to an orthonormal tangent frame $(e_1, \cdots, e_n)$ at each point of U as follows:

$$A_{E_1}=\lambda_1^1 I_1\oplus\cdots\oplus\lambda_n^1 I_1, \cdots, A_{E_r}=\lambda_1^r I_1\oplus\cdots\oplus\lambda_n^r I_1.$$

By our assumptions and Lemma 2.2, we have

$$K_{ij}(\lambda_i^a-\lambda_j^a)^2=0 \quad (1\leqq a\leqq r,\ 1\leqq i\neq j\leqq n). \tag{2.1}$$

In the first place, suppose that p is a geodesic point, that is, $A_{E_1}=\cdots=A_{E_r}=0$ at p. Since M is isoparametric, $A_{E_1}=\cdots=A_{E_r}=0$ on U. Thus $(E_1, \cdots, E_r)$ satisfies the property (#).

In the next place, we consider the case where p is not a geodesic point. Since p is not a geodesic point, we may assume that $\lambda_1^1\neq 0$, $K_{1i}\neq 0$ $(2\leqq i\leqq l_1)$ and $K_{1j}=0$ $(l_1+1\leqq j\leqq n)$. From (2.1), we have

(2.2) $$\lambda_1^a=\lambda_i^a \quad (2\leqq i\leqq l_1,\ 1\leqq a\leqq r).$$

We set

$$E'_1:=\Big(\sum_{a=1}^{r}\lambda_1^a E_a\Big)/\lambda_1,$$

$$\bar{E}_b:=(\lambda_1^1 E_b-\lambda_1^b E_1)/((\lambda_1^1)^2+(\lambda_1^b)^2)^{1/2} \quad (2\leqq b\leqq r),$$

where $\lambda_1=\Big(\sum_{a=1}^{r}(\lambda_1^a)^2\Big)^{1/2}$. It is clear that

$$\langle E'_1, E'_1\rangle=\pm 1,\quad \langle E'_1, \bar{E}_b\rangle=0,\quad \langle \bar{E}_b, \bar{E}_b\rangle=\pm 1,\quad \nabla^{\perp}E'_1=\nabla^{\perp}\bar{E}_b=0.$$

Because of (2.2), $A_{E'_1}$ and $A_{\bar{E}_b}$ $(2\leqq b\leqq r)$ are expressed as follows:

$$A_{E'_1}=\lambda_1 I_{l_1}\oplus\lambda'^1_{l_1+1}I_1\oplus\cdots\oplus\lambda'^1_n I_1$$

$$A_{\bar{E}_b}=0_{l_1}\oplus\bar{\lambda}^b_{l_1+1}I_1\oplus\cdots\oplus\bar{\lambda}^b_n I_1 \quad (2\leqq b\leqq r).$$

Let $(E'_2, \cdots, E'_r)$ be an orthonormal normal system given by applying Gram-Schmidt orthogonalization to $(\bar{E}_2, \cdots, \bar{E}_r)$. It is clear that E'_b $(2\leqq b\leqq r)$ are parallel and $A_{E'_b}$ $(2\leqq b\leqq r)$ are expressed as follows:

$$A_{E'_b}=0_{l_1}\oplus\lambda'^b_{l_1+1}I_1\oplus\cdots\oplus\lambda'^b_n b_1 \quad (2\leqq b\leqq r).$$

By the assumption that $K_{1i}=0$ $(l_1+1\leqq i\leqq n)$ and the equation (1.4), we have

$$\begin{aligned}0=K_{1i}&=\langle e_1, e_1\rangle\langle e_i, e_i\rangle\langle R(e_1, e_i)e_i, e_1\rangle\\&=\langle e_1, e_1\rangle\langle e_i, e_i\rangle\langle \pm\sum_{a=1}^{r}(A_{E'_a}e_1\wedge A_{E'_a}e_i)e_i, e_1\rangle\\&=\pm\lambda_1\lambda'^1_i,\end{aligned}$$

that is, $\lambda'^1_i=0$ $(l_1+1\leqq i\leqq n)$. After all, we obtain $A_{E'_1}=\lambda_1 I_{l_1}\oplus 0_{n-l_1}$. Thus if $A_{E'_2}=\cdots=A_{E'_r}=0$, $(E'_1, \cdots, E'_r)$ satisfy the property (#). So we consider the case where there exists $b\geqq 2$ such that $A_{E'_b}\neq 0$. We may assume that $\lambda'^2_{l_1+1}\neq 0$, $K_{l_1+1,i}\neq 0$ $(l_1+2\leqq i\leqq l_1+l_2)$ and $K_{l_1+1,j}=0$ $(l_1+l_2+1\leqq j\leqq n)$. By the same process as the above, we can obtain a parallel orthonormal normal system $(E''_2, \cdots, E''_r)$ such that

$$A_{E''_2}=0_{l_1}\oplus\lambda_2 I_{l_2}\oplus 0_{n-l_1-l_2},$$

$$A_{E''_b}=0_{l_1+l_2}\oplus\lambda''^b_{l_1+l_2+1}I_1\oplus\cdots\oplus\lambda''^b_n I_1 \quad (3\leqq b\leqq r).$$

In the sequel, by repeating the same process, we reach the conclusion. Q.E.D.

