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REMARKS ON d-GONAL CURVES

By

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§ 0. Introduction.

Let M be a compact Riemann surface and f be a meromorphic function on M. We denote the principal divisor associated to f by (f) and the polar divisor of f by $(f)_{\infty}$. If d=degree of the divisor $(f)_{\infty}$, we call f a meromorphic function of degree d. If d is the minimal integer in which a non-trivial meromorphic function f of degree d exists on M, then we call M a d-gonal curve. In this case the complete linear system $|(f)_{\infty}|$ has projective dimension one. Moreover if f defines a cyclic covering $M\rightarrow \boldsymbol{P}_1$ over a Riemann sphere $\boldsymbol{P}_1$, then we call M a cyclic d-gonal curve.

Now we assume that M is a p-gonal curve of genus g with a prime number p. Then Namba has shown that M has a unique linear system g_p^1 of projective dimension one and degree p provided $g>(p-1)^2$ ([6]). For example if M is defined by an equation $y^p-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0$ with $(p, r_i)=1$, $\Sigma r_i\equiv 0 \pmod p$ and $s\geqq 2p+1$, then M is p-gonal and having a unique g_p^1 ([7]).

In this paper we treat a compact Riemann surface M defined by an equation;

$$y^d-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0 \qquad *)$$

$$\text{with}\quad \Sigma r_i\equiv 0 \bmod d \quad\text{and}\quad 1\leqq r_i<d,$$

where d is not necessarily a prime number.

In § 2, we will show that M is d-gonal with the function x of degree d if there are enough r_i's relatively prime to p for each prime number p dividing d. In this case we call M a cyclic d-gonal curve. We will also show that M has a unique g_d^1 if there are more sufficient such r_i's as above (§ 2).

In § 3, let M be a cyclic d-gonal curve defined by $*$) having a unique g_d^1 and M' be a compact Riemann surface defined by $y^d-(x-b_1)^{t_1}\cdots(x-b_s)^{t_s}=0$. We will study the relations among a_i, b_i, r_i and t_i $(1\leqq i\leqq s)$ in the case M and M' are conformaly equivalent. Namba [7] and Kato [5] have already studied this problem in the case d is a prime number. We will give similar results for an arbitrary d (§ 3).

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In §4, we consider a covering map $\pi': M'\to M$, where M is a cyclic d-gonal curve with a unique g_d^1 and M' is a d'-gonal curve. In the case $d=d'$, we can apply the same methods in [3], and we will see that M' is also cyclic d-gonal. Moreover if π' is normal and $d=d'$, then the covering group of π' is isomorphic to *cyclic, dehedral, tetrahedral, octahedral* or *icosahedral*. For a general case $d\leqq d'$, we will show some relations between d and d' (§4).

In §5, we will give some remarks about coverings $M\to N$ with a cyclic d-gonal curve M having a unique g_d^1.

Finally we determine the equation $*$), which defines the curve M (with a unique g_d^1) having an automorphism V ($\notin\langle T\rangle$) of order N, where T is the automorphism defined by $T^*x=x$ and $T^*y=e^{2\pi i/d}y$ (§6).

§1. Preliminaries

At first we give several results on the existence of meromorphic functions on a compact Riemann surface M of genus g following Accola and Namba.

LEMMA 1.1. (Accola [1]) *Let M be a compact Riemann surface of genus g. Let f_1 and f_2 be two meromorphic functions on M of degree n_1 and n_2 respectively. If f_1 and f_2 generate the full field $C(M)$ of meromorphic functions on M, then* $g\leqq(n_1-1)(n_2-1)$.

The following lemma by Namba is easily obtained from Lemma 1.1.

LEMMA 1.2 (Namba [6]) *Let M be a compact Riemann surface of genus g and f be a meromorphic function of degree p on M with a prime number p.*

(1) *If h is a meromorphic function of degree n on M satisfying $(p-1)(n-1)\leqq g-1$, then p divides n and $h=r(f)$, where $r(x)$ is a rational function of degree n/p.*

(2) *If $(p-1)^2\leqq g-1$, then M is p-gonal and having a unique linear system g_p^1 of degree p and dimension* 1.

PROOF. (1) By lemma 1.1, the subfield $C(f, g)$ of $C(M)$ generated by f and h is not equal to $C(M)$. As $p=[C(M):C(f)]$ is a prime number, $C(f)=C(f, g)$. (2) If h is any meromorphic function of degree p, then $C(h)=C(f)$ by (1). □

Next we give some results concerning covering maps. Let $\pi: M'\to M$ be an arbitrary covering with compact Riemann surfaces M and M'. For a divisor $D=\Sigma n_iQ_i$ ($n_i\in Z$, $Q_i\in M'$) we define a divisor $Nm_\pi D=NmD$ by $\Sigma n_i\pi(Q_i)$. On the other hand, for a meromorphic function f on M' we denote by $Nm[f]$ the meromorphic function on M obtained by the norm map $Nm: \boldsymbol{C}(M') \to \boldsymbol{C}(M)$. It is well known that the equation of principal divisors $Nm_{\pi}(f) = (Nm[f])$ holds ([2]). When the divisor $Nm(f)$ is trivial, we can choose a constant c such that the divisor $Nm(f+c)$ is non trivial. This means that $d' \geqq d$ if M' and M are d'-gonal and d-gonal respectively.

When M and M' are both d-gonal, we have the following lemma:

LEMMA 1.3 (Ishii [3]) *Let $\pi': M' \to M$ be a covering map that both M and M' are d-gonal. Then;*

(1) *there exists a covering map $\pi: \boldsymbol{P}_1' \to \boldsymbol{P}_1$ with Riemann spheres $\boldsymbol{P}_1'$ and $\boldsymbol{P}_1$ satisfying the following diagram;*

$$\begin{array}{ccc} M' & \xrightarrow{\phi'} & \boldsymbol{P}_1' \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[Nm[\phi']]{} & \boldsymbol{P}_1 \end{array}, \quad \boldsymbol{C}(M') = \boldsymbol{C}(M) \underset{\boldsymbol{C}(\boldsymbol{P}_1)}{\otimes} \boldsymbol{C}(\boldsymbol{P}_1'), \quad \boldsymbol{C}(M) \cap \boldsymbol{C}(\boldsymbol{P}_1') = \boldsymbol{C}(\boldsymbol{P}_1),$$

where ϕ' is a morphism of degree d,

(2) *if M' has a unique g_d^1 and π' is normal, then π is also normal and $Gal(M'/M) \cong Gal(\boldsymbol{P}_1'/\boldsymbol{P}_1)$ (i.e., cyclic, dehedral, tetrahedral, octahedral, or isosahedral).*

§ 2.

