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ASYMPTOTIC BEHAVIOUR OF DENSITIES OF MULTI-DIMENSIONAL STABLE DISTRIBUTIONS

By

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Abstract. In one-dimension asymptotic behaviour of densities of stable distributions is well-known. However, in multi-dimensional cases it is very difficult to investigate asymptotic behaviour of densities of non-degenerate stable distributions in general. In the present paper we give the following two results: If the Lévy measure of the stable distribution has mass at a half-line, then the density decreases along the half-line with the same order as in one-dimensional case. If the Lévy measure is supported only on finitely many halflines, then we can determine asymptotic behaviour along each direction starting at 0.

Keywords: multi-dimensional stable distribution, Lévy-Ito decomposition of Lévy processes.

1. Introduction and results

Let $\mu(dx)$ be a *stable distribution* on $\boldsymbol{R}^d$ with exponent $0<\alpha<2$. Then its log-characteristic function $\Psi(z)$ is given as follows: For $z=|z|\xi$, $\xi\in S^{d-1}=\{x\in\boldsymbol{R}^d : |x|=1\}$,

$$\Psi(z)=-|z|^{\alpha}\int_{S^{d-1}}|\langle\xi, \theta\rangle|^{\alpha}\left[1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\xi, \theta\rangle\right]\lambda(d\theta)+i\langle z, b\rangle \quad \text{if } \alpha\neq 1,$$

$$-|z|\int_{S^{d-1}}|\langle\xi, \theta\rangle|\left[1+i\frac{2}{\pi}\operatorname{sgn}\langle\xi, \theta\rangle\log|\langle z, \theta\rangle|\right]\lambda(d\theta)+i\langle z, b\rangle \quad \text{if } \alpha=1,$$

where $\lambda(d\theta)$ is a finite measure on S^{d-1} and $b\in\boldsymbol{R}^d$. If $b=0$ $(\alpha\neq 1)$ or $\int\theta\lambda(d\theta)=0$ $(\alpha=1)$, then μ satisfies the *scaling property*: $\mu^{t*}(dx)=t^{-d/\alpha}\mu(t^{-1/\alpha}dx)$, in this case μ is called *strictly stable*. Note that the Lévy measure $n(dx)$ of μ is given by

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$$n(dx)=\int_{S^{d-1}}\lambda(d\theta)\int_0^\infty 1_{dx}(r\theta)r^{-1-\alpha}dr \qquad \text{on} \quad \boldsymbol{R}^d\setminus\{0\}.$$

We say that μ is non-degenerate if the support of μ spans $\boldsymbol{R}^d$, or equivalently the support of λ spans $\boldsymbol{R}^d$. Write this condition **Span Spt** $\lambda=\boldsymbol{R}^d$.

Throughout the present paper we always assume that μ is a non-degenerate stable distribution on $\boldsymbol{R}^d$. It is then well-known that $\mu(dx)$ is absolutely continuous and has a density $p(x)$ with respect to the Lebesgue measure dx on $\boldsymbol{R}^d$, which is expressed as

$$p(x)=\frac{1}{(2\pi)^d}\int_{R^d}\exp[-i\langle x, z\rangle+\Psi(z)]dz. \tag{1.1}$$

Furthermore $p(x)$ is a C^∞-function with derivatives of all orders vanishing at infinity (cf. [6], [7], [8] and [9]).

If we write $p(x)=p(x;b)$, then $p(x;b)=p(x-b;0)$. Henceforth, we assume $b=0$. Then note that μ is strictly stable except $\alpha=1$.

