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DEGREE OF THE STANDARD ISOMETRIC MINIMAL IMMERSIONS OF COMPLEX PROJECTIVE SPACES INTO SPHERES

By

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1. Introduction.

Let (M^m, g) be an irreducible symmetric space of compact type, and Δ be the Laplace-Beltrami operator of (M, g) acting on C^∞ functions. We denote by λ_k the k-th eigen-value of Δ, $0=\lambda_0<\lambda_1<\cdots$, and by V^k the corresponding eigen-space.

For each $k\geqq 1$, an orthonormal base of V^k defines the standard isometric minimal immersion x_k of $(M, (\lambda_k/m)g)$ into the unit hypersphere in V^k centered at the origin. do Carmo and Wallach [2] showed that the standard minimal immersion x_k of the sphere S_c^n with constant sectional curvature $c=n/k(n+1)$ into a unit sphere of dimension $m(k)=(2k+n-1)(k+n-2)!/k!(n-1)!-1$ has degree k (cf. §3, about the definition of the degree). Every homogeneous harmonic polynomial of degree k on $\boldsymbol{R}^{n+1}$ induces a harmonic function on S^n by restriction. Such a function just belongs to V^k. Conversely every function in V^k is obtained in this way. So the degree of $x_k: S_c^n\rightarrow S_1^{m(k)}$ is equal to the (algebraic) degree of the polynomials.

Wallach says [8], without proof, that the standard minimal immersion x_1 of complex projective space CP_h^n, $n\geqq 2$, of constant holomorphic sectional curvature $h=2n/(n+1)$ into $S_1^{n(n+2)-1}$ has degree 2. Let $\pi: S^{2n+1}\rightarrow CP^n$ be the Hopf fibration, where we consider S^{2n+1} as the unit hypersphere in $\boldsymbol{C}^{n+1}$ with respect to the standard Hermitian product. A complex valued homogeneous polynomial f on $\boldsymbol{C}^{n+1}$ of $2n+2$ variables $z_1, \cdots, z_{n+1}, \bar{z}_1, \cdots, \bar{z}_{n+1}$ is said to be of type (p, q) when f satisfies

$$f(cz_1, \cdots, cz_{n+1}, \bar{c}\bar{z}_1, \cdots, \bar{c}\bar{z}_{n+1})$$
$$=c^p\bar{c}^q f(z_1, \cdots, z_{n+1}, \bar{z}_1, \cdots, \bar{z}_{n+1}),$$
$$c\in\boldsymbol{C},\ (z_1, \cdots, z_{n+1})\in\boldsymbol{C}^{n+1},$$

or in short

$$f(cZ)=c^p\bar{c}^q f(Z), \quad c\in\boldsymbol{C}, \quad Z\in\boldsymbol{C}^{n+1}.$$

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Every real valued homogeneous harmonic polynomial on C^{n+1} of type (k, k) induces a harmonic function on CP^n through π. Such a function belongs to V^k. Conversely every function in V^k is obtained in this way [1]. In this paper we show the following:

THEOREM. *Let x_k be the standard minimal immersion of CP^n_h, $n\geqq 2$, of constant holomorphic sectional curvature $h=2n/k(n+k)$ into a unit sphere $S_1^{m(k)}$, where*

$$m(k)=n(n+2k)((n+1)(n+2)\cdots(n+k-1))^2/(k!)^2-1\,.$$

Then x_k has degree $2k$.

From our Theorem the (geometric) degree of $x_k: CP^n_h \to S_1^{m(k)}$ coincides with the (algebraic) degree of the polynomials on C^{n+1} which induce the functions in V^k.

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2. The standard minimal immersions

In this section we define the standard minimal immersions of a compact irreducible symmetric space. We refer to do Carmo and Wallach [2] for details.

Let (M^m, g) be an irreducible symmetric space of compact type, and V^k the eigen-space of $\Delta^{(M,g)}$ corresponding the k-th eigen-value λ_k. We define the L^2-inner product $(\ ,\ )$ on V^k by

$$(f, h)=\int_M f\cdot h d\mu\,, \quad f, h\in V^k\,.$$

For simplicity, we normalize the canonical measure $d\mu$ of (M, g) in such a way that $\int_M d\mu=\dim. V^k=m(k)+1$. An orthonormal base $f_0, f_1, \cdots, f_{m(k)}$ of V^k defines naturally a mapping x_k of M into $R^{m(k)+1}$. Let (G, K) be a symmetric pair corresponding to M so that $M=G/K$. Then G acts on V^k as a group of orthogonal transformations by

$$(\sigma\cdot f)(p)=f(\sigma^{-1}\cdot p)\,, \quad \sigma\in G\,, \quad p\in M\,. \tag{2.1}$$