Let M be a compact Riemann surface of genus g that has two meromorphic functions h and h' of degree d and d' respectively. Let $\boldsymbol{C}(h, h')$ be a subfield of $\boldsymbol{C}(M)$ generated by h and h', and $\tilde{M}$ be the compact Riemann surface of genus $\tilde{g}$ whose function field is isomorphic to $\boldsymbol{C}(h, h')$. Put $[\boldsymbol{C}(M): \boldsymbol{C}(h, h')] = t$. Then $\tilde{M}$ has meromorphic functions of degree d/t and d'/t induced by h and h' respectively. By Lemma 1.1 we have;

LEMMA 2.1. $\tilde{g} \leqq (d/t-1)(d'/t-1)$.

From now on we assume;

M is defined by the equation $*$), T is the automorphism of M defined by $(x, y) \longmapsto (x, \zeta_d y)$, where $\zeta_d = \exp(2\pi i/d)$, and h is the canonical map $M \to M/\langle T \rangle = \boldsymbol{P}_1$.

We denote by g_k the genus of the quotient compact Riemann surface $M/\langle T^k \rangle$ for a positive integer k dividing d and $k \neq d$. Moreover if $k = q$ is a prime number, we denote by s_q the number of branch points of the canonical map $M/\langle T^q\rangle \to M/\langle T\rangle \cong \boldsymbol{P}_1$. s_q is equal to the number of r_i's prime to q and we have $g_q=(q-1)(s_q-2)/2$ ($\because\ \Sigma r_i \equiv 0 \mod d$).

LEMMA 2.2. *Assume that M has a meromorphic function h' of degree d'. Let q_0 be the smallest prime number dividing G.C.D. $(d, d')=(d, d')$. If d' satisfies the inequalities:*

$$g_q>(d/q_0-1)(d'/q_0-1)\cdots\cdots\cdots **)$$

for any prime q dividing G.C.D.(d, d'),

then $t=d$ or 1. Especially when $(r_i, d)=1$ for all $1\leqq i\leqq s$, $t=d$ or 1 provided $g_{q_0}>(d/q_0-1)(d'/q_0-1)$.

PROOF. Assume $t\neq d$, 1. As $\langle T^{d/t}\rangle$ is a unique subgroup of order t in $\langle T\rangle$, $\tilde{M}$ should be isomorphic to $M/\langle T^{d/t}\rangle$ and $\tilde{g}=g_{d/t}$. For any prime number q dividing $d/t\,(\neq 1)$, we have $\langle T^q\rangle \supset \langle T^{d/t}\rangle$ and $\tilde{g}-1\geqq g_q-1\geqq (d/q_0-1)(d'/q_0-1) \geqq (d/t-1)(d'/t-1)$. This contradicts to Lemma 2.1. If $(r_i, d)=1$ for all $i=1, \cdots, s$, then $s=s_q=s_{q_0}$ and $g_q\geqq g_{q_0}$ for any prime number q dividing (d, d'). Thus the latter part of this lemma is reduced to the first part. □

PROPOSITION 2.3. *Assume M is a compact Riemann surface of genus g defined by the equation $*$). Let d' be a positive integer satisfying the inequalities $**$) in lemma 2.2 and $(d-1)(d'-1)\leqq g-1$. Then;*

(1) *If d does not divide d', then there is no meromorphic function of degree d'.*

(2) *If d divides d', then every meromorphic function h' of degree d' is obtained by $r(h)$, where r is some rational function of degree d'/d and h is the canonical map $M\to M/\langle T\rangle$.*

PROOF. Let h' be a meromorphic function of degree d'. $(d-1)(d'-1)\leqq g-1$ means $t\neq 1$ by lemma 1.1. Thus $\boldsymbol{C}(h, h')=\boldsymbol{C}(h)$ by lemma 2.2 and $h'=r(h)$ for some rational function r. □

REMARK. If $d=p$ is a prime number, this proposition is exactly same as Lemma 1.2(1).

THEOREM 2.4. *Let M be a compact Riemann surface of genus g defined by $*$) and q_0 be the smallest prime number dividing d.*

(1) *Assume $(d-1)(d-2)\leqq g-1$ and $(d/q_0-1)(d/q_0-2)\leqq g_q-1$ for any prime q dividing d. Then M is d-gonal.*

(2) *Assume* $(d-1)^2 \leqq g-1$ *and* $(d/q_0-1)^2 \leqq g_q-1$ *for any prime* q *dividing* d. *Then* M *is* d-*gonal and having a unique* g_d^1.

PROOF. (1) Assume that there is a meromorphic function h' of degree d' with $d' \leqq d-1$. By $(d-1)(d-2) \leqq g-1$ and lemma 1.1, $t=[C(M):C(h, h)] \neq 1$. As $t|(d, d')$ and $d'<d$, we have $d' \leqq d-t$. Thus $d'/q_0 \leqq d/q_0-1$ and $(d/q_0-1)(d'/q_0-1) \leqq (d/q_0-1)(d/q_0-2) \leqq g_q-1$ for any prime number q dividing d. Hence the assumptions in Proposition 2.3 are satisfied. This is a contradiction. (2) Let h' be a meromorphic function of degree d. By the same way as in (1) and Proposition 2.3(2), we have $C(h, h')=C(h)$. Thus M has a unique g_d^1. □

When $(r_i, d)=1$ for all $i=1, \cdots, s$, we can restate Theorem 2.4 as follows;

THEOREM 2.4′. (1) *If* $(d-1)(d-2) \leqq g-1$ *and* $(d/q_0-1)(d/q_0-2) \leqq g_{q_0}-1$, *then* M *is* d-*gonal.*

(2) *If* $(d-1)^2 \leqq g-1$ *and* $(d/q_0-1)^2 \leqq g_{q_0}-1$, *then* M *is* d-*gonal and having a unique* g_d^1.