In general, if M is simply connected and the normal connection is flat, then there exists a parallel orthonormal normal frame field on M. By using this fact, we can obtain the following.

THEOREM 2.4. *Under the same hypothesis as in Theorem* 2.3, *if M is simply connected, then there exists a parallel orthonormal normal frame field* $(E_1, \cdots, E_r)$ *on M with the property* (#) *in Theorem* 2.3.

§3. Eigendistributions of the shape operator.

Let M be a semi-Riemannian manifold equipped with a metric $\langle\,,\rangle$ and D a distribution on M, that is, a subbundle of the tangent bundle TM. If $\nabla_X Y \in D$ for any $X \in TM$ and any $Y \in \Gamma(D)$, then D is said to be *parallel*, where $\Gamma(D)$ is the space of all cross sections of D. If $\langle\,,\rangle|_D$ is non-degenerate at each point of M, then D is said to be *non-degenerate*. We have

LEMMA A. *Let D be a non-degenerate parallel distribution on a semi-Riemannian manifold M. Let M' be the maximal integral manifold of D through a point of M. Then M' is a totally geodesic semi-Riemannian submanifold of M. If M is complete, then so is M'.*

Let Q be a (1, 1)-tensor field on M. If Q is proper at each point of M, then Q is said to be *proper*. The following result is stated in [1].

LEMMA B. *Let Q be a proper* (1, 1)*-tensor field on M which has exactly two mutually distinct constant eigenvalues λ_1 and λ_2. Suppose that $(\nabla_X Q)Y = (\nabla_Y Q)X$ holds for any $X, Y \in T_pM$ $(p \in M)$. Then $D_{\lambda_i} = Ker(Q - \lambda_i I)$ $(i=1, 2)$ are non-degenerate parallel distributions on M.*

By using these results, we obtain the following theorem.

THEOREM 3.1. *Let M^n be a semi-Riemannian submanifold of R_ν^{n+r}. Suppose that for each point p of M, there exists a parallel orthonormal normal frame field $(E_1, \cdots, E_r)$ on a neighborhood U of p with the property* (#) *in Theorem* 2.3. *Then*

(i) *$D_a = Ker(A_{E_a} - \lambda_a I)$ $(1 \leqq a \leqq s)$ and $D_0 = (D_1 \oplus \cdots \oplus D_s)^\perp$ are parallel on U respectively, where $(D_1 \oplus \cdots \oplus D_s)^\perp$ is the orthogonal complement of $D_1 \oplus \cdots \oplus D_s$ in TU,*

(ii) *the each maximal integral manifold of D_a is a totally umbilical submanifold of R_ν^{n+r} with the mean curvature vector $\varepsilon_a^\perp \lambda_a E_a$ $(\varepsilon_a^\perp = \langle E_a, E_a\rangle)$ $(1 \leqq a \leqq s)$ and that of D_0 is a totally geodesic semi-Riemannian submanifold of R_ν^{n+r}.*

PROOF. Let us restrict ourselves to the neighborhood U.

(i) By applying Lemma B to A_{E_a}, we see that each D_a is parallel on U

$(1 \leqq a \leqq s)$. Since $D_1 \oplus \cdots \oplus D_s$ is parallel on U, so is the orthogonal complement D_0.

(ii) Let $U_{(a)}$ be the maximal integral manifold of D_a through a point of U $(1 \leqq a \leqq s)$. We denote the second fundamental form of U (resp. $U_{(a)}$) in R_ν^{n+r} by h (resp. h_a). Take $X, Y \in T_q U_{(a)}$ $(q \in U_{(a)})$. Since $U_{(a)}$ is totally geodesic in U, $h_a(X, Y) = h(X, Y)$ holds. Also, by the assumption, we have

$$\begin{aligned} h(X, Y) &= \sum_{b=1}^{r} \varepsilon_b^{\perp} \langle h(X, Y), E_b \rangle E_b \\ &= \sum_{b=1}^{r} \varepsilon_b^{\perp} \langle A_{E_b} X, Y \rangle E_b \\ &= \langle X, Y \rangle \varepsilon_a^{\perp} \lambda_a E_a . \end{aligned}$$

Thus we obtain that $h_a(X, Y) = \langle X, Y \rangle \varepsilon_a^{\perp} \lambda_a E_a$, that is, $U_{(a)}$ is a totally umbilical submanifold of R_ν^{n+r} with the mean curvature vector $\varepsilon_a^{\perp} \lambda_a E_a$. Similarly, we can show that the each maximal integral manifold of D_0 is a totally geodesic semi-Riemannian submanifold of R_ν^{n+r}. Q. E. D.