We are concerned with asymptotic behaviour of the density function $p(x)$ as $|x|\to+\infty$. In one-dimension it is well-known that $p(x)$ decreases like $|x|^{-1-\alpha}$ as $x\to+\infty$ if λ has mass at $\{+1\}$. In addition, if λ has no mass at $\{-1\}$, then $p(x)=0$ for $x<0$ when $0<\alpha<1$, and $p(x)>0$ for $x<0$ and decreases exponentially fast as $x\to-\infty$ when $1\leqq\alpha<2$ (see § 2). In multi-dimensional cases Pruitt and Taylor [6] give an upper estimate $p(x)\leqq K|x|^{-1-\alpha}$ for a strictly stable density. When λ is absolutely continuous and has a continuous density with respect to the uniform measure on S^{d-1}, Dziubanski [2] investigates an asymptotic behaviour $p(r\sigma)\sim cr^{-d-\alpha}$ as $r\to+\infty$, where $\sigma\in S^{d-1}$, $c=c(\sigma)\geqq 0$ and $a\sim b$ means that $a/b\to 1$. Furthermore Arkhipov [1] gives an asymptotic expansion of $p(r\sigma)$ under some additional regularity condition on the density of λ. On the other hand one can easily deduce that if λ is supported only on the orthonormal basis of $\boldsymbol{R}^d$, then $p(x)=\prod_{j=1}^d p_j(x_j)$, where $x=(x_1, \cdots, x_d)$ and p_j is a one-dimensional density corresponding to e_j. Therefore if $\sigma\in S^{d-1}\cap\{x_j>0, j=1, \cdots, d\}$, then we have $p(r\sigma)\sim cr^{-d(1+\alpha)}$ as $r\uparrow+\infty$, where $c=c(\sigma)>0$.

From these results it would be expected that a general α-stable density $p(r\sigma)$ on $\boldsymbol{R}^d$ has the following asymptotic property: For each $\sigma\in S^{d-1}$ there exist $c=c(\sigma)>0$ and $k=k(\alpha, \sigma)\geqq 1+\alpha$ such that

$$p(r\sigma)\sim cr^{-k} \qquad \text{as} \quad r\to+\infty.$$

In this paper we first discuss a lower estimate for a general stable density $p(r\sigma)$ and we show that a lower estimate coincides with that of the upper estimate when λ has mass at σ. Furthermore we show that the above asymptotic relation is valid when λ is a discrete measure whose support consists of only finitely many points in S^{d-1}.

Our first result is the following: Let μ be a non-degenerate stable distribution on R^d and **Con Spt** λ be the smallest convex hull in S^{d-1} containing all elements of **Spt** λ, and **Int** S denotes the interior of a set S in S^{d-1}. Recall that $b=0$.

THEOREM 1. *Suppose that* λ *has mass at* $\sigma_0 \in S^{d-1}$, *i.e.,* $\lambda(\{\sigma_0\})>0$. *If* $0<\alpha<1$ *and* $\sigma_0 \in$ **Int** (**Con Spt** λ), *or if* $1 \leqq \alpha < 2$, *then there exist positive constants* $C_1=C_1(\alpha, \sigma_0)$ *and* $r_0=r_0(\alpha, \sigma_0)$ *such that* $0<C_1 \leqq r^{1+\alpha} p(r\sigma_0)$ *for all* $r \geqq r_0$, *where* C_1 *is independent of* $r \geqq r_0$.

REMARK 1. By the result of [6], assuming thst $\int \theta \lambda(d\theta)=0$ when $\alpha=1$, it holds that $0<C_1 \leqq r^{1+\alpha} p(r\sigma_0) \leqq C_2 < \infty$ for all $r \geqq r_0$ where the constant C_2 is independent of $r \geqq 0$ and σ_0 (the upper estimate seems valid without the restriction $\int \theta \lambda(d\theta)=0$ when $\alpha=1$, but we have no proof for it).

Now we assume that λ has mass at only finitely many points in S^{d-1} (of course we also assume that $b=0$ and **Span Spt** $\lambda = R^d$). To state the next result we define the following subsets of R^d: For each $1 \leqq k \leqq d$

(i) $S^0(k)$ is a union of closed convex cones with the origin as vertex, the cones which are subtended by every linearly independent k-elements of **Spt** λ,

(ii) $S(k)=S^0(k) \cap S^{d-1}$, $S(0)=\emptyset$ and $T(k)=S(k)-S(k-1)$.

Now our main result in the present paper is the following:

THEOREM 2. *Let* $d \leqq 3$. *Suppose that* **Spt** λ *is a finite set of* S^{d-1}. *Let* $\sigma \in S^{d-1}$.

a) *Let* $0<\alpha<1$.