PROOF. Use the latter part of Lemma 2.2. □

EXAMPLE 2.5. Let M be a compact Riemann surface defined by $y^4-x(x-a_1)(x-a_2)(x-a_3)\{(x-a_4)(x-a_5)(x-a_6)(x-a_7)\}^2=0$, where a_i $(1 \leqq i \leqq 7)$ are distinct non-zero numbers, then $g=7$. Put $N=M/\langle T^2\rangle$. N is defined by $y^2-x(x-a_1)(x-a_2)(x-a_3)=0$, *i.e.*, $g_2=1$. M satisfies the conditions of Theorem 2.4(1), and then M is 4-*gonal*. On the other hand M has infinitely many g_4^1. In fact if g_2^1 and $g_2^{1\prime}$ are two distinct linear systems on N, then $\pi^*g_2^1$ and $\pi^*g_2^{1\prime}$ are distinct linear systems of degree 4 and dimension 1 on M, where $\pi: M \to N$ is a canonical map. Thus M has infinitely many g_4^1.

EXAMPLE 2.6. For prime numbers p and q with $p \geqq q$, let M be defined by $y^{pq}-(x-a_1)^{r_1}(x-a_2)^{r_2} \cdots (x-a_s)^{r_s}=0$ with $\Sigma r_i \equiv 0 \bmod pq$ and $(r_i, pq)=1$, $1 \leqq i \leqq s$. If s satisfies $s \geqq 2pq-1$ and $(p-1)(p-2)<(q-1)(s-2)/2$, then M is pq-gonal. If s satisfies $s \geqq 2pq+1$ and $(p-1)^2<(q-1)(s-2)/2$, then M is pq-gonal and having a unique g_{pq}^1.

PROOF. These results are easily from $g=(pq-1)(s-2)/2$, $g_p=(p-1)(s-2)/2$, $g_q=(q-1)(s-2)/2$, and Theorem 2.4′. □

EXAMPLE 2.7. Let M be defined by $y^4-x^2(x-a_1)(x-a_2)(x-a_3)=0$, where a_1, a_2, a_3 are distinct non-zero numbers. The covering map $x: M \to \boldsymbol{P}_1$ is completely ramified at A_1, A_2, A_3 and Q with $x(A_i)=a_i$ $(i=1, 2, 3)$ and $x(Q)=\infty$ respectively. Also x is ramified at two points P_1 and P_2 with ramification index 2 and $x(P_1)=x(P_2)=0$. Thus $g=4(<(4-1)(4-2))$ and $g_2=1$. Then this M does not satisfy the conditions in Theorem 2.4(1). In fact M is trigonal with a principal divisor $(x/y)=P_1+P_2+Q-A_1-A_2-A_3$, and not a hyperelliptic curve by Lemma 1.2(1).

REMARK. M in Example 2.7 does not satisfy the condition of Lemma 1.2(2) for $p=3$. But M has unique g_3^1, because M has a canonical divisor $(dx/y)=2A_1+2A_2+2A_3$ and by [4] (III. 8.7).

§ 3.

In the following sections we give some applications of our results in § 2. At first we will prove the following Theorem, which have been obtained by Namba [7] and improved by Kato [5] in the case $d=p$ a prime number.

THEOREM 3.1. *Let M and M' be defined by the following equations;*

$$y^d-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0 \cdots\cdots\cdots\cdots\cdots \text{i)}$$

and

$$\tilde{y}^d-(\tilde{x}-b_1)^{t_1}\cdots(\tilde{x}-b_s)^{t_s}=0 \quad \cdots\cdots\cdots\cdots\cdots \text{ii)}$$

respectively, where $1\leqq r_i\leqq d-1$, $1\leqq t_i\leqq d-1$, $\Sigma r_i\equiv\Sigma t_i\equiv 0 \mod d$. *Assume M satisfies the conditions in Theorem 2.4(2), and M and M' are birationally equivalent. Then, by changing the indeces suitably, we have;*

(1) *there exists $A\in Aut(\boldsymbol{P}_1)$ satisfying $b_i=Aa_i$ $(1\leqq i\leqq s)$, and*

$$\#)\begin{cases} ord_p t_i = ord_p r_i & \text{if} \quad ord_p r_i < ord_p d \qquad \text{or} \\ ord_p t_i \geqq ord_p d & \text{if} \quad ord_p r_i \geqq ord_p d \quad (1\leqq i\leqq s) \end{cases}$$

for each prime number p dividing d.

(2) *if $(r_1, d)=1$, then $r_1/t_1\in(Z/dZ)^\times$ and $(r_1/t_1)t_i\equiv r_i \mod d$ $(1\leqq i\leqq s)$.*

(3) *if d is square free, then $r_1t_i\equiv t_1r_i \mod d$ $(2\leqq i\leqq s)$.*

PROOF. (1) The proof owes to the uniqueness of g_d^1 (Theorem 2.4(2)), and goes almost same way as in the proof of Theorem 1.1 in [6]. Let $\varphi: M\to M'$ be the birational map. As M has unique g_d^1, there exists $A\in Aut\,\boldsymbol{P}_1$ satisfying a commutative diagram;

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ x\downarrow & & \downarrow\tilde{x} \\ \boldsymbol{P}_1 & \xrightarrow[A]{} & \boldsymbol{P}_1 . \end{array}$$

Thus we may assume $Aa_i=b_i$ for $i=1, \cdots, s$. Let M'' be a curve defined by $z^d-(u-A^{-1}b_1)^{t_1}\cdots(u-A^{-1}b_s)^{t_s}=0$ and $\psi_A=\psi$ be a birational map from M' to M'' defined by $(\tilde{x}, \tilde{y})\to(u, z)=(A^{-1}\tilde{x}, c\tilde{y}/(\tilde{x}-\gamma)^{k'})$, where c is a suitable constant, $\gamma=A(\infty)$ and $k'=(\Sigma t_\nu)/d$ ([6]). Put $w=z\cdot\psi\cdot\varphi$, which is a meromorphic function on M. Then M is also defined by

$$w^d-(x-a_1)^{t_1}\cdots(x-a_s)^{t_s}=0 \quad\cdots\cdots\cdots\cdots \text{i}').$$

As both i) and i$'$) define the ramification type of the same cyclic covering $x: M\to P_1$, we can see #) by considering a covering map $M/\langle T^{p^{\mathrm{ord}_p d}}\rangle\to P_1$ induced by x.