§4. Proper isoparametric semi-Riemannian submanifolds in a semi-Euclidean space.

In this section, we characterize proper isoparametric semi-Riemannian submanifolds in a semi-Euclidean space under the hypothesis as in Theorem 2.3. Now we prepare the following lemma.

LEMMA 4.1. *Let M^n be a semi-Riemannian submanifold of R_ν^{n+r} with the second fundamental form h and $D_1, \cdots, D_t$ non-degenerate parallel distributions on M such that $TM = D_1 \oplus \cdots \oplus D_t$. Suppose that $h(X, Y) = 0$ holds for any $X \in (D_a)_p$ and any $Y \in (D_b)_p$ $(a \neq b,\ p \in M)$ and the each maximal integral manifold of D_a $(1 \leqq a \leqq t)$ is a totally umbilical submanifold of R_ν^{n+r} with the mean curvature vector η_a. Then*

(i) $\tilde{\nabla}_X Y \in D_b$ *for any* $X \in D_a$ *and any* $Y \in \Gamma(D_b)$ $(a \neq b)$,

(ii) $\tilde{\nabla}_X \eta_b = 0$ *for any* $X \in D_a$ $(a \neq b)$,

(iii) $\langle \eta_a, \eta_b \rangle = 0$ $(a \neq b)$.

PROOF. It is sufficient to prove the case where $t = 2$.

(i) Take $X \in (D_1)_p$ and $Y \in \Gamma(D_2)$ $(p \in M)$. Let $(U, x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2})$ be a coordinate neighborhood of p in M such that $\partial/\partial x_i \in D_1$ and $\partial/\partial y_j \in D_2$ $(1 \leqq i \leqq n_1,\ 1 \leqq j \leqq n_2)$, where $n_a = \dim D_a$ $(a = 1, 2)$. Choose constants X^i $(1 \leqq i \leqq n_1)$

and smooth functions Y^j $(1\leqq j\leqq n_2)$ such that $X=\sum_{i=1}^{n_1}X^i\partial/\partial x_i$ and $Y=\sum_{j=1}^{n_2}Y^j\partial/\partial y_j$. Since D_1, D_2 are parallel on M and $\nabla_{\partial/\partial x_i}\partial/\partial y_j=\nabla_{\partial/\partial y_j}\partial/\partial x_i$, we have $\nabla_{\partial/\partial x_i}\partial/\partial y_j=0$. Therefore, the assumption on h implies $\tilde{\nabla}_{\partial/\partial x_i}\partial/\partial y_j=0$ and hence $\tilde{\nabla}_X Y=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}X^i(\partial/\partial x_i Y^j)\partial/\partial y_j\in(D_2)_p$.

(ii) Take $X\in\Gamma(D_1)$. By the Weingarten formula (1.2), we have

$$\tilde{\nabla}_X\eta_2=-A_{\eta_2}X+\nabla_X^{\perp}\eta_2\,, \tag{4.1}$$

where A and $\nabla^{\perp}$ are the shape operator and the normal connection of M, respectively. For $Y\in T_pM$, we have

$$\begin{aligned}\langle A_{\eta_2}X,\,Y\rangle&=\langle h(X,\,Y),\,\eta_2\rangle\\&=(1/n_2)\sum_{j=1}^{n_2}\varepsilon_j\langle h(X,\,Y),\,h(e_j,\,e_j)\rangle\,,\end{aligned} \tag{4.2}$$

where $(e_1,\cdots,e_{n_2})$ is a local orthonormal frame field of D_2 about p and $\varepsilon_j=\langle e_j,\,e_j\rangle$ $(1\leqq j\leqq n_2)$. On the other hand, from the equations (1.3) and (1.4), it follows that

$$\langle h\langle X,\,Y\rangle,\,h(e_j,\,e_j)\rangle=\langle R(Y,\,e_j)e_j,\,X\rangle+\langle h(X,\,e_j),\,h(Y,\,e_j)\rangle\,, \tag{4.3}$$

where R is the curvature tensor of M. Moreover, by the assumption, the right hand side of (4.3, is equal to zero. Therefore, the equation (4.2) implies $A_{\eta_2}X=0$. Also, by the assumptions and the equations (1.3) and (1.5), we have

$$\begin{aligned}\nabla_X^{\perp}\eta_2&=(1/n_2)\sum_{j=1}^{n_2}\varepsilon_j\nabla_X^{\perp}(h(e_j,\,e_j))\\&=(1/n_2)\sum_{j=1}^{n_2}\varepsilon_j\{\nabla_{e_j}^{\perp}(h(X,\,e_j))-h(\nabla_{e_j}X,\,e_j)\\&\qquad-h(X,\,\nabla_{e_j}e_j)+2h(\nabla_Xe_j,\,e_j)\}\\&=(2/n_2)\sum_{j=1}^{n_2}\varepsilon_jh(\nabla_Xe_j,\,e_j)\,.\end{aligned}$$

Moreover, since the each maximal integral manifold of D_2 is totally geodesic in M and totally umbilic in R_ν^{n+r}, $h(\nabla_Xe_j,\,e_j)=\langle\nabla_Xe_j,\,e_j\rangle\eta_2=0$ holds. Therefore, $\nabla_X^{\perp}\eta_2=0$ is induced. Finally, we obtain $\tilde{\nabla}_X\eta_2=0$.