If $\sigma \in T(k) \cap$ **Int** $S(d)$ *for some* $1 \leqq k \leqq d$, *then* $p(r\sigma) \sim c_1 r^{-k(1+\alpha)}$ *as* $r \to +\infty$.

If $\sigma \notin$ **Int** $S(d)$, *then* $p(r\sigma)=0$.

b) *Let* $1 \leqq \alpha < 2$

If $\sigma \in T(k)$ *for some* $1 \leqq k \leqq d$, *then* $p(r\sigma) \sim c_2 r^{-k(1+\alpha)}$ *as* $r \to +\infty$.

If $\sigma \notin S(d)$, then $p(r\sigma)$ decreases faster than any negative order of r, that is, $p(r\sigma)$ is a rapidly decreasing function of $r \geqq 0$.

Here constants $c_1, c_2>0$ are independent of r and can be determined explicitly by the expression of $\Psi(z)$.

For $d \geqq 4$ this theorem could be also proved in a similar way to our proof. However, it seems to be so tedious to describe the proof in general. So we treat the case of $d=2$ and 3. This theorem is proved by using the rotation of contour of integration as is similar to the one-dimensional case. Lemmas 2 and 4 are essential to the proof of this theorem (see § 3).

In the first cases of (a) and (b) in Theorem 2 we can give more concrete information. We say that λ *has mass at* $(m+1)$-*directions* $\sigma_j \in S^{d-1}$, $j=0, 1, 2, \cdots, m$, if λ has mass at σ_j and/or $-\sigma_j$ for each $j=0, 1, 2, \cdots, m$ (of course we assume $\sigma_j \neq \sigma_k$ if $j \neq k$). Now suppose that λ has mass at only $(m+1)$-directions σ_j, $j=0, 1, 2, \cdots, m$. When $\sigma \in T(k)$ for some $1 \leqq k \leqq d$, we define a vertex set $V_k(\sigma)$ of $\{\sigma_j, j=0, 1, \cdots, m\}$ and an index set $I_k(\sigma)$ as follows;

$\{\sigma_{j_1}, \cdots, \sigma_{j_k}\} \in V_k(\sigma)$ if $\{\sigma_{j_1}, \cdots, \sigma_{j_k}\}$ is linearly independent and σ is contained in the interior of **Span** $\{\sigma_{j_1}, \cdots, \sigma_{j_k}\}$,

$$j(k) \equiv \{j_1, \cdots, j_k\} \in I_k(\sigma) \quad \text{if} \quad \{\sigma_{j_1}, \cdots, \sigma_{j_k}\} \in V_k(\sigma).$$

Moreover for $j(k) \in I_k(\sigma)$ set $H_{j(k)} = \mathbf{Span}\, \{\sigma_{j_1}, \cdots, \sigma_{j_k}\}$ and fix an orthonormal basis $\{e_{j_1}, \cdots, e_{j_k}\}$ of $H_{j(k)}$. Now let

(i) $p_{j(k)}$ be a k-dimensional density on $H_{j(k)}$ with a log-characteristic function $\Psi|_{H_{j(k)}}$,

(ii) $p_{j(k)}^{\perp}$ be a $(d-k)$-dimensional density on $H_{j(k)}^{\perp}$ with a log-characteristic function $\Psi|_{H_{j(k)}^{\perp}}$ (if $k=d$, set $p_{j(k)}^{\perp}=1$).

In particular we write $p_j = p_{j(1)}$: a one-dimensional density on $H_{j(1)}$, when $j(1) = \{j\}$.

THEOREM 3. *Let* $d \leqq 3$. *Suppose that* $\sigma \in T(k) \cap \mathbf{Int}\, S(d)$ *in case of* $0<\alpha<1$ *and that* $\sigma \in T(k)$ *in case of* $1 \leqq \alpha < 2$ *for some* $1 \leqq k \leqq d$. *Then*

$$p(r\sigma) \sim \sum_{j(k) \in I_k(\sigma)} p_{j(k)}(r\sigma(j(k)))\, p_{j(k)}^{\perp}(0) \qquad as \quad r \to +\infty,$$

$$= \sum_{j(k) \in I_k(\sigma)} g(j(k)) \prod_{s=1}^{k} p_{j_s}(r h_{j_s})\, p_{j(k)}^{\perp}(0),$$

where $\sigma(j(k)) = \sum_{s=1}^{k} h_{j_s} \sigma_{j_s} = \sigma|_{H_{j(k)}}$ *and* $g(j(k)) = |\det Q_{j(k)}|$ *with a* $k \times k$*-matrix* $Q_{j(k)}$ *such that* $Q_{j(k)} \sigma_{j_s} = e_{j_s}$ *for every* $s=1, 2, \cdots, k$.