(2), (3) Put $v=w^{r_1}/y^{t_1}$, then we have;

$$v^d-(x-a_2)^{r_1t_2-r_2t_1}\cdots(x-a_s)^{r_1t_s-r_st_1}=0\cdots\cdot\text{iii}).$$

Put $[\boldsymbol{C}(M):\boldsymbol{C}(x, v)]=t$. As $\boldsymbol{C}(M)\supset\boldsymbol{C}(x, v)\supset\boldsymbol{C}(x)$ are cyclic extensions, $v^{d/t}$ is in $\boldsymbol{C}(x)$ and $r_1t_i-t_1r_i\equiv0 \bmod t$ $(2\leqq i\leqq s)$ by iii). Moreover we can see that s numbers $(r_1t_i-t_1r_i)/t$ $(2\leqq i\leqq s)$ and d/t have no common divisor and $G.C.D.(r_1, t_1, d)=(r_1, t_1, d)$ divides t. On the other hand $\boldsymbol{C}(x, v)$ is the function field of the curve $M/\langle T^{d/t}\rangle$. Assume $d\neq t$, and take a prime number q dividing d/t. Then the curve $M/\langle T^q\rangle$ is defined by the following two equations simultaneously;

$$y^q-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0 \cdots\cdots\cdots\cdots\cdots\cdots\text{A})$$

and

$$v^q-(x-a_2)^{(r_1t_2-r_2t_1)/t}\cdots(x-a_s)^{(r_1t_s-r_st_1)/t}=0 \cdots\cdot\text{B}).$$

Now we will show $r_1\not\equiv0 \bmod q$. In fact this is obvious when $(r_1, d)=1$. Next we consider the case d is square free. From #) we have $(r_1, t_1, d)=(r_1, d)$. As d is square free and $(r_1, t_1, d)|t$, $(d/t, r_1, d)=(d/t, r_1)=1$ and $(r_1, q)=1$. Thus a_1 is a branch point of the covering $x: M/\langle T^q\rangle\to\boldsymbol{P}_1$ by A). But this contradicts to B). So we have $t=d$ and

$$r_1t_i-t_1r_i\equiv0 \bmod d \quad (2\leqq i\leqq s).$$

When $(r_1, d)=1$, then $(t_1, d)=1$ by #, and we get (2). □

Remark. Conversely if there exists $A\in Aut(\boldsymbol{P}_1)$ as in (1) and we have $(r_1/t_1)t_i\equiv r_i \bmod d$ $(2\leqq i\leqq s)$, then M and M' are birationally equivalent ([6]).

§4.

Next we consider a covering map $\pi': M' \to M$ with a cyclic d-gonal curve M defined by $*$) of genus g and a d'-gonal curve M' of genus g'.

THEOREM 4.1. *Assume $d=d'$. Then;*

(1) *M' is also a cyclic d-gonal curve.*

(2) *If M satisfies the conditions of Theorem* 2.4(2) *aud π' is normal, then the Galois group af π' is cyclic, dehedral, tetrahedral, octahedral or isosahedral.*

PROOF. (1) Easily from Lemma 1.3(1). (2) Let T (resp. T') be the automorphism of order d on M (resp. M') as in §2. By the commutative diagram in Lemma 1.3 and the uniqueness of g_d^1 on M we may assume that T' induces T. For each prime number q dividing d, we have a commutative diagram;

$$\begin{array}{ccc} M' & \longrightarrow & M'/\langle T'^q\rangle \\ \downarrow & & \downarrow \\ M & \longrightarrow & M/\langle T^q\rangle. \end{array}$$

Let g_q' be genus of $M'/\langle T'^q\rangle$. As $g \leqq g'$ and $g_q \leqq g_q'$, M' is also satisfying the conditions in Theorem 2.4(2). Then M' has a unique g_d^1. By Lemma 1.3(2) we have our results. □

THEOREM 4.2. *Assume $d \leqq d'$. If d and d' satisfy the conditions of Proposition* 2.3. *on M, then d divides d'.*

PROOF. Let D' be a positive divisor of degree d' on M' such that $|D'|$ has projective dimension 1. Assume $Nm_\pi D'$ has some common point with $Nm_\pi E$ for each $E \in |D'|$. Then each $E \in |D'|$ has some common point with $\pi^* NmD'$. On the other hand if E and E' in $|D'|$ have common points, then $E=E'$ by the minimality of d'. Hence $|D'|$ should be a finite set. This is a contradiction. Thus there is a meromorphic function h of degree d' on M' and $Nm[h]$ is also of degree d' on M'. By Proposition 2.3 we have $d \mid d'$. □

COROLLARY 4.3. *Let $\pi': M' \to M$ be an unramified covering of degree q with a cyclic p-gonal curve M of genus g, where p and q are distinct prime numbers. Assume $g > p^2q - 2p + 1$. Then;*

(a) *M' is a pq-gonal curve with a unique g_{pq}^1.*

(b) *Let $\phi: M' \to \boldsymbol{P}_1'$ be the covering map defined by g_{pq}^1 in* a), *then;*

(b-i) *ϕ is not cyclic (i.e., M' is not a cyclic pq-gonal curve).*

(b-ii) *if $p \nmid q-1$, then ϕ is not normal.*

PROOF. (a) Let $h: M \to \boldsymbol{P}_1$ be the covering map of degree p, then $h \circ \pi'$ is a meromorphic function of degree pq on M'. For $g > p^2q - 2p + 1 > (pq-1)(p-1)$, M' is pm-gonal $(1 \leqq m \leqq q-1)$ or pq-gonal by Theorem 4.2. (see the remark of Proposition 2.3). Now we assume that M' is pq-gonal. Let ψ be a meromorphic function of degree pq on M'. Put $K = \boldsymbol{C}(\psi, h \circ \pi')$ and $[\boldsymbol{C}(M'): K] = t$. As the genus g' of M' is $q(g-1)+1$, we have $g' > (pq-1)^2$ and $t \neq 1$. Consider the following diagram;

$$\begin{array}{ccc} \boldsymbol{C}(M') \supset & K & \supset \boldsymbol{C}(\psi) \\ \cup & & \cup \\ \boldsymbol{C}(M) & \supset & \boldsymbol{C}(h \circ \pi'). \end{array}$$

If $t = q$, then $[K: \boldsymbol{C}(h \circ \pi')] = p$ and genus of $K = g$ ($\because$ π' is unramified and $(p, q) = 1$). For $g > (p-1)^2$, $K = \boldsymbol{C}(h \circ \pi')$. This is a contradiction. If $t = p$, then $K \supset \boldsymbol{C}(h \circ \pi')$ is an unramified extension. As $\boldsymbol{C}(h \circ \pi')$ is of genus 0, this is a contradiction. Thus we have $t = pq$ and M' has a unique g^1_{pq}. If M' is pm-gonal $(1 \leqq m \leqq q-1)$ and ψ is a meromorphic function of degree pm on M', then $[\boldsymbol{C}(M'): \boldsymbol{C}(\psi, h \circ \pi')] = p$ by $(p, q) = 1$ and $g' > (pm-1)(pq-1)$. This is a contradiction.