(iii) Let $(\bar{e}_1,\cdots,\bar{e}_{n_1})$ (resp. $(e_1,\cdots,e_{n_2})$) be an orthonormal frame of $(D_1)_p$ (resp. $(D_2)_p$) $(p\in M)$. By the equation (1.4), we have

$$\begin{aligned}\langle\eta_1,\,\eta_2\rangle&=(1/n_1n_2)\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\bar{\varepsilon}_i\varepsilon_j\langle h(\bar{e}_i,\,\bar{e}_i),\,h(e_j,\,e_j)\rangle\\&=(1/n_1n_2)\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\bar{\varepsilon}_i\varepsilon_j(\langle R(\hat{e}_i,\,e_j)e_j,\,\bar{e}_i\rangle+\langle h(\bar{e}_i,\,e_j),\,h(\bar{e}_i,\,e_j)\rangle)\,.\end{aligned}$$

Moreover, the right hand side of this equation is equal to zero by the assumptions. Hence, we obtain $\langle\eta_1, \eta_2\rangle=0$. Q. E. D.

For a semi-Riemannian submanifold M, we define the *first normal space* N_p^1 at p as follows:

$$N_p^1=Span\,\{h(X, Y)\mid X, Y\in T_pM\}.$$

A subbundle N of $T^{\perp}M$ is said to be *normal parallel* is $\nabla_X^{\perp}E\in N$ for any $X\in TM$ and any $E\in\Gamma(N)$. The following reduction theorem was proved by Magid [6].

THEOREM C. *Let M^n be a semi-Riemannian submanifold isometrically immersed into R_ν^{n+r} by f. If the first normal spaces constitute a normal parallel subbundle, then there exists a complete $(n+s)$-dimensional totally geodesic submanifold $\overline{M}$ of R_ν^{n+r} such that $f(M)\subset\overline{M}$, where s is the dimension of the first normal spaces.*

By using this theorem, he obtained the following result [6], where he also treated the case $\langle\eta, \eta\rangle=0$.

THEOREM D. *Let M^n be a totally umbilical submanifold isometrically immersed into R_ν^{n+r} by f. Suppose that the mean curvature vector η satisfies $\langle\eta, \eta\rangle\neq 0$. Then*

(I) *If $\langle\eta, \eta\rangle>0$, then $f(M)\subset S_\mu^n$*

(II) *If $\langle\eta, \eta\rangle<0$, then $f(M)\subset H_\mu^n$,*

where μ is the index of M.

By using Theorem C, D and Lemma 4.1, we can show the following lemma.

LEMMA 4.2. *Under the same hypothesis as in Lemma 4.1, moreover suppose that η_a $(1\leqq a\leqq t)$ are non-null and $\langle\eta_a, \eta_a\rangle>0$ $(1\leqq a\leqq u)$, $\langle\eta_a, \eta_a\rangle<0$ $(u+1\leqq a\leqq s)$ and $\eta_a=0$ $(s+1\leqq a\leqq t)$. Then*

$$\begin{aligned} f(M)&\subset S_{\nu_1}^{n_1}(c_1)\times\cdots\times S_{\nu_u}^{n_u}(c_u)\times H_{\nu_{u+1}}^{n_{u+1}}(c_{u+1})\times\cdots\times H_{\nu_s}^{n_s}(c_s)\times R_{\nu_0}^{n_0}\\ &\subset R_{\nu_1}^{n_1+1}\times\cdots\times R_{\nu_u}^{n_u+1}\times R_{\nu_{u+1}+1}^{n_{u+1}+1}\times\cdots\times R_{\nu_s+1}^{n_s+1}\times R_{\nu_0}^{n_0}\subset R_\nu^{n+r}, \end{aligned}$$

where $c_a=\langle\eta_a, \eta_a\rangle$, $(\nu_a, n_a-\nu_a)$ is the signature of D_a $(1\leqq a\leqq s)$ and $(\nu_0, n_0-\nu_0)$ is that of $D_{s+1}\oplus\cdots\oplus D_t$.