Note that the assumption of Theorem 3 implies that there is at least one $j(k) = \{j_1, \cdots, j_k\} \in I_k(\sigma)$ such that $p_{j(k)}^{\perp}(0) > 0$ and $p_{j_s}(r\sigma_{j_s}) \sim c(j_s) r^{-1-\alpha}$ as $r \to +\infty$ with a positive constant $c(j_s)$ for each $s=1, \cdots, k$.

REMARK 2. a) Note that $S(d) = \mathbf{Con\ Spt}\, \lambda$ and $T(1) = \mathbf{Spt}\, \lambda$.

b) In a similar way to the proof of Theorem 2 we can show that if μ is *rotation invariant*, that is, $\Psi(z) = -c|z|^{\alpha} (c>0)$, then

$$p(x) \approx \sum_{n=1}^{\infty} c_n |x|^{-d-n\alpha} \qquad \text{as} \quad |x| \to +\infty,$$

where

$$c_n=\pi^{-d/2-1}\alpha\frac{(-1)^{n-1}}{(n-1)!}2^{n\alpha-1}c^n\sin\frac{\pi n\alpha}{2}\Gamma\Big(\frac{n\alpha+d}{2}\Big)\Gamma\Big(\frac{n\alpha}{2}\Big).$$

This expansion means that

(1.2) $$p(x)=\sum_{n=1}^{N}c_n|x|^{-d-n\alpha}+O(|x|^{-d-(N+1)\alpha})\quad\text{as}\quad |x|\to+\infty \text{ for all } N.$$

In particular, if $0<\alpha<1$, then $p(x)=\sum_{n=1}^{\infty}c_n|x|^{-d-n\alpha}$.

This result was shown by S. C. Port (A. 13 in [5]) by making use of a subordination technique.

2. Some Preliminary Results

For the proof of Theorem 2, we mention some results in the one-dimensional case which are well-known in [3].

a) $\alpha\neq 1$. In this case $p(x)$ is expressed with some constants $c_0>0$ and $|\beta_0|\leqq 1$ as follows:

(2.1) $$p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big[-ixz-c_0|z|^{\alpha}\Big(1-i\beta_0\tan\frac{\pi\alpha}{2}\operatorname{sgn}z\Big)\Big]dz$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-ixz-c|z|^{\alpha}e^{-i\theta}\operatorname{sgn}z\right]dz$$

(2.2) $$\approx\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}x^{-1-n\alpha}\Gamma(n\alpha+1)c^n\sin n\eta\quad\text{as}\quad 0<x\to+\infty,$$

where

(2.3) $$c=c_0\sec\theta,\quad \theta=\theta(\beta)=\pi L(\alpha)\beta/2\quad\text{and}\quad \eta=\eta(\theta)=\theta+\pi\alpha/2$$

$$=\pi(\alpha+L(\alpha)\beta)/2\quad\text{with}\quad L(\alpha)=\alpha(0<\alpha<1),\ =\alpha-2(1<\alpha<2)$$

and $\beta=2\pi^{-1}L(\alpha)^{-1}\arctan(\beta_0\tan\pi\alpha/2)$.

Note that $|\theta|<\pi/2$, $c>0$, $0\leqq\eta\leqq\pi$, $|\beta|\leqq 1$ and

(2.4) $\beta_0=\pm 1$ if and only if $\beta=\pm 1$ and then λ has mass at only $\{\pm 1\}$ respectively.