The irreducibility of the linear isotropy action of K and the G-invariance of the metric g guarantees that x_k is an isometric immersion of (M^m, c^2g) into $R^{m(k)+1}$ for some constant $c>0$. A Theorem of T. Takahashi [7] implies that x_k is an isometric minimal immersion of (M, c^2g) into a sphere of radius $c(m/\lambda_k)^{1/2}$ where $m=\dim. M$. Since there exists an orthogonal matrix $(\sigma_{ij})_{0\leq i,j\leq m(k)}$ such that $\sigma\cdot f_j=\sum_{i=0}^{m(k)}\sigma_{ij}f_i$ for each $\sigma\in G$, we have

$$\sum_{j=0}^{m(k)} f_j^2(\sigma^{-1}\cdot K)=\sum_{j=0}^{m(k)}(\sigma\cdot f_j)^2(eK)=\sum_{j=0}^{m(k)} f_j^2(eK)\,. \tag{2.2}$$

Integrating right and left hand sides of (2.2) on M, we obtain

$$\begin{aligned}\sum_{j=0}^{m(k)}(f_j, f_j)=m(k)+1&=\Big(\sum_{j=0}^{m(k)} f_j^2(e\cdot K)\Big)\int_M d\mu \\ &=\Big(\sum_{j=0}^{m(k)} f_j^2(e\cdot K)\Big)(m(k)+1)\,.\end{aligned} \tag{2.3}$$

So we obtain

$$\sum_{j=0}^{m(k)} f_j^2(e\cdot K)=1\,. \tag{2.4}$$

(2.2) and (2.4) show that $x_k(M)$ is contained in the unit sphere in $R^{m(k)+1}$ centered at the origin, hence we get $c=(\lambda_k/m)^{1/2}$. We shall call this isometric minimal immersion x_k of $(M, (\lambda_k/m)g)$ into $S_1^{m(k)}$ the *k-th standard minimal immersion* of M.

The standard minimal immersion can be described in another words as follows. Take an orthonormal base $e_0, e_1, \cdots, e_{m(k)}$ of $R^{m(k)+1}$ such that $e_0=x_k(e\cdot K)=(f_0(e\cdot K), \cdots, f_{m(k)}(e\cdot K))$. Let A be an isometry of V^k into $R^{m(k)+1}$ such that $A(f_j)=e_j$, $j=0, 1, \cdots m(k)$. Let G act on $R^{m(k)+1}$ so that A is a G-isomorphism. Then by a simple computation we get

$$x_k(\sigma\cdot K)=A(\sigma\cdot f_0)\,,\quad \sigma\in G\,. \tag{2.5}$$

Since A is an isometry, we can consider x_k as an isometric minimal immersion of $(M, (\lambda_k/m)g)$ into a unit hypersphere in V^k defined by

$$x_k(\sigma\cdot K)=\sigma\cdot f_0\,,\quad \sigma\in G\,. \tag{2.6}$$

Hereafter we take the standard minimal immersions in the latter sense.

3. Degree of an equivariant isometric immersions

In this section we define the higher fundamental forms and the degree of an equivariant isometric immersion.

Let $x: M^m\to\tilde{M}^{m+q}(c)$ be an isometric immersion of a Riemannian homogeneous space $M=G/K$ into a space of constant curvature c. Such an immersion x is said to be *equivariant*, if there exists a continuous homomorphism ρ of G into the isometry group $I(\tilde{M})$ of $\tilde{M}=\tilde{M}^{m+q}(c)$ such that

$$x(\sigma\cdot p)=\rho(\sigma)\cdot x(p)\,,\quad p\in M\,,\quad \sigma\in G\,. \tag{3.1}$$

It is easily seen that the standard minimal immersion in §. 2 are naturally equivariant.

Let $B_{2|p}$ be the second fundamental form of x at $p\in M$, and $O_p^2(M)$ be the linear span of Image $B_{2|p}$ in the normal space $N_p(M)$ of the immersion x at $p\in M$. Because of the equivariance of x, $\bigcup_{p\in M}O_p^2(M)$ has the structure of a subbundle of the normal bundle $N(M)$. The orthogonal projection $N_{2|p}: N_p(M)\rightarrow(O_p^2(M))^{\perp}$ at each point $p\in M$ defines a differentiable homomorphism $N_2: N(M)\rightarrow N(M)$. We define the *third fundamental form* $B_{3|p}$ at $p\in M$ by

$$B_{3|p}(u, v, w)=[(DB_2)(u, v, w)]^{N_{2|p}}, \quad u, v, w\in T_pM, \tag{3.2}$$

where DB_2 is the covariant derivative of van der Waerden-Bortolotti of B_2. Inductively we define $O_p^j(M)$ as the linear span of Image $B_{j|p}$, $N_{j|p}$ as the orthogonal projection $N_p(M)\rightarrow(O_p^2(M)+\cdots+O_p^j(M))^{\perp}$, and $B_{j+1|p}$ by

$$B_{j+1|p}(u_1, \cdots, u_{j+1})=[(DB_j)(u_1, \cdots, u_{j+1})]^{N_{j|p}}, \quad u_1, \cdots, u_{j+1}\in T_pM. \tag{3.3}$$

By the following Lemma 3.1, $\bigcup_{p\in M}O_p^j(M)$ has the structure of a subbundle of $N(M)$ and we can define N_j and the *higher fundamental forms* B_{j+1} on M inductively. We can express B_{j+1} using the Riemannian connection $\tilde{\nabla}$ in $\tilde{M}$ as follows. We extend $N_{j|p}$ to T_pM by putting $N_{j|p}(T_pM)=0$. Then

$$B_{j+1|p}(u_1, \cdots, u_{j+1})=[\tilde{\nabla}_{U_1}(B_j(U_2, \cdots, U_{j+1}))]^{N_{j|p}}, \tag{3.4}$$

where $U_1, \cdots, U_{j+1}$ are local extensions of $u_1, \cdots, u_{j+1}$.