PROOF. We shall prove in the case where $t=3$, $u=1$ and $s=2$. We denote the maximal integral manifold of D_a (resp. $D_a^{\perp}$) through $p\in M$ by $(L_a)_p$ (resp. $(L_a^{\perp})_p$) $(1\leqq a\leqq 3)$, where $D_a^{\perp}$ is the orthogonal complement of D_a in TM. Since

$(L_1)_p$ is a totally umbilical submanifold of R^{n+r}_ν with the mean curvature vector η_1, it is contained in the affine subspace $(\bar{L}_1)_p = T_p((L_1)_p) \oplus R(\eta_1)_p$ through $f(p)$ by Theorem C, where $R(\eta_1)_p$ is the line tangent to $(\eta_1)_p$. Now we shall show that $(\bar{L}_1)_p$ and $(\bar{L}_1)_q$ are parallel in R^{n+r}_ν for any $p, q \in M$. First we consider the case where p and q are contained in a cubic coordinate neighborhood V with respect to $D_1 \oplus D_1^\dagger$. Then it is clear that $(L_1^\dagger)_p \cap (L_1)_q \neq \emptyset$. Take $q' \in (L_1^\dagger)_p \cap (L_1)_q$. Since $(L_1^\dagger)_p = (L_1^\dagger)_{q'}$, $(\bar{L}_1)_p$ and $(\bar{L}_1)_{q'} (= (\bar{L}_1)_q)$ are parallel in R^{n+r}_ν by (i), (ii) of Lemma 4.1. Next we consider a general case for p and q. Take a curve $\sigma: [0, 1] \to M$ with $\sigma(0) = p$, $\sigma(1) = q$. Since $\sigma([0, 1])$ is compact, there exists a finite open covering $\{V_i \mid 1 \leqq i \leqq k\}$ of $\sigma([0, 1])$ by cubic coordinate neighborhoods such that $V_i \cap V_{i+1} \neq \emptyset$ $(1 \leqq i \leqq k-1)$, $p \in V_1$ and $q \in V_k$. Take $p_i \in V_i \cap V_{i+1}$ $(1 \leqq i \leqq k-1)$. Since p_{i-1} and p_i is contained in a cubic coordinate neighborhood, $(\bar{L}_1)_{p_{i-1}}$ and $(\bar{L}_1)_{p_i}$ are parallel in R^{n+r}_ν. Similarly, so are $(\bar{L}_1)_p$ and $(\bar{L}_1)_{p_1}$ (resp. $(\bar{L}_1)_{p_{k-1}}$ and $(\bar{L}_1)_q$). Therefore, $(\bar{L}_1)_p$ and $(\bar{L}_1)_q$ are parallel in R^{n+r}_ν. Similarly, $(\bar{L}_a)_p$ and $(\bar{L}_a)_q$ $(a=2, 3)$ are parallel in R^{n+r}_ν for any $p, q \in M$, where $(\bar{L}_2)_p = T_p((L_2)_p) \oplus R(\eta_2)_p$, $(\bar{L}_3)_p = T_p((L_3)_p)$. Also, by (iii) of Lemma 4.1, $(\bar{L}_a)_p \perp (\bar{L}_b)_p$ holds for any $p \in M$ $(a \neq b)$.

Let $R_{(a)}$ $(1 \leqq a \leqq 3)$ be the subspace of R^{n+r}_ν spanned by all tangent vectors of $(\bar{L}_a)_p$. Note that $R_{(a)}$ $(1 \leqq a \leqq 3)$ are well-defined and orthogonal to one another by the above facts. Let $R_{(0)}$ be the orthogonal complement of $R_{(1)} \oplus R_{(2)} \oplus R_{(3)}$. We regard $R_{(a)}$ $(0 \leqq a \leqq 3)$ as the affine subspace through the origin of R^{n+r}_ν. It is clear that $R^{n+r}_\nu = R_{(0)} \times \cdots \times R_{(3)}$. Let ψ_a $(0 \leqq a \leqq 3)$ be the natural projections of R^{n+r}_ν onto $R_{(a)}$. It is easy to show that $\psi_0 \circ f$ is a constant map. Suppose that $(L_1^\dagger)_p = (L_1^\dagger)_q$. Then we have $(\psi_1 \circ f)(p) = (\psi_1 \circ f)(q)$. Since $(\eta_1)_p$ and $(\eta_1)_q$ are parallel in R^{n+r}_ν by (ii) of Lemma 4.1, $(\psi_1)_*(\eta_1)_p = (\psi_1)_*(\eta_1)_q$. Therefore, from Theorem D and $\langle \eta_1, \eta_1 \rangle > 0$, if follows that there exists a hypersurface $S^{n_1}_{\nu_1}$ of $R_{(1)}$ which contains both $(\psi_1 \circ f)((L_1)_p)$ and $(\psi_1 \circ f)((L_1)_q)$. By the same method as used in the proof of parallelism between $(\bar{L}_a)_p$ and $(\bar{L}_a)_q$, we can show that $(\psi_1 \circ f)((L_1)_p)$ is contained in this hypersurface for any $p \in M$. This fact implies that $(\psi_1 \circ f)(M) \subset S^{n_1}_{\nu_1}$. Similar arguements on $(\psi_2 \circ f)(M)$ and $(\psi_3 \circ f)(M)$ lead to

$$f(M) \subset (\psi_1 \circ f)(M) \times (\psi_2 \circ f)(M) \times (\psi_3 \circ f)(M) \subset S^{n_1}_{\nu_1} \times H^{n_2}_{\nu_2} \times R^{n_0}_{\nu_0}$$
$$\subset R_{(1)} \times R_{(2)} \times R_{(3)}.$$