(b-i) We may assume $h \circ \pi' = \psi$ by (a). If ψ is cyclic, then there exists an automorphism T' on M' of order p, and we have a commutative diagram;

$$\begin{array}{ccc} M' & \longrightarrow & M'/\langle T' \rangle \\ {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\ M & \xrightarrow[h]{} & M/\langle T \rangle = \boldsymbol{P}_1, \end{array}$$ where π' is unramified.

For $(p, q) = 1$, π is unramified. This is a contradiction. (b-ii) Assume ψ is normal with galois group G. If $p < q$ and $p \nmid q-1$, it is well known that G is cyclic. But this can not be happened by (a). If $p > q$, then G has a unique normal subgroup $\langle T' \rangle$ of index q generated by T'. Thus we have a same commutative diagram as in the proof of (b-i). This is also a contradiction. □

§5.

We consider a covering $\pi': M \to N$, where M is cyclic d-gonal and N is e-gonal. Put $deg\, \pi = n$ and $d' = ne$.

THEOREM 5.1. *Assume d and d' satisfy the conditions of Proposition 2.3. Then e divides d. Moreover if $u: M \to M/\langle T^{d/e} \rangle$ is the canonical map, then there exists a covering map $v: M/\langle T^{d/e} \rangle \to N$ satisfying $\pi' = v \circ u$. Especially when $d = d' = ne$, N is isomorphic to $M/\langle T^{d/e} \rangle$.*

Proof. Let $\phi_N: N \to \tilde{P}_1$ be the covering over Riemann sphere $\tilde{P}_1$ of degree e. Then $\phi_N \circ \pi'$ is a meromorphic function on M of degree $d'=ne$. By Proposition 2.3, d divides $ne=d'$, and we have a commutative diagram;

$$\begin{array}{ccc} M' & \xrightarrow{\ h\ } & P_1 = M/\langle T\rangle \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ N & \xrightarrow[\ \phi_N\ ]{} & \tilde{P}_1\ , \end{array}$$

with a rational function $\tilde{\pi}$ of degree d'/d and the canonical map h. The function fields $C(N)$ and $C(P_1)$ are linearly independent over $C(\tilde{P}_1)$ for the minimality of e. Then there exists a e-gonal curve $\tilde{M}$ with a function field $C(\tilde{M})$ isomorphic to $C(P_1) \underset{C(\tilde{P}_1)}{\otimes} C(N)$. By the universal property of $C(\tilde{M})$ we have the following commutative diagram;

$$\begin{array}{cccc} M & \searrow & \overset{h}{\longrightarrow} & \searrow \\ \downarrow{\scriptstyle \pi} & \tilde{M} & \xrightarrow{\ \tilde{\phi}\ } & P_1 = M/\langle T\rangle \\ \searrow & \downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ & N & \xrightarrow[\ \phi_N\ ]{} & \tilde{P}_1 \end{array}$$

where $deg\,\tilde{\phi}=e$ and $deg\,\tilde{\pi}=ne/d$. We can see that e divides d. As h is a cyclic extension, $\tilde{M} \cong M/\langle T^{d/e}\rangle$. □

Example 5.2. Let M be the cyclic pq-gonal curve defined in Example 2.6 with $p \geqq q$, $s \geqq 2pq+1$ and $(p-1)^2 < (q-1)(s-2)/2$. Then any covering $\pi: M \to N$ of degree p (resp. q) with a q (resp. p)-gonal curve N is birational to the cyclic q (resp. p)-gonal curve defined by $y^q-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0$ (resp. $y^p-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0$).

§6.

Let M be a cyclic d-gonal curve with a unique g_d^1 defined by

$$y^d-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s}=0,\quad \Sigma r_i \equiv 0 \bmod d, \cdots\cdots *)$$

$(r_i, d)=1$ for all i, here we can take ∞ as one of a_i's.

Let T be the automorphism of order d as in §2, and $\phi: M \to M/\langle T\rangle$ be the canonical map. We will determine the equation $*$), which defines M having an automorphism V ($\notin \langle T\rangle$) of order N.

For the uniqueness of g_d^1, we have $V\langle T\rangle V^{-1}=\langle T\rangle$ and V induces an automorphism $\tilde{V}$ on $M/\langle T\rangle=\boldsymbol{P}_1(x)$. Let $\boldsymbol{C}(x)$ and $\boldsymbol{C}(u)$ be the function fields of $M/\langle T\rangle$ and $M/\langle V, T\rangle$ respectively. Then $\pi'\colon M/\langle T\rangle\rightarrow M/\langle T, V\rangle$ is a cyclic covering of order N' $(N'|N)$ and we may assume $\pi'^*u=x^{N'}$.

Before considering generally, we study the following two cases;

$$\text{Case 1)}\quad \langle T\rangle\cap\langle V\rangle=\langle T\rangle,\qquad \text{Case 2)}\quad \langle T\rangle\cap\langle V\rangle=\{1\}.$$

Case 1) $\langle T\rangle\cap\langle V\rangle=\langle T\rangle$

We can see that $d|N$ and $N'=N/d$. We may assume $V^{N/d}=T$ and $\tilde{V}^*x=\zeta'x$ with a primitive N'-th root ζ' of 1. We denote the set {fixed point of $\tilde{V}$} by $F(\tilde{V})$. Then $\#F(\tilde{V})=2$.