LEMMA 3.1. *Let $x: M^m\rightarrow\tilde{M}^{m+q}(c)$ be an equivariant isometric immersion of a Riemannian homogeneons space $M=G/K$ into a space of constant cnrvature c. Then*

(1) *B_j is G-invariant and commutes with $\rho(\sigma)$.*

$$B_{j|\sigma\cdot p}(\sigma\cdot u_1, \cdots, \sigma\cdot u_j)=\rho(\sigma)\cdot B_{j|p}(u_1, \cdots, u_j), \tag{3.5}$$
$$\rho(\sigma)\cdot O_p^j(M)=O_{\sigma\cdot p}^j(M), \quad \sigma\in G.$$

$$N_j\circ\rho(\sigma)=\rho(\sigma)\circ N_j, \quad \sigma\in G. \tag{3.6}$$

(2) *B_j is a symmetric $C^\infty(M)$ multilinear mapping,*

$$B_j: \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{j\text{-times}}\longrightarrow N(M).$$

PROOF. We prove (3.5) and (3.6) by induction on j. From (3.1) we get

$$x_{*|\sigma\cdot p}\sigma\cdot u=\rho(\sigma)\cdot x_{*|p}u, \quad \sigma\in G, \quad u\in T_pM.$$

Since σ and $\rho(\sigma)$ are isometries of M and $\tilde{M}$, we have

$$\begin{aligned}B_{2|\sigma\cdot p}(\sigma\cdot u_1, \sigma\cdot u_2)&=\tilde{\nabla}_{x_*\sigma\cdot u_1}x_*\sigma\cdot U_2-x_*\nabla_{\sigma\cdot u_1}\sigma\cdot U_2\\&=\tilde{\nabla}_{\rho(\sigma)\cdot x_*u_1}\rho(\sigma)\cdot x_*U_2-x_*\sigma\cdot\nabla_{u_1}U_2\\&=\rho(\sigma)\cdot\tilde{\nabla}_{x_*u_1}x_*U_2-\rho(\sigma)\cdot x_*\nabla_{u_1}U_2\end{aligned}$$

$$=\rho(\rho)\cdot B_{2|p}(u_1, u_2).$$

Then we get

$$\rho(\sigma)\cdot O_p^2(M)=O_{\sigma\cdot p}^2(M).$$

Since $\rho(\sigma)$ induces an isometry of $N_p(M)$ to $N_{\sigma\cdot p}(M)$, we get

$$N_{2|\sigma\cdot p}\circ\rho(\sigma)=\rho(\sigma)\circ N_{2|p}, \quad \sigma\in G, \quad p\in M.$$

Suppose that (3.5) and (3.6) are valid for $j=2, 3, \cdots, k$. Then by (3.4), (3.5) and (3.6), we have

$$\begin{aligned} B_{k+1|\sigma\cdot p}(\sigma\cdot u_1, \cdots, \sigma\cdot u_{k+1}) &=[\tilde{\nabla}_{x_*\sigma\cdot u_1}\rho(\sigma)\cdot B_k(U_2, \cdots, U_{k+1})]^{N_k} \\ &=[\rho(\sigma)\cdot\tilde{\nabla}_{x_*u_1}B_k(U_2, \cdots, U_{k+1})]^{N_k} \\ &=\rho(\sigma)\cdot[\tilde{\nabla}_{x_*u_1}B_k(U_2, \cdots, U_{k+1})]^{N_k} \\ &=\rho(\sigma)\cdot B_{k+1}(u_1, \cdots, u_{k+1}) \end{aligned}$$

From this we get

$$\rho(\sigma)\cdot O_p^{k+1}(M)=O_{\sigma\cdot p}^{k+1}(M).$$

(3.6) for $j=k+1$ is easily verified.