Q. E. D.

REMARK. From the assumption of Lemma 4.2, we can show that the second fundamental form is parallel and the normal connection is flat. In [6], he characterized a complete Riemannian submanifold M^n of R_ν^{n+r} with parallel second fundamental form and flat normal connection. The proof depends on Satz 2 of [12], which uses the Moore's lemma [8]. We can show that they are generally valid for proper semi-Riemannian submanifolds. On the other hand, Moore treats the case where M is a product manifold. If M is complete, then we can use the Moore's lemma for the universal covering of M. However, if M is not complete, then the lemma is not valid for this arguement at least globally. The lemma assures that each product neighborhood V of M is contained in a product manifold $\bar{M}$ of semi-Riemannian space forms as an open submanifold. However, we have to show that the manifolds $\bar{M}$ can be tahen in common for all V as in Lemma 4.2.

The distributions D_a $(0 \leqq a \leqq s)$ of Theorem 3.1 satisfy the conditions of Lemma 4.2. Hence we have the following proposition.

PROPOSITION 4.3. *Let M^n be a semi-Riemannian submanifold isometrically immersed into R_ν^{n+r} by f. Suppose that there exists a parallel orthonormal normal frame field $(E_1, \cdots, E_r)$ on M with the property (#) in Theorem 2.3. Then*

$$f(M) \subset S_{\nu_1}^{n_1}(c_1) \times \cdots \times S_{\nu_u}^{n_u}(c_u) \times H_{\nu_{u+1}}^{n_{u+1}}(c_{u+1}) \times \cdots \times H_{\nu_s}^{n_s}(c_s) \times R_{\nu_0}^{n_0}$$

$$\subset R_{\nu_1}^{n_1+1} \times \cdots \times R_{\nu_u}^{n_u+1} \times R_{\nu_{u+1}+1}^{n_{u+1}+1} \times \cdots \times R_{\nu_s+1}^{n_s+1} \times R_{\nu_0}^{n_0} \subset R_\nu^{n+r},$$

where u is the number of $+1$ in $\{\langle E_1, E_1\rangle, \cdots, \langle E_s, E_s\rangle\}$ and $n = n_0 + \cdots + n_s$.

By taking the universal semi-Riemannian covering manifold of M if necessary, this proposition together with Theorem 2.4 gives the following main theorem.

TNEOREM 4.4. *Let M^n be a proper isoparametric semi-Riemannian submanifold isometrically immersed into R_ν^{n+r} by f with parallel mean curvature vector and $\langle \nabla' A, \nabla' A\rangle \geqq 0$. Furthermore, suppose that all sectional curvatures of M are non-negative (resp. non-positive), $\langle\ ,\ \rangle|_{T^\perp M}$ is positive definite (resp. negative definite). Then*

$$f(M) \subset S_{\nu_1}^{n_1} \times \cdots \times S_{\nu_s}^{n_s} \times R_{\nu_0}^{n_0} \subset R_{\nu_1}^{n_1+1} \times \cdots \times R_{\nu_s}^{n_s+1} \times R_{\nu_0}^{n_0} \subset R_\nu^{n+r}$$

(resp. $f(M) \subset H_{\nu_1}^{n_1} \times \cdots \times H_{\nu_s}^{n_s} \times R_{\nu_0}^{n_0} \subset R_{\nu_1+1}^{n_1+1} \times \cdots \times R_{\nu_s+1}^{n_s+1} \times R_{\nu_0}^{n_0} \subset R_\nu^{n+r}$), where $n = n_0 + \cdots + n_s$.

§ 5. Proper isoparametric semi-Riemannian submanifolds in $S_\nu^{n+r}(c)$ or $H_\nu^{n+r}(\tilde{c})$.

In this section we shall show the results corresponding to § 4 in the case where the ambient space is $H_\nu^{n+r}(\tilde{c})$ (or $S_\nu^{n+r}(\tilde{c})$).

LEMMA 5.1. *Let M^n be a proper isoparametric semi-Riemannian submanifold of $H_\nu^{n+r}(\tilde{c})$ such that*

(i) *the mean curvature vector is parallel,*

(ii) $\langle\nabla' A, \nabla' A\rangle \geqq 0$.

Then, if we consider M as isometrically immersed into $R_{\nu+1}^{n+r+1}$, M also is a proper isoparametric semi-Riemannian submanifold with (i) *and* (ii).