Case 1-a) $\#F(\tilde{V})\cap\{a_1, \cdots, a_s\}=2$

We may assume that two elements of the above set are $a_{s-1}=0$ and $a_s=\infty$. As $\tilde{V}$ acts on $\{a_1, \cdots, a_{s-2}\}$ faithfully, M can be defined by;

$$\text{A)}\qquad y^d=x\Big\{\prod_{t=1}^{k}\prod_{j=1}^{N/d}(x-\zeta'^{j-1}c_t)^{m_{N/d\cdot(t-1)+j}}\Big\},$$

$$1+\sum_{t=1}^{k}\sum_{j=1}^{N/d}m_{N/d\cdot(t-1)+j}\not\equiv 0 \bmod d,$$

where $(m_*, d)=1$, and c_t $(\neq 0)$ are distinct complex numbers satisfying

$$\{\zeta'^{j-1}c_t\,|\,1\leqq j\leqq N/d\}\cap\{\zeta'^{j-1}c_s\,|\,1\leqq j\leqq N/d\}=\emptyset \qquad \text{for}\quad t\neq s.$$

By acting V^* on both sides of A), we have;

$$\text{B)}\qquad (T^*y)^d=\zeta'^M\Big\{\prod_{t=1}^{k}\prod_{j=1}^{N/d}(x-\zeta'^{j-2}c_t)^{m_{N/d\cdot(t-1)+j}}\Big\}x,$$

where $M=1+\sum_{t=1}^{k}\sum_{j=1}^{N/d}m_{N/d\cdot(t-1)+j}$.

By the proof of Theorem 3.1 and comparing A) with B), there exists a positive integer v $(1\leqq v<d,\ (v, d)=1)$ satisfying $v\cdot m_{N/d\cdot(t-1)+j}\equiv m_{N/d\cdot(t-1)+j+1} \bmod d$ $(1\leqq j\leqq N/d-1)$, and $vm_{N/d\cdot t}\equiv m_{N/d\cdot(t-1)+1} \bmod d$. But in this case, $v\cdot 1\equiv 1 \bmod d$. Thus we have $v=1$ and $m_{N/d\cdot(t-1)+1}=\cdots=m_{N/d\cdot t}\overset{put}{=}r_t$ $(t=1\leqq t\leqq k)$. The equation A) is;

$$\text{I)}\qquad y^d=x\Big\{\prod_{t=1}^{k}\prod_{j=1}^{N/d}(x-\zeta'^{j-1}c_t)^{r_t}\Big\}=x\cdot\prod_{t=1}^{k}(x^{N/d}-b_t)^{r_t},$$

As $V^*y^d=\zeta'y^d$ and V is of order N, we have $V^*y=\eta y$, where η satisfies $\eta^d=\zeta'$ and $\eta^{N'}$ is a primitive N/N' $(=d)$-th root of 1.

PROPOSITION 6.1a). *Case* 1-a *happens if and only if* M *is defined by* I) *with* $d|N$, $(r_t, d)=1$ $(t=1, \cdots, k)$ *and* $N/d\sum_{t=1}^{k} r_t+1 \not\equiv 0 \bmod d$. V *is defined by*

$$V^*x=\zeta' x \quad \text{and} \quad V^*y=\eta y, \cdots\cdots\cdots\cdots\cdots\cdots 1)$$

where ζ' *is a primitive* N'*-th root of* 1, η *satisfies* $\eta^d=\zeta'$ *and* $\eta^{N'}$ *is a primitive* d*-th root of* 1 (*for example*, $\eta=e^{2\pi i/N}$ *and* $\zeta'=e^{2\pi i/N'}$ *satisfy these conditions*).

Case 1-b) $\#F(\tilde{V})\cap\{a_1, \cdots, a_s\}=1$

We may assume that the element of the above set is a_s. There exists a point $P\in M$ such that $\phi(P)\notin\{a_1, \cdots, a_s\}$ and $V(P)\in\langle T\rangle P=\langle V^{N/d}\rangle P$. Then $V^d(P)=P$. If $(d, N/d)=r\neq 1$, then $T^{d/r}P=V^{N/d\cdot d/r}P=P$. This contradicts to $\phi(P)\notin\{a_1, \cdots, a_s\}$. Thus $(d, N/d)=1$ and $\langle V^d\rangle\cap\langle V^{N/d}\rangle=\{1\}$. We have $C(M)=C(M/\langle V^{N/d}\rangle)\underset{C(M/\langle V\rangle}{\otimes} C(M/V^d)$, Assume $\phi(P)=\infty$, $a_s=0$ and $\pi'^*u=x^{N/d}$. As $M/\langle V^d\rangle\to M/\langle V\rangle=\boldsymbol{P}_1(u)$ is cyclic of degree d, $C(M/\langle V^d\rangle)$ is defined by $y^d=u\prod_{t=1}^{k}(u-b_t)^{n_t}$, with $(n_t, d)=1$ $(t=1, \cdots, e)$ and $1+n_1+\cdots+n_k\not\equiv 0 \bmod d$. Then M is defined by $y^d=x^{N/d}(x^{N/d}-b_1)^{n_1}\cdots(x^{N/d}-b_k)^{n_k}$. For $(d, N/d)=1$, M can be defined by the following equation;

II) $$y^d=x\cdot(x^{N/d}-b_1)^{r_1}\cdots(x^{N/d}-b_k)^{r_k}, \quad \text{with} \quad 1+\Sigma r_t\not\equiv 0 \bmod d.$$

After all, we have;

PROPOSITION 6.1b). *Case* 1-b) *happens if and only if* $(N/d, d)=1$ *and* M *is defined by* II) *with* $(r_t, d)=1$ *and* $1+\sum_{t=1}^{e} r_t\not\equiv 0 \bmod d$. V *is defined by*;

$$V^*x=\zeta' x \quad \text{and} \quad V^*y=\eta y, \cdots\cdots\cdots\cdots\cdots\cdots 2)$$

where ζ' *is a primitive* N'*-th root of* 1, η *satisfies* $\eta^d=\zeta'$ *and* $\eta^{N'}$ *is a primitive* d*-th root of* 1.

Case 1-c) $\#F(\tilde{V})\cap\{a_1, \cdots, a_s\}=\emptyset$

By the same way as in Case 1-b), we have;

PROPOSITION 6.1c). *Case* 1-c) *happens if and only if* $(N/d, d)=1$ *and* M *is defined by*;

III) $$y^d=(x^{N/d}-b_1)^{r_1}\cdots(x^{N/d}-b_k)^{r_k}$$

with $(r_t, d)=1$ *and* $\sum_{t=1}^{k} r_t\equiv 0 \bmod d$. V *is defined by*;

$$V^*x=\zeta' x \quad and \quad V^*y=\zeta'' y, \quad \cdots\cdots\cdots\cdots\cdots\cdots 3)$$

where ζ' *(resp.* ζ''*) is a primitive* N' *(resp.* d*)-th root of* 1.

Case 2) $\langle T\rangle\cap\langle V\rangle=\{1\}$

The automorphism $\tilde{V}$ on $M/\langle T\rangle$ induced by V is of order N, and we may assume that $\tilde{V}^*x=\zeta x$ with a primitive N-th root ζ of 1.