Let $e_1^{(r)}, \cdots, e_{s(r)}^{(r)}$ be a local orthonormal frame field, such that it spans $O_p^r(M)$ at each point around the origin $o=eK$, $r\geqq 1$, where we mean $O_p^1(M)=x_*(T_pM)$. Then $B_{j-1}(U_2, \cdots, U_j)$, $j\geqq 3$, can be written in the following form by C^∞ functions $f_i^{(r)}$

$$B_{j-1}(U_2, \cdots, U_j)=\tilde{\nabla}_{U_2}B_{j-2}(U_3, \cdots, U_j) -\sum_{r=1}^{j-2}\sum_{i=1}^{s(r)} f_i^{(r)}e_i^{(r)}. \tag{3.7}$$

Differentiating both sides of (3.7) with respect to U_1, we obtain

$$\tilde{\nabla}_{U_1}(B_{j-1}(U_2, \cdots, U_j))=\tilde{\nabla}_{U_1}(\tilde{\nabla}_{U_2}B_{j-2}(U_3, \cdots, U_j)) -\sum_{r=1}^{j-2}\sum_{i=1}^{s(r)}(U_1\cdot f_i^{(r)})e_i^{(r)}-\sum_{r=1}^{j-2}\sum_{i=1}^{s(r)} f_i^{(r)}(\tilde{\nabla}_{U_1}e_i^{(r)}). \tag{3.8}$$

Since the second and third term of the right hand side of (3.8) is contained in the kernel of N_j, (3.4) and (3.8) imply

$$B_j(U_1, \cdots, U_j)=[\tilde{\nabla}_{U_1}(\tilde{\nabla}_{U_2}(B_{j-2}(U_3, \cdots, U_j)))]^{N_{j-1}}.$$

Obviously (2) is true for $j=2, 3$ by the equation of Coddazzi. So we assume (2) is true for $j-1$, $j\geqq 4$. Since $\tilde{M}$ is a space of constant curvature, we have

$$\begin{aligned}\tilde{\nabla}_{U_1}\tilde{\nabla}_{U_2}B_{j-1}(U_3, \cdots, U_j)-\tilde{\nabla}_{U_2}\tilde{\nabla}_{U_1}B_{j-2}(U_3, \cdots, U_j)&\\ -\tilde{\nabla}_{[U_1,U_2]}B_{j-2}(U_3, \cdots, U_j)&\\ =\tilde{R}(U_1, U_2)B_{j-2}(U_3, \cdots, U_j)&\\ =c(\langle U_2, B_{j-2}(U_3, \cdots, U_j)\rangle U_1-\langle U_1, B_{j-2}(U_3, \cdots, U_j)\rangle U_2)&\\ =0\,.&\end{aligned}$$

We operate N_{j-1} on the above equation. Then we get

$$B_j(U_1, U_2, U_3, \cdots, U_j)=B_j(U_2, U_1, U_3, \cdots, U_j)\,.$$

Hence by induction hypothesis, (2) is als true for j. Q. E. D.

We call *degree* of x the first integer d such that $B_{d|p}\neq 0$, $B_{d+1|p}=0$ at some point $p\in M$. It is obvious that the above definition of degree is independent of the choice of p.

Now we confine our consideration to the standard minimal immersions of an irreducible symmetric space $M=G/K$ of compact type. We regard $O_{eK}^j(M)$ as a subspace in V^k in a natural manner. Let $S^j(T_{eK}M)$ be the j-fold symmetric power of $T_{eK}M$. We extend the linear isotropy action of K on $S^j(T_{eK}M)$ in a natural manner. Since $B_{j|eK}$ is a symmetric multilinear form by Lemma 3.1 (2), we extend this to a linear map of $S^j(T_{eK}M)$ to $O_{eK}^j(M)$, and denote it also by B_j.

LEMMA 3.2. *Let $x: M\rightarrow S_1^{m(k)}\subset V^k$ be the k-th standard minimal immersion of a compact irreducible symmetric space M. Then*
(1) *the j-th fundamental form B_j is a K-homomorphism,*

$$B_j:\ S^j(T_{eK}M)\longrightarrow O_{eK}^j(M)\,.$$

(2) *V^k admits the following orthogonal direct sum decomposition*

$$V^k=\boldsymbol{R}f_0+T_{eK}M+O_{eK}^2(M)+\cdots+O_{eK}^d(M)\,,$$

where d is the degree of x_k.
(3) *Let $e_1, \cdots, e_m$ be an orthonormal frame of $T_{eK}M$. Put $r=\sum_{i=1}^m e_i^2\in S^2(T_{eK}M)$, then*

$$\text{Ker}\, B_j\supset r\cdot S^{j-2}(T_{eK}M)\,,\quad j\geqq 2\,. \tag{3.9}$$

PROOF. (1) holds by (3.5).

It is easy to see that $x_k(M)$ is not contained in any totally geodesic submanifold in $S_1^{m(k)}$. Then (2) is a direct consequence of a Theorem of J. Erbacher [3].

Let $E_1, \cdots, E_m$ be a local orthonormal frame field around $o=eK$, such that $E_{i|eK}=e_i$. Since x_k is a minimal immersion, we have $\sum_{i=1}^m B_2(E_i, E_i)=0$. This implies $B_2(r)=0$. Assume that $\sum_{i=1}^m B_{j+2}(E_{k_1}, \cdots, E_{k_j}, E_i, E_i)=0$, $j\geqq 0$. Then, by (3.4), we have

$$\sum_{i=1}^{m} B_{j+3}(E_{k_0}, E_{k_1}, \cdots, E_{k_j}, E_i, E_i)=0\,.$$

This proves (3.9). Q.E.D.