PROOF. Let A and $\nabla^\perp$ (resp. $\tilde{A}$ and $\tilde{\nabla}^\perp$) be the shape operator and the normal connection of M in $H_\nu^{n+r}(\tilde{c})$ (resp. $R_{\nu+1}^{n+r+1}$). By the Gauss formula (1.1) and the Weingaten formula (1.2), we have

$$\tilde{A}_E X = A_E X\,, \qquad \tilde{\nabla}_X^\perp E = \nabla_X^\perp E\,, \tag{5.1}$$

$$\tilde{A}_{\bar{E}} X = \pm\sqrt{-\tilde{c}}\, X\,, \qquad \tilde{\nabla}_X^\perp \bar{E} = 0 \tag{5.2}$$

for any $X \in TM$ and any $E \in \Gamma(T^\perp M)$, where $\bar{E}$ is a unit normal vector field of $H_\nu^{n+r}(\tilde{c})$ in $R_{\nu+1}^{n+r+1}$ and $T^\perp M$ is the normal bundle of M in $H_\nu^{n+r}(\tilde{c})$. By (5.1), (5.2) and the assumption, we see that M is a proper isoparametric semi-Riemannian submanifold of $R_{\nu+1}^{n+r+1}$.

Let η (resp. $\tilde{\eta}$) be the mean curvature vector of M in $H_\nu^{n+r}(\tilde{c})$ (resp. $R_{\nu+1}^{n+r+1}$) and $\bar{\eta}$ that of $H_\nu^{n+r}(\tilde{c})$ in $R_{\nu+1}^{n+r+1}$. Since $H_\nu^{n+r}(\tilde{c})$ is a totally umbilical submanifold of $R_{\nu+1}^{n+r+1}$, $\tilde{\eta} = \eta + \bar{\eta}$ holds. Moreover, the equation (5.1) and the assumption (resp. the equation (5.2) and $\bar{\eta} = \pm\sqrt{-\tilde{c}}\,\bar{E}$) imply $\tilde{\nabla}_X^\perp \eta = 0$ (resp. $\tilde{\nabla}_X^\perp \bar{\eta} = 0$) for any $X \in TM$. Thus $\tilde{\nabla}_X^\perp \tilde{\eta} = 0$.

By (5.1), (5.2) and the assumption, we can show $\langle\tilde{\nabla}'\tilde{A}, \tilde{\nabla}'\tilde{A}\rangle = \langle\nabla' A, \nabla' A\rangle \geqq 0$, where $(\tilde{\nabla}'_X \tilde{A})_E Y = \nabla_X(\tilde{A}_E Y) - \tilde{A}_{\tilde{\nabla}_X^\perp E} Y - \tilde{A}_E(\nabla_X Y)$ for any $X \in TM$, any $Y \in \Gamma(TM)$ and any $E \in \Gamma(T^\perp M \oplus T^\perp H_\nu^{n+r}(\tilde{c}))$. Q. E. D.

This lemma together with Theorem 4.4 gives the following theorem.

THEOREM 5.2. *Let M^n be a proper isoparametric semi-Riemannian submanifold isometrically immersed into $H_\nu^{n+r}(\tilde{c})$ by f with parallel mean curvature vector and $\langle\nabla' A, \nabla' A\rangle \geqq 0$. Furthermore, suppose that all sectional curvatures of M are non-positive, $\langle\,,\rangle|_{T^\perp M}$ is negative definite. Then*

$$(i\circ f)(M)\subset H^{n_1}_{\nu_1}(c_1)\times\cdots\times H^{n_s}_{\nu_s}(c_s)\subset H^{n+s-1}_{\nu+s-r-1}(\bar{c})\subset H^{n+r}_{\nu}(\tilde{c})\subset R^{n+r+1}_{\nu+1},$$

where $n=n_1+\cdots+n_s$, $1/c_1+\cdots+1/c_s=1/\bar{c}\geqq 1/\tilde{c}$ *and* i *is the inclusion mapping of* $H^{n+r}_{\nu}(\tilde{c})$ *into* $R^{n+r+1}_{\nu+1}$.

PROOF. By Theorem 4.4 and Lemma 5.1, we have

$$(i\circ f)(M)\subset H^{n_1}_{\nu_1}(c_1)\times\cdots\times H^{n_s}_{\nu_s}(c_s)\times R^{n_0}_{\nu_0}\times\{x\}$$

$$\subset R^{n_1+1}_{\nu_1+1}\times\cdots\times R^{n_s+1}_{\nu_s+1}\times R^{n_0}_{\nu_0}\times R^{r-s+1}_{r-s+1}=R^{n+r+1}_{\nu+1}.$$

Take $p\in(i\circ f)(M)$. We denote the leaf of $R^{n_0}_{\nu_0}$ through p by L_p and $L_p\cap(i\circ f)(M)$ by $\hat{L}_p$. Suppose $n_0>1$. Since $\hat{L}_p$ is totally geodesic in $R^{n+r+1}_{\nu+1}$, it is also totally geodəsic in $H^{n+r}_{\nu}(\tilde{c})$. Hence $\hat{L}_p$ is of constant curvature $\tilde{c}$. This fact contradicts the flatness of L_p. Therefore, we have $n_0\leqq 1$. If $n_0=1$, then $\hat{L}_p$ is a family of non-null curves of $H^{n+r}_{\nu}(\tilde{c})$. By the way, all line segments of $R^{n+r+1}_{\nu+1}$ contained in $H^{n+r}_{\nu}(\tilde{c})$ are null. Hence each component of $\hat{L}_p$ is not a line segment. This fact contradicts that L_p is totally geodesic in $R^{n+r+1}_{\nu+1}$. Thus we see that $n_0=0$.