In particular if $\beta_0=-1$, then $\eta=0(0<\alpha<1)$, $=\pi(1<\alpha<2)$ and it holds that

(2.5) $$p(x)=0\qquad\text{for}\quad x\geqq 0\quad\text{if}\quad 0<\alpha<1,$$

$$\sim\frac{1}{\sqrt{2\pi(\alpha-1)}}(c_0\alpha)^{-1/(2\alpha-2)}x^{(2-\alpha)/(2\alpha-2)}\exp\left[-(\alpha-1)\alpha^{-\alpha/(\alpha-1)}c_0^{-1/(\alpha-1)}x^{\alpha/(\alpha-1)}\right]$$

as $0<x\to+\infty$ if $1<\alpha<2$.

b) $\alpha=1$.

(2.6) $$p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big[-ixz-c\Big(|z|+i\frac{2\beta}{\pi}z\log|z|\Big)\Big]dz,\ c>0,\ |\beta|\leqq 1,$$

$$\approx\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{c^n}{n!}x^{-1-n}\int_0^{\infty}e^{-z}z^n\,\mathrm{Im}\Big[i(1+\beta)-\frac{2\beta}{\pi}\log\frac{z}{x}\Big]^n dz \text{ as } 0<x\to+\infty.$$

In this case (2.4) also holds. Moreover, if $\beta=-1$ (i.e., **Spt** $\lambda=\{-1\}$), then

(2.7) $$p(x)\sim\frac{1}{2\sqrt{ce}}\exp\Big[\frac{\pi}{4c}x-\frac{2}{\pi e}ce^{-\pi/(2c)}\Big] \quad \text{as} \quad 0<x\to+\infty.$$

c) The asymptotic behaviour of each derivative of $p(x)$ is obtained by differentiating the above formulae.

d) Moreover (cf. [9])

(2.8) $p(x)=0$ if and only if $0<\alpha<1$ and either $x\geqq 0$, $\beta=-1$ or $x\leqq 0$, $\beta=1$.
In particular if $\alpha\neq 1$ then

(2.9) $$p(0)=\pi^{-1}c^{-1/\alpha}\Gamma(\alpha^{-1}+1)\cos\Big(\frac{\pi}{2\alpha}L(\alpha)\beta\Big).$$

REMARK 3. In the case $0>x\to-\infty$, we obtain the same results by changing x, β_0 and β to $|x|$, $-\beta_0$ and $-\beta$ (thus, θ to $-\theta$) respectively. Because if we write $p(x;\alpha,\beta)=p(x)$ as $p(x)$ depends on (α,β), then $p(-x;\alpha,\beta)=p(x;\alpha,-\beta)$ holds.

3. Proof of Results

Before proceeding to the proof of Theorem 1, we present a general fact on multidimensional stable distributions, which is interesting in its own right. Let $p(x)$ be a density function of non-degenerate stable distribution μ of exponent $0<\alpha<2$. Recall that $b=0$ in $\Psi(z)$ and $S^0(d)$ is the smallest closed convex cone with vertex 0, which contains **Spt** λ. Note that **Int** $S^0(d)\neq\emptyset$ because of **Span Spt** $\lambda=\boldsymbol{R}^d$, where **Int** V denotes interior of a set V in $\boldsymbol{R}^d$.

LEMMA 1. *$p(x)=0$ if and only if $0<\alpha<1$ and $x\notin$* **Int** *$S^0(d)$.*

PROOF. Let (X_t, P) be a Lévy process on $\boldsymbol{R}^d$ corresponding to μ, then $P(X_t\in dx)=\mu^{t*}(dx)$. Of course for each $t>0$, $\mu^{t*}(dx)$ has a C^∞-density $p_t(x)$ with respect to the Lebesgue measure on $\boldsymbol{R}^d$, and $p_1=p$. We divide the proof into three cases: $\alpha=1$, $1<\alpha<2$ and $0<\alpha<1$, and use the Lévy-Ito decomposition of Lévy processes (see [4], [8]).