4. Proof of our Theorem

In this section we prove our Theorem stated in the introduction. For this we need some results about representation of the special unitary group $SU(n+1)$. First we explain the notations.

We denote by $P_{p,q}^{n+1}$ the complex vector space of all homogeneous polynomials of type (p, q) on $\boldsymbol{C}^{n+1}$. Let $C^\infty(\boldsymbol{C}^{n+1}, \boldsymbol{C})$ be the space of all complex valued C^∞ functions on $\boldsymbol{C}^{n+1}$. We denote by D the Laplace-Beltrami operator of $C^\infty(\boldsymbol{C}^{n+1}, \boldsymbol{C})$. Then D can be written as

$$D=-4\sum_{i=1}^{n+1}\partial^2/\partial z^i\partial\bar{z}^i\,.$$

We put $H_{p,q}^{n+1}=\{f\in P_{p,q}^{n+1}\,;\, Df=0\}$ and $r=\sum_{i=1}^{n+1} z^i\bar{z}^i\in P_{1,1}^{n+1}$.

Let $\mathfrak{h}$ be the space of all diagonal matrices in the Lie algebra $\mathfrak{su}(n+1)$ of $SU(n+1)$. Since $\mathfrak{su}(n+1)$ is a compact semisimple Lie algebra and $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{su}(n+1)$, $\mathfrak{h}^C$ is a Cartan subalgebra of $(\mathfrak{su}(n+1))^C=\mathfrak{sl}(n+1)$. We define $\lambda_1, \cdots, \lambda_n\in\mathfrak{h}^*$ by

$$\lambda_i\left(\begin{pmatrix} (-1)^{1/2}x_1 & & 0 \\ & \ddots & \\ 0 & & (-1)^{1/2}x_{n+1}\end{pmatrix}\right)=x_i,\ 1\leqq i\leqq n+1,\ x_1, \cdots, x_{n+1}\in\boldsymbol{R}$$

and fix the following lexicographic order in $\mathfrak{h}^*$

$$\lambda_1>\lambda_2>\cdots>\lambda_n>0>\lambda_{n+1}$$

We define an action of $SU(n+1)$ on $C^\infty(\boldsymbol{C}^{n+1}, \boldsymbol{C})$ by

$$(\sigma\cdot f)(Z)=f(\sigma^{-1}\cdot Z)\,,\quad Z\in\boldsymbol{C}^{n+1}\,,\quad \sigma\in G\,.$$

It can easily be seen that $P_{p,q}^{n+1}$ and $H_{p,q}^{n+1}$ are $SU(n+1)$-invariant subspaces of $C^\infty(\boldsymbol{C}^{n+1}, \boldsymbol{C})$. Furthermore we have the following:

THEOREM. 4.1. ([6], §. 14)

(1) $$P_{p,q}^{n+1}=\begin{cases} H_{p,q}^{n+1}\,, & \text{if } (p, q)=(0, 0), (1, 0), (0, 1)\,, \\ H_{p,q}^{n+1}+r\cdot P_{p-1,q-1}^{n+1}\ (\textit{direct sum})\,, & \text{if otherwise.}\end{cases}$$

(2) *$H^{n+1}_{p,q}$ is an $SU(n+1)$-irreducible subspace of $C^\infty(C^{n+1}, C)$ with highest weight $p\lambda_1 - q\lambda_{n+1}$.*

From now on we employ the following notations;

$$G = SU(n+1)$$

$$K = S(U(1)\times U(n)) = \left\{\begin{bmatrix} 1/\det\sigma & 0 \\ 0 & \sigma \end{bmatrix};\ \sigma\in U(n)\right\}$$

$$L = \left\{\begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix};\ \sigma\in SU(n)\right\}$$

$$\mathfrak{g} = \mathfrak{su}(n+1) = \{X\in M_{n+1}(\boldsymbol{C});\ {}^tX+\bar{X}=0,\ \text{trace}\,X=0\}$$

$$\mathfrak{k} = \left\{\begin{bmatrix} -\text{trace}\,X & 0 \\ 0 & X \end{bmatrix};\ X\in M_n(\boldsymbol{C}),\ {}^tX+\bar{X}=0\right\}$$

$$\mathfrak{l} = \left\{\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix};\ X\in M_n(\boldsymbol{C}),\ \text{trace}\,X=0,\ {}^tX+\bar{X}=0\right\}$$

$$\mathfrak{h} = \left\{\begin{pmatrix} (-1)^{1/2}x_1 & & 0 \\ & \ddots & \\ 0 & & (-1)^{1/2}x_{n+1} \end{pmatrix};\ x_1, \cdots, x_{n+1}\in\boldsymbol{R},\ x_1+\cdots+x_{n+1}=0\right\}$$

$$\mathfrak{h}' = \left\{\begin{pmatrix} 0 & & & \\ & (-1)^{1/2}x_2 & & \\ & & \ddots & \\ & & & (-1)^{1/2}x_{n+1} \end{pmatrix};\ x_2, \cdots, x_{n+1}\in\boldsymbol{R},\ x_2+\cdots+x_{n+1}=0\right\}$$