Let o_a be the center of $H^{n_a}_{\nu_a}(c_a)$ $(1\leqq a\leqq s)$. Take $p\in(i\circ f)(M)$. We can uniquely decompose p into $p=p_1+\cdots+p_s+x$, where $p_a\in R^{n_a+1}_{\nu_a+1}$ $(1\leqq a\leqq s)$. From $\langle p_a-o_a, p_a-o_a\rangle=1/c_a$, it follows that

$$\langle p_a, p_a\rangle=\langle o_a+(p_a-o_a), o_a+(p_a-o_a)\rangle$$

$$=\langle o_a, 2p_a-o_a\rangle+1/c_a$$

$$=\langle o_a, 2p-o\rangle+1/c_a,$$

where $o=o_1+\cdots+o_s$. Hence we have

$$1/\tilde{c}=\langle p, p\rangle=\langle p_1, p_1\rangle+\cdots+\langle p_s, p_s\rangle+\langle x, x\rangle$$

$$=\langle o, 2p-o\rangle+1/c_1+\cdots+1/c_s+\langle x, x\rangle.$$

Thus $\langle o, 2p-o\rangle=1/\tilde{c}-(1/c_1+\cdots+1/c_s+\langle x, x\rangle)$ holds. This equality implies that $\langle p, o\rangle$ is independent of $p\in(i\circ f)(M)$. Hence, if o is a non-zero vector, then $(i\circ f)(M)$ is contained in the hyperplane orthogonal to o in $R^{n_1+1}_{\nu_1+1}\times\cdots\times R^{n_s+1}_{\nu_s+1}\times\{x\}$. This fact contradicts that $(i\circ f)(M)$ is full in $R^{n_1+1}_{\nu_1+1}\times\cdots\times R^{n_s+1}_{\nu_s+1}\times\{x\}$. Therefore, we see that o is the zero vector and $1/\tilde{c}=1/c_1+\cdots+1/c_s+\langle x, x\rangle$. These facts imply that

$$H^{n_1}_{\nu_1}(c_1)\times\cdots\times H^{n_s}_{\nu_s}(c_s)\times\{x\}\subset H^{n+r}_{\nu}(\tilde{c})$$

and hence

$$H^{n_1}_{\nu_1}(c_1)\times\cdots\times H^{n_s}_{\nu_s}(c_s)\times\{x\}\subset H^{n+r}_{\nu}(\tilde{c})\cap(R^{n_1+1}_{\nu_1+1}\times\cdots\times R^{n_s+1}_{\nu_s+1}\times\{x\})$$
$$=H^{n+s-1}_{\nu+s-r-1}(\bar{c})\times\{x\}.$$

Here $1/\bar{c}=1/c_1+\cdots+1/c_s$ because

$$1/\tilde{c}=\langle q, q\rangle=\langle x+(q-x), x+(q-x)\rangle=\langle x, x\rangle+1/\bar{c}$$

for $q\in H^{n+s-1}_{\nu+s-r-1}(\bar{c})\times\{x\}$. Therefore, we obtain

$$(i\circ f)(M)\subset H^{n_1}_{\nu_1}(c_1)\times\cdots\times H^{n_s}_{\nu_s}(c_s)\times\{x\}\subset H^{n+s-1}_{\nu+s-r-1}(\bar{c})\times\{x\}$$
$$\subset H^{n+r}_{\nu}(\tilde{c})\subset R^{n+r+1}_{\nu+1}.$$

Q. E. D.

Similarly, in the case where the ambient space is $S^{n+r}_{\nu}(\tilde{c})$, we have the following theorem.

THEOREM 5.3. *Let M^n be a proper isoparametric semi-Riemannian submanifold isometrically immersed into $S^{n+r}_{\nu}(\tilde{c})$ by f with parallel mean curvature vector and $\langle\nabla' A, \nabla' A\rangle\geqq 0$. Furthermore, suppose that all sectional curvatures of M are non-negative, $\langle\,,\rangle|_{T^\perp M}$ is positive definite. Then*

$$(i\circ f)(M)\subset S^{n_1}_{\nu_1}(c_1)\times\cdots\times S^{n_s}_{\nu_s}(c_s)\subset S^{n+s-1}_{\nu}(\bar{c})\subset S^{n+r}_{\nu}(\tilde{c})\subset R^{n+r+1}_{\nu},$$

where $n=n_1+\cdots+n_s$, $1/c_1+\cdots+1/c_s=1/\bar{c}\leqq 1/\tilde{c}$ and i is the inclusion mapping of $S^{n+r}_{\nu}(\tilde{c})$ into R^{n+r+1}_{ν}.

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