Case 2-a) $\#\{a_1, \cdots, a_s\}\cap F(\tilde{V})=2$ and

Case 2-b) $\#\{a_1, \cdots, a_s\}\cap F(\tilde{V})=1$

By the same way as in Case 1-a), M can be defined by

$$\text{IV)} \qquad y^d=x\prod_{t=1}^{k}(x^N-b_t)^{r_t}, \quad \text{with} \quad (r_t, N)=1.$$

In Case 2-a) (resp. 2-b), $N\sum_{t=1}^{k} r_t+1\not\equiv 0$ (resp. $\equiv 0$) mod d. As V satisfies $V^*y^d=\zeta\cdot y^d$ and V is of order N, V is defined by;

$$V^*x=\zeta x \quad \text{and} \quad V^*y=\xi\cdot y, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots 4)$$

where ξ is a N-th root of 1 satisfying $\xi^d=\zeta$. $\therefore$ $(d, N)=1$ and ξ is also a primitive N-th root of 1. After all we have;

PROPOSITION 6.2. *Case* 2-a) *(resp.* 2-b)) *happens if and only if* $(N, d)=1$ *and* M *is birational to the curve defined by* IV) *with* $(r_t, N)=1$ *and* $N\sum_{t=1}^{k} r_t+1\not\equiv 0$ *(resp.* $\equiv 0$) mod d. V *is defined by* 4) *with a primitive* N*-th root* ξ *of* 1 *and* $\zeta=\xi^d$.

Case 2-c) $\#\{a_1, \cdots, a_s\}\cap F(\tilde{V})=\emptyset$

By the same way as in Case 1-a), M is birational to the curve defined by

$$y^d=\left\{\prod_{t=1}^{k}\prod_{j=1}^{N}(x-\zeta^{j-1}b_t)^{m_{N(t-1)+j}}\right\} \quad \text{with} \quad \sum_{t=1}^{k}\sum_{j=1}^{N} m_{N/d\cdot(t-1)+j}\equiv 0 \bmod d$$

and $(m_*, d)=1$. Moreover there exists a positive integer v $(1\leqq v\leqq d-1, (v, d)=1)$ satisfying $vm_{N(t-1)+j}\equiv m_{N(t-1)+j+1}$ mod d $(1\leqq j\leqq N-1)$, and $vm_{N\cdot t}\equiv m_{N(t-1)+1}$ mod d. We see $v^N\equiv 1$ mod d. Thus M is defined by

$$\text{V)} \qquad y^d=\prod_{t=1}^{k}\prod_{j=1}^{N}(x-\zeta^{j-1}b_t)^{n_t v^{j-1}}$$

with positive integers n_t satisfying $\sum_{t=1}^{k}\sum_{j=1}^{N} n_t v^{j-1}\equiv 0$ mod d and $(n_*, d)=1$. Put $R=\Sigma n_t$ and $S=\Sigma v^{j-1}$. Then $RS\equiv 0$ mod d. By acting V^* on the both sides of V again, we have

$$
\begin{aligned}
(V^*y)^d &= \prod_{t=1}^{k}\prod_{j=1}^{N}(\zeta x-\zeta^{j-1}b_t)^{n_t v^{j-1}} \\
&= \zeta^{RS}\prod_{t=1}^{k}\prod_{j=1}^{N}(x-\zeta^{j-2}b_t)^{n_t v^{j-1}} \\
&= \begin{cases} \zeta^{RS}y^{vd}/\prod_{t=1}^{k}(x-\zeta^{N-1}b_t)^{n_t(v^N-1)},\ \zeta^{RS}\neq 1 & (\text{if } RS\not\equiv 0 \bmod N). \\ \text{or} & \\ y^{vd}/\prod_{t=1}^{k}(x-\zeta^{N-1}b_t)^{n_t(v^N-1)} & (\text{if } RS\equiv 0 \bmod N). \end{cases}
\end{aligned}
$$

Then we have;

$$
V^*y=\begin{cases} \eta\zeta^{RS/d}y^v/\prod_{t=1}^{k}(x-\zeta^{N-1}b_t)^{n_t(v^N-1)/d}, & (\text{if } RS\not\equiv 0 \bmod N)\cdots\cdot\text{V-i)} \\ \text{or} & \\ \eta y^v/\prod_{t=1}^{k}(x-\zeta^{N-1}b_t)^{n_t(v^N-1)/d}, & (\text{if } RS\equiv 0 \bmod N)\cdots\cdot\text{V-ii)}. \end{cases}
$$

where η is some d-th root (not necessarily primitive) of 1.

Assume $RS\not\equiv 0 \bmod N$. Using V-i) repeatedly, we have;

$$
\begin{aligned}
V^{*N}y &= \eta^S\zeta^{(RS/d)S}y^{v^N}\Big/\left[\left\{\prod_{l=0}^{N-1}\prod_{t=1}^{k}(\zeta^l x-\zeta^{N-1}b_t)^{n_t}\right\}^{v^{N-1-l}}\right]^{(v^N-1)/d} \\
&= \eta^S\zeta^{(RS/d)S}y^{v^N}/\zeta^{R(v^{N-2}+2v^{N-3}+\cdots(N-)v^0)}\left[\left\{\prod_{l=0}^{N-1}\prod_{t=1}^{k}(x-\zeta^{N-l-1}b_t)^{n_t}\right\}^{v^{N-1-l}}\right]^{(v^N-1)/d} \\
&= \eta^S\zeta^{(RS/d)S-R(S^2-NS)/d}y^{v^N}/(y^d)^{(v^N-1)/d}=\eta^S\zeta^{RNS/d}y=\eta^S y \quad (\because RS\equiv 0 \bmod d).
\end{aligned}
$$

For $V^{*N}y=y$, $\eta^S=1$ should be held.

When $RS\equiv 0 \bmod N$, by the same way as above, we have;

$$
V^{*N}y=\eta^S\zeta^{-R(S^2-NS)/d}y^{v^N}(y^d)^{(v^N-1)/d}=\eta^S\zeta^{-RS^2/d}y\,.
$$

Thus η should satisfy $\eta^S=\zeta^{RS^2/d}$.