Then G/K is identified with CP^n in a natural way, and (G, K) is a Riemannian symmetric pair corresponding to CP^n. $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{l}$ are Lie algebras of G, K and L respectively. We define $\lambda'_1, \cdots, \lambda'_n\in\mathfrak{h}'^*$ by

$$\lambda'_i\left(\begin{pmatrix} 0 & & & \\ & (-1)^{1/2}x_2 & & \\ & & \ddots & \\ & & & (-1)^{1/2}x_{n+1} \end{pmatrix}\right) = x_{i+1},\ 1\leq i\leq n\,,$$

and fix the following lexicographic order in $\mathfrak{h}'^*$

$$\lambda'_1 > \lambda'_2 > \cdots > \lambda'_{n-1} > 0 > \lambda'_n\,.$$

It is well-known that the k-th eigen-space V^k of Δ^{CP^n} is G-isomorphic with the subspace $H^{n+1}_{k,k}\cap C^\infty(\boldsymbol{C}^{n+1}, \boldsymbol{R})$ of $H^{n+1}_{k,k}$ through the Hopf fibration $\pi: S^{2n+1}\to CP^n$, where $C^\infty(\boldsymbol{C}^{n+1}, \boldsymbol{R})$ denotes the space of all real valued C^∞ functions on $\boldsymbol{C}^{n+1}$. By Theorem 4.1 $(V^k)^C$ is an irreducible G-module with highest weight $k(\lambda_1-\lambda_{n+1})$.

We denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Precisely

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & -\bar{z}_1\cdots-\bar{z}_n \\ z_1 & \\ \vdots & 0 \\ z_n & \end{pmatrix};\ z_1, \cdots, z_n\in \boldsymbol{C}\right\}$$

We identify $\mathfrak{p}$ with $T_{eK}(CP^n)$ in a usual manner. As the base of $\mathfrak{p}$ we take the following one;

$$X_i=E_{1,i+1}-E_{i+1,1}, \quad Y_i=(-1)^{1/2}(E_{1,i+1}+E_{i+1,1}), \quad 1\leqq i\leqq n,$$

where $E_{i,j}$ is a matrix unit of which (i, j)-component is 1 and other components are 0. We put

$$Z_i=X_i-(-1)^{1/2}Y_i, \quad \bar{Z}_i=X_i+(-1)^{1/2}Y_i, \quad 1\leqq i\leqq n,$$

then $Z_1, \cdots, Z_n, \bar{Z}_1, \cdots, \bar{Z}_n$ forms a base of $\mathfrak{p}^C$ over $\boldsymbol{C}$. Let $z^1, \cdots, z^n$ be the usual complex coordinate functions on $\boldsymbol{C}^n$, and $\bar{z}^1, \cdots, \bar{z}^n$ be their complex conjugate functions. Let $S(\mathfrak{p}^C)=\sum_{j=0}^{\infty} S^j(\mathfrak{p}^C)$ be the symmetric algebra of $\mathfrak{p}^C$. We identify L with $SU(n)$ canonically. Then $SU(n)$ acts on $\mathfrak{p}$ as a subgroup of the linear isotropy group K. Extend this action to $S(\mathfrak{p}^C)$ in a usual manner. Let $P(\boldsymbol{C}^n)=\sum_{j=0}^{\infty} P_j$ be the polynomial algebra in 2n-variables $z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n$. Then we have

LEMMA 4.2. *There exists a graded algebra isomorphism* $f: S(\mathfrak{p}^C)\rightarrow P(\boldsymbol{C}^n)$ *such that* $f(Z_i)=z^i$ *and* $f(\bar{Z}_i)=\bar{z}^i$. *Furthermore* f *commutes with the action of* $SU(n)$.

PROOF. About the first half of the Lemma we refer to [5], p. 428. We remark that f carries the element $Z_1^{i_1}\cdots Z_n^{i_n}\bar{Z}_1^{j_1}\cdots\bar{Z}_n^{j_n}\in S(\mathfrak{p}^C)$ to $(z^1)^{i_1}\cdots(z^n)^{i_n}(\bar{z}^1)^{j_1}\cdots(\bar{z}^n)^{j_n}\in P(\boldsymbol{C}^n)$.