(1) $\alpha=1$. In this case $\Psi(z)$ is expressed by

$$\Psi(z)=-|z|\int_{S^{d-1}}|\langle\xi,\ \theta\rangle|\left[1+i\frac{2}{\pi}\operatorname{sgn}\langle\xi,\ \theta\rangle\log|\langle z,\ \theta\rangle|\right]\lambda(d\theta)$$

$$=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^{\infty}[e^{i\langle z,\,r\theta\rangle}-1-i\langle z,\ r\theta\rangle 1_{(0,1)}(r)]r^{-2}dr+\langle b_0,\ z\rangle,$$

where $\lambda_0=2\pi^{-1}\lambda$ and $b_0=-2\pi^{-1}c_0\int\theta\lambda(d\theta)$ with

$$c_0=\int_1^{\infty}r^{-2}\sin r dr+\int_0^1 r^{-2}(\sin r-r)dr\,.$$

Then by the Lévy-Ito decomposition we see that

$$X_t=\int_0^t\int_{0<|x|<1}x\tilde{N}(ds\ dx)+\int_0^t\int_{1\leq|x|<\infty}xN(ds\ dx)+tb_0\,,$$

where $N(ds\ dx)=\#\{s\in ds:\ X_S-X_{S-}\in dx\}$ is a Poisson random measure corresponding to a Poisson point process with characteristic measure

$$n(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^{\infty}1_{dx}(r\theta)r^{-2}dr \qquad \text{on} \quad \boldsymbol{R}^d\setminus\{0\}$$

and $\tilde{N}(ds\ dx)=N(ds\ dx)-ds\ n(dx)$. Now for each $0<\varepsilon<1$ we define

$$X_t^{\varepsilon}=\int_0^t\int_{\varepsilon\leq|x|<1}x\tilde{N}(ds\ dx)+\int_0^t\int_{1\leq|x|<\infty}xN(ds\ dx)+tb_0$$

$$=\int_0^t\int_{\varepsilon\leq|x|<\infty}xN(ds\ dx)-tb^{\varepsilon}$$

with

$$b^{\varepsilon}=(-\log\varepsilon+c_0)\frac{2}{\pi}\int_{S^{d-1}}\theta\lambda(d\theta)\,.$$

Then $X_t^{\varepsilon}+tb^{\varepsilon}$ is a compound Poisson Process with Lévy measure

$$n^{\varepsilon}(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_{\varepsilon}^{\infty}1_{dx}(r\theta)r^{-2}dr\,.$$

Thus, if we set $F_0^{\varepsilon}=\{0\}$, $F_1^{\varepsilon}=\mathbf{Spt}\ n^{\varepsilon}$, $F_{n+1}^{\varepsilon}=F_n^{\varepsilon}+F_1^{\varepsilon}\ (n\geqq 1)$, then it holds that $\mathbf{Spt}\ X_t^{\varepsilon}+tb^{\varepsilon}=\mathbf{CL}(\cup_{n=0}^{\infty}F_n^{\varepsilon})$ for all $t>0$ and that $\uparrow\lim_{\varepsilon\downarrow 0}\mathbf{CL}(\cup_{n=0}^{\infty}F_n^{\varepsilon})=S^0(d)$, where $\mathbf{Spt}\ X_t^{\varepsilon}$ denotes a support of a distribution of X_t^{ε} under P and $\mathbf{CL}\ V$ denotes closure of a set V in $\boldsymbol{R}^d$. From these results we can easily see that $\mathbf{Spt}\ \mu=\boldsymbol{R}^d$. In fact, if $\int\theta\lambda(d\theta)=0$ then $S^0(d)=\boldsymbol{R}^d$ because of $\mathbf{Span\ Spt}\ \lambda=\boldsymbol{R}^d$. Hence $\mathbf{Spt}\ X_t=\uparrow\lim_{\varepsilon\downarrow 0}\mathbf{Spt}\ X_t^{\varepsilon}=S^0(d)=\boldsymbol{R}^d$ for all $t>0$. Therefore $\mathbf{Spt}\ \mu=\mathbf{Spt}\ X_1=\boldsymbol{R}^d$. If $\int\theta\lambda(d\theta)\neq 0$ then $|b^{\varepsilon}|\to+\infty$ as $\varepsilon\to 0$ and $b^{\varepsilon}\in\mathbf{Int}\ S^0(d)$ for small ε because of $\int\theta\lambda(d\theta)\in\mathbf{Int}\ S^0(d)$. Thus for each $x\in\boldsymbol{R}^d$ we have $x+b^{\varepsilon}\in\mathbf{Int}\ S^0(d)$ if $0<\varepsilon<1$ is sufficiently small. Hence there is an $0<\varepsilon<1$ such that $x+b^{\varepsilon}\in\mathbf{CL}(\cup_{n=0}^{\infty}F_n^{\varepsilon})$, that is, $x\in\mathbf{Spt}\ X_t^{\varepsilon}\subset\mathbf{Spt}\ X_t$ for all $t>0$. Therefore $\mathbf{Spt}\ \mu=\mathbf{Spt}\ X_1=\boldsymbol{R}^d$. Now if we assume that $p(x)=0$ for some $x\in\boldsymbol{R}^d$, then $L^*p(x)=(\partial/\partial_t)p_t(x)|_{t=1}=0$, where L^* is a Lévy generator of $-X_t$:

$$L^*p(x)=\int_{S^{d-1}}\lambda_0^*(d\theta)\int_0^\infty[p(x+r\theta)-p(x)-\langle r\theta,\nabla p(x)\rangle 1_{(0,1)}(r)]r^{-2}dr+\langle b_0,\nabla p(x)\rangle$$

with $\lambda_0^*(d\theta)=\lambda_0(-d\theta)$. Hence noting that $\nabla p(x)=0$, we have $p(x-r\theta)=0$ for a.e. $r\geqq 0$ and λ-a.e. $\theta\in\mathbf{Spt}\ \lambda$. By the continuity of p it holds that $p(x-r\theta)=0$ for all $r\geqq 0$, $\theta\in\mathbf{Spt}\ \lambda$. Furthermore we easily deduce that

$$p(x-r\theta)=0 \qquad \text{for all} \quad r\geqq 0,\ \theta\in\mathbf{Con\ Spt}\ \lambda.$$

This implies that $\mu(x-\mathbf{Int}\ S^0(d))=0$, but which is contrary to $\mathbf{Spt}\ \mu=\boldsymbol{R}^d$ and $\mathbf{Int}\ S^0(d)\neq\emptyset$. Therefore we get $p(x)>0$ for all $x\in\boldsymbol{R}^d$.

(2) $1<\alpha<2$. In this case $p>0$ on $\boldsymbol{R}^d$ has been already proved in [9] by using the scaling property of $p_t(x)$. We here give an alternative proof by the same way as in (1). In this case the previous arguments work replacing $\Psi(z)$, $n^\varepsilon(dx)$ and L^* by the following:

$$\Psi(z)=-|z|^\alpha\int_{S^{d-1}}|\langle\xi,\theta\rangle|^\alpha\left[1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\xi,\theta\rangle\right]\lambda(d\theta)$$

$$=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^\infty[e^{i\langle z,r\theta\rangle}-1-i\langle z,r\theta\rangle]r^{-1-\alpha}dr,$$

where $\lambda_0=c(\alpha)\lambda$ with $c(\alpha)=2\Gamma(\alpha+1)\sin(\pi\alpha/2)/\pi$.

$$X_t=\int_0^t\int_{0<|x|<\infty}x\tilde{N}(ds\ dx)$$

with Lévy measure

$$n(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^\infty 1_{dx}(r\theta)r^{-1-\alpha}dr \qquad \text{on} \quad \boldsymbol{R}^d\setminus\{0\}.$$

For each $0<\varepsilon<1$,

$$X_t^\varepsilon=\int_0^t\int_{\varepsilon\leqq|x|<\infty}x\tilde{N}(ds\ dx)=\int_0^t\int_{\varepsilon\leqq|x|<\infty}xN(ds\ dx)-tb^\varepsilon$$

where

$$b^\varepsilon=\varepsilon^{-\alpha}(\alpha-1)^{-1}\int_{S^{d-1}}\theta\lambda_0(d\theta),$$

and its Lévy measure is given by

$$n^\varepsilon(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_\varepsilon^\infty 1_{dx}(r\theta)r^{-1-\alpha}dr.$$

The Lévy generator L^* of $-X_t$:

$$L^*p(x)=\int_{S^{d-1}}\lambda_0^*(d\theta)\int_0^\infty[p(x+r\theta)-p(x)-\langle r\theta,\nabla p(x)\rangle]r^{-1-\alpha}dr.$$