PROPOSITION 6.3. *Case* 2-c) *happens if and only if M is birational to the curve defined by* V) *with $v^N\equiv 1 \bmod d$ and $RS\equiv 0 \bmod d$. If $RS\not\equiv 0$ (resp. $RS\equiv 0$) $\bmod N$, V is defined by $V^*x=\zeta x$ and* V-i) *(resp.* V-ii)*) with d-th root η of* 1 *satisfying $\eta^S=1$ (resp. $\eta^S=\zeta^{RS^2/d}$), here η is not necessarily primitive (for example, $\eta=1$ (resp. $\eta=\zeta^{RS/d}$) satisfies $\eta^S=1$ (resp. $\eta^S=\zeta^{RS^2/d}$)).*

General case $\langle T\rangle\cap\langle V'\rangle=\langle V^{N'}\rangle=\langle T^{d'}\rangle$.

We can obtain the equations of M and V as follows. We may assume that $N'|N$ and $d'|d$, then $d/d'=N/N'$. The case $d'=1$ is exactly same as the case 1) (Propositions 6-1a$\sim$c)).

When $d'>1$, put $M'=M/\langle T\rangle\cap\langle V\rangle$. Then M' is d'-gonal with a unique $g^1_{d'}$ having an automorphism V' $(=V \bmod \langle V^{d'}\rangle)$ of order d'. We can apply Proposition 6.2 or 6.3, and M' is defined by an equation of type IV) or V).

For example, assume M' is defined by;

$$y'^{d'}=\prod_{t=1}^{k'}\prod_{j=1}^{N'}(x-\zeta'^{j-1}b'_t)^{n'_t v'^{j-1}} \qquad \text{(cf. V)}$$

with $(n'_*, d')=(v', d')=1$, $1\leqq v'\leqq d'-1$, and $R'S'\equiv 0 \bmod d'$, where $R'=\sum_{t=1}^{k'} n'_t$, $S'=\sum_{j=1}^{N'} v'^{j-1}$ and a primitive N'-th root ζ' of 1. Moreover, assume $R'S'\not\equiv 0 \bmod N'$. Then V' is defined by;

$$\begin{cases} V'^*x=\zeta' x \\ V'^*y'=\eta'\zeta'^{R'S'/d'}y'^{v'}/\prod_{t=1}^{k'}(x-\zeta'^{N'-1}b'_t)^{n'_t(v'^{N'}-1)/d'} \end{cases} \qquad \text{(cf. V-i)},$$

with d'-th root η' (not necessarily primitive) of 1 satisfying $\eta'^{s'}=1$. Put $y'=y^{d/d'}$, we can have the equation of M;

$$y^d=\prod_{t=1}^{k'}\prod_{j=1}^{N'}(x-\zeta'^{j-1}b'_t)^{n'_t v'^{j-1}}. \quad \cdots\cdots\cdots \text{VI)}$$

As M is defined by $*$), we have $R'S'\equiv 0 \bmod d$, $(n'_*, d)=(v', d)=1$ and $v'^N\equiv 1 \bmod d$. Thus V on M is defined by;

$$\begin{cases} V^*x=\zeta' x \\ V^*y=\eta\zeta'^{R'S'/d}y^{v'}/\prod_{t=1}^{k'}(x-\zeta'^{N'-1}b'_t)^{n'_t(v'^{N'}-1)/d}, \end{cases}$$

where η satisfies $\eta^{d/d'}=\eta'$. We can see $V^{*N'}y=\eta^{S'}y$. As V is of order N, $\eta'^{S'}$ should be a primitive N/N' $(=d/d')$ root of 1. When $(S', d/d')=1$, $\eta'=1$, and $\eta=\exp(2\pi i d'/d)$ satisfies these conditions,

Considering the other cases, we finally have;

THEOREM 6.4. *Let M be a cyclic d-gonal curve with a unique g^1_d defined by* $*$) *with an automorphism V $(\notin\langle T\rangle)$ or order N. Then M and V are determined as the following types*;

I) *Let d' (>1) and N' (>1) be two integers satisfying $d'|d$, $N'|N$ and $d/d'=N/N'\neq 1$.*

I-i) *M is a curve defined by the equation*

$$y^d=\prod_{t=1}^{k'}\prod_{j=1}^{N'}(x-\zeta'^{j-1}b_t)^{n'_t v'^{j-1}} \quad \cdots\cdots\cdots \text{VI)}$$

with $1\leqq v'\leqq d'-1$, $(n'_*, d)=(v', d)=1$ *and* $S'R'\equiv 0 \bmod d$.

If $S'R'\not\equiv 0 \bmod N'$, *then* V *is defined by*

$$\begin{cases} V^*x=\zeta' x \\ V^*y=\eta\zeta'^{R'S'/d}y^{v'}/\prod_{t=1}^{k'}(x-\zeta'^{N'-1}b'_t)^{n'_t(v'^{N'}-1)/d}, \end{cases}$$

where η *is a d-th root (not necessarily primitive) of* 1 *such that* $\eta^{S'}$ *is a primitive* d/d'*-th root of* 1. (*for example, when* $(S', d/d')=1$, $e^{2\pi i d'/d}$ *can be taken as* η).

If $S'R'\equiv 0 \bmod N'$, V *is defined by*

$$\begin{cases} V^*x=\zeta' x \\ V^*y=\eta y^{v'}/\prod_{t=1}^{k'}(x-\zeta'^{N'-1}b'_t)^{n'_t(v'^{N'}-1)/d}, \end{cases}$$

where η *is a d-th root (not necessarily primitive) of* 1 *such that* $\eta\zeta'^{-R'S'^2/d}$ *is a primitive* d/d'*-th root of* 1. (*for example, when* $(S', d/d')=1$, *we can take* $\zeta'^{R'S'/d}\zeta_{d/d'}$ *as* η, *where* $\zeta_{d/d'}$ *is a primitive* d/d'*-th root of* 1). (*cf. Prop.* 6.3)

I-ii) *If* $(d', N')=1$, *we have an additional type*;

$$y^d=x\prod_{t=1}^{k}(x^{N'}-b_t)^{r_t}$$

with $(r_t, N)=1$. *In this case* V *is defined by*;

$$V^*y=\xi y \quad \text{and} \quad V^*x=\xi^d x,$$

where ξ *is a primitive N-th root of* 1. (*cf. Prop.* 6.2)

II) *In case of* $d|N$, *in addition to* 1), *we have other types of M and V as follows*;

II-i) *M and V in Proposition* 6.1a).

II-ii) *In addition to* II-i), *M and V in Proposition* 6.1b) *and* 6.1c), *provided* $((d, N/d)=1$.

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