We will prove that $f|S^1(\mathfrak{p}^C)$ commutes with the action of $SU(n)$. Then by the definition of the action of $SU(n)$ on $S(\mathfrak{p}^C)$ and on $P(\boldsymbol{C}^n)$ and by the above remark, we can see that f commutes with the action of $SU(n)$. We identify $\sigma=(\sigma_{ij})_{1\leqq i,j\leqq n}\in SU(n)$ with $\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}\in L$. Since the linear isotropy action σ on $\mathfrak{p}$ is $\mathrm{Ad}(\sigma)$, we have

$$\sigma\cdot Z_i=\mathrm{Ad}(\sigma)X_i-(-1)^{1/2}\,\mathrm{Ad}(\sigma)Y_i=\sum_{j=1}^{n}\bar{\sigma}_{ji}Z_j,$$

$$\sigma\cdot\bar{Z}_i=\mathrm{Ad}(\sigma)X_i+(-1)^{1/2}\,\mathrm{Ad}(\sigma)Y_i=\sum_{j=1}^{n}\sigma_{ji}\bar{Z}_j.$$

On the other hand we have

$$\sigma\cdot z^i=z^i\circ\sigma^{-1}=\sum_{j=1}^{n}\bar{\sigma}_{ji}z^j$$

$$\sigma\cdot\bar{z}^i=\bar{z}^i\circ\sigma^{-1}=\sum_{j=1}^{n}\sigma_{ji}\bar{z}^j .$$

So $f|S^1(\mathfrak{p}^C)$ commutes with the action of $SU(n)$. Q. E. D.

By the above Lemma $S^j(\mathfrak{p}^C)$ is $SU(n)$-isomorphic with $P_j=\sum_{p+q=j}P^n_{p,q}$. We identify $S^j(\mathfrak{p}^C)$ with P_j under f.

As we showed in §. 3, B_j is a K-homomorphism, *a fortiori* L-homomorphism of $S^j(\mathfrak{p})$ into V^k. There exists a unique extension B^C_j of B_j, $B^C_j:(S^j(\mathfrak{p}))^C\rightarrow(V^k)^C$, which is also an L-homomorphism. Since $(S^j(\mathfrak{p}))^C$ is L-isomorphic with $S^j(\mathfrak{p}^C)$, we have an L-homomorphism

$$B^C_j:\ S^j(\mathfrak{p}^C)=\sum_{p+q=j}P^n_{p,q}\longrightarrow(V^k)^C .$$

Now we apply Theorem 4.1 to $SU(n)$-module $P^n_{p,q}$, then we have

$$P^n_{p,q}=H^n_{p,q}+r\cdot P^n_{p-1,q-1} . \tag{4.1}$$

By Lemma 3.2 (3), we can easily obtain

$$B^C_j(r\cdot P^n_{p-1,q-1})=0 . \tag{4.2}$$

So we get

$$B^C_j\Big(\sum_{p+q=j}P^n_{p,q}\Big)=\sum_{p+q=j}B^C_j(H^n_{p,q}) .$$

Since $H^n_{p,q}$ is $SU(n)$-irreducible, $B^C_j|H^n_{p,q}$ is zero or an isomorphism. We denote by I the set of all indices (p, q) such that $B^C_j|H^n_{p,q}$ is an isomorphism. Then we have

$$(O^j_{eK}(CP^n))^C=\sum_{p+q=j,\,(p,q)\in I}B^C_j(H^n_{p,q}) . \tag{4.3}$$

Let d be the degree of the k-th standard isometric minimal immersion of CP^n. Then by Lemma 3.2 (2), we have

$$(V^k)^C=Cf_0+\mathfrak{p}^C+(O^2_{eK}(CP^n))^C+\cdots+(O^d_{eK}(CP^n))^C . \tag{4.4}$$

Since f_0 is a K-fixed, *a fortiori* L-fixed vector, Cf_0 is an irreducible L-module with highest weight 0. Hence Cf_0 is $SU(n)$-isomorphic with $H^n_{0,0}$, $\mathfrak{p}^C=S^1(\mathfrak{p}^C)$ is $SU(n)$-isomorphic with $H^n_{1,0}+H^n_{0,1}$ by Theorem 4.1 (applied to $SU(n)$-modules $P^n_{1,0}$ and $P^n_{0,1}$) and Lemma 4.2. Therefore we have the following direct sum decomposition of $(V^k)^C$ into $SU(n)$-irreducible subspaces by (4.3) and (4.4)

$$(V^k)^C=H^n_{0,0}+H^n_{1,0}+H^n_{0,1}+\sum_{j=2}^{d}\Big(\sum_{(p,q)\in I,\,p+q=j}H^n_{p,q}\Big) . \tag{4.5}$$

We see that $\max_{(p,q)\in I}(p+q)=d$ by (4.3). Using a Proposition of Ikeda and Taniguchi ([4], p. 50), we can show that $d=2k$. But we give here another proof. First we show the following:

LEMMA 4.3. *d is not less than $2k$.*

PROOF. We denote by $\exp tH$ the one parameter subgroup in L generated by $H\in\mathfrak{h}'$. For the non-zero element $v=(\bar{z}^2)^k(z^{n+1})^k\in(V^k)^C$ and for any

$$H=\begin{pmatrix} 0 & & & \\ & (-1)^{1/2}x_2 & & \\ & & \ddots & \\ & & & (-1)^{1/2}x_{n+1} \end{pmatrix}$$

we have

$$\begin{aligned} H\cdot v &= d/dt\,|_{t=0}\exp tH\cdot v \\ &= d/dt\,|_{t=0}(e^{(-1)^{1/2}x_2t}\bar{z}^2)^k(e^{-(-1)^{1/2}x_{n+1}t}z^{n+1})^k \\ &= k((-1)^{1/2}x_2\bar{z}^2)(\bar{z}^2)^{k-1}-k((-1)^{1/2}x_{n+1}z^{n+1})(z^{n+1})^{k-1} \\ &= (-1)^{1/2}k(\lambda_1'-\lambda_n')(H)\cdot v\,. \end{aligned}$$

Let $\pi_{p,q}$: $(V^k)^C\to H^n_{p,q}$ be the projection with respect to the decomposition (4.5). Then there exists a pair $(p, q)\in I$ such that $\pi_{p,q}(v)\neq 0$. Since $\pi_{p,q}$ is an $SU(n)$-homomorphism we have

$$H\cdot\pi_{p,q}(v)=\pi_{p,q}(H\cdot v)=(-1)^{1/2}k(\lambda_1'-\lambda_n')(H)\pi_{p,q}(v)\,.$$

Then

$$k(\lambda_1'-\lambda_n')=2k\lambda_1'+k\lambda_2'+\cdots k\lambda_{n-1}'$$

is a weight of the $SU(n)$-module $H^n_{p,q}$. Since the highest weight of $H^n_{p,q}$ is equal to

$$p\lambda_1'-q\lambda_n'=(p+q)\lambda_1'+q\lambda_2'+\cdots+q\lambda_{n-1}',$$

we have

$$2k\leqq p+q\leqq\max_{(p,q)\in I}(p+q)=d\,. \qquad \text{Q.E.D.}$$

To prove our Theorem, we have only to show the following:

LEMMA 4.4. *d is not greater than $2k$.*

PROOF. Let Λ [resp. M] be the set of all weights of $(V^k)^C$ as representation of G [resp. L] and $\tilde{V}_\lambda$, $\lambda\in\Lambda$, [resp. $\tilde{\tilde{V}}_\mu$, $\mu\in$M] be the corresponding weight spaces. Then we have two weight space decompositions of $(V^k)^C$

$$(V^k)^C=\sum_{\lambda\in\Lambda}\tilde{V}_\lambda=\sum_{\mu\in\mathrm{M}}\tilde{\tilde{V}}_\mu\,. \tag{4.6}$$

It is easily seen that $\lambda|\mathfrak{h}'$ is contained in M for any $\lambda\in\Lambda$ and $\tilde{V}_{\lambda}\subset\tilde{\tilde{V}}_{\lambda|\mathfrak{h}'}$. So for every weight $\mu\in$M there exists $\lambda\in\Lambda$ such that $\lambda|\mathfrak{h}'=\mu$. Otherwise $\tilde{\tilde{V}}_{\mu}$ cannot be contained in $\sum_{\lambda\in\Lambda}\tilde{\tilde{V}}_{\lambda}$, which is a contradiction.

Put $\alpha_i=\lambda_i-\lambda_{i+1}$, $1\leqq i\leqq n$. Then it is well-known that every weight $\lambda\in\Lambda$ can be written in the following form

$$\lambda=\lambda_0-\sum_{i=1}^{n} m_i\alpha_i\,, \tag{4.7}$$

where $\lambda_0=k(\lambda_1-\lambda_{n+1})$ and m_i's are nonegative integers.

Let (p, q) be a pair in I. We choose $\lambda=\lambda_0-\sum_{i=1}^{n} m_i\alpha_i$ such that

$$\lambda|\mathfrak{h}'=p\lambda_1'-q\lambda_n'=(p+q)\lambda_1'+q\lambda_2'+\cdots+q\lambda_{n-1}'\,.$$

Then we have

$$\begin{aligned}\lambda|\mathfrak{h}'&=\Big(\lambda_0-\sum_{i=1}^{n} m_i\alpha_i\Big)|\mathfrak{h}'\\ &=\sum_{i=1}^{n}(k-m_i)(\alpha_i|\mathfrak{h}')\\ &=(k+m_1-m_2-m_n)\lambda_1'+\sum_{i=2}^{n-1}(k+m_i-m_{i+1}+m_n)\lambda_i\,.\end{aligned} \tag{4.8}$$

By the definition of λ we have

$$k+m_1-m_2-m_n=p+q\,. \tag{4.9}$$

Let S_{α_1} be the reflection of $\mathfrak{h}^*$ with respect to α_1. Then S_{α_1} is an element of the Weyl group of $\mathfrak{g}$. We get by a simple computation

$$S_{\alpha_1}(\lambda)=\lambda_0-(k-m_1+m_2)\alpha_1-\sum_{i=2}^{n} m_i\alpha_i\,.$$

Since Λ is invariant under the Weyl group, $S_{\alpha_1}(\lambda)$ must be contained in Λ, and hence

$$k-m_1+m_2\geqq 0\,.$$

This and (4.9) imply that

$$2k\geqq p+q\,,\quad \text{for any } (p, q)\in I\,.$$

So the Lemma is proved. Q. E. D.

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