(3) $0<\alpha<1$. We show that $p(x)=0$ if and only if $x\notin \mathbf{Int}\, S^0(d)$. In this case $\Psi(z)$ and X_t are expressed by the following:

$$\Psi(z)=-|z|^{\alpha}\int_{S^{d-1}}|\langle\xi,\theta\rangle|^{\alpha}\left[1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\xi,\theta\rangle\right]\lambda(d\theta)$$

$$=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^{\infty}[e^{i\langle z,r\theta\rangle}-1]r^{-1-\alpha}dr,$$

where λ_0 is the same as in (2), and

$$X_t=\int_0^t\int_{0<|x|<\infty}xN(ds\,dx).$$

Moreover for each $0<\varepsilon<1$ we define

$$X_t^{\varepsilon}=\int_0^t\int_{\varepsilon\leq|x|<\infty}xN(ds\,dx),$$

then $\mathbf{Spt}\, X_t^{\varepsilon}=\mathbf{CL}(\cup_{n=0}^{\infty}F_n^{\varepsilon})$. Hence by limiting $\varepsilon\to 0$ we have $\mathbf{Spt}\, X_t=S^0(d)$, that is, $p(x)=0$ if $x\notin\mathbf{Int}\, S^0(d)$. Furthermore by a similar argument to (1) we can see that $p(x)>0$ if $x\in\mathbf{Int}\, S^0(d)$. In fact, if $p(x)=0$ for some $x\in\mathbf{Int}\, S^0(d)$, then $L^*p(x)=(\partial/\partial t)p_t(x)|_{t=1}=0$, where L^* is given by

$$L^*p(x)=\int_{S^{d-1}}\lambda_0^*(d\theta)\int_0^{\infty}[p(x+r\theta)-p(x)]r^{-1-\alpha}dr$$

with $\lambda_0^*(d\theta)=\lambda_0(-d\theta)$. Hence we have $\mu(x-\mathbf{Int}\, S^0(d))=0$, but this is contrary to $\mathbf{Spt}\,\mu=S^0(d)$. Therefore we get $p>0$ on $\mathbf{Int}\, S^0(d)$. **Q.E.D.**

We also mention the following result: To emphasize the dependence on λ we write $\Psi(z)=\Psi_{\lambda}(z)$ and $p(x)=p_{\lambda}(x)$. Let Q be a linear transformation on $\boldsymbol{R}^d$ and set $\lambda_Q(d\theta)=\lambda(Q^{-1}d\theta)$ on $Q(S^{d-1})$. Then by the definition of $\Psi(z)$ we have $\Psi_{\lambda_Q}(z)=\Psi_{\lambda}({}^tQz)$, where ${}^tQ$ denotes a transposed matrix of Q. Moreover by using (1.1) we can easily deduce that if Q is invertible, then p_{λ_Q} is well-defined and

$$p_{\lambda}(x)=|\det Q|\,p_{\lambda_Q}(Qx) \tag{3.1}$$

holds.

PROOF OF THEOREM 1. First assume that $\lambda(\{\sigma_0\})>0$ for some $\sigma_0\in\boldsymbol{S}^{d-1}$, and also that $\sigma_0\in\mathbf{Int}\,(\mathbf{Con\ Spt}\,\lambda)$ if $0<\alpha<1$. For simplicity we write $\sigma_0=\sigma$. In (3.1) let Q be an orthogonal transformation, then $p_{\lambda}(x)=p_{\lambda_Q}(Qx)$. From this we may assume that $\sigma=(1,0,\cdots,0)$. Moreover it is easily deduced that $p(r\sigma)$ is expressed by

$$p(r\sigma)=cp_1(r)p_{d-1}(0,\cdots,0) \tag{3.2}$$

or

$$(3.3)\qquad p(r\sigma)=\int_{-\infty}^{\infty} p_1(r-y)p_d(y, 0, \cdots, 0)dy,$$

where p_j is a j-dimensional density ($j=1, d-1, d$) and $c>0$. In fact, we define λ^{σ} by $\lambda=\delta_{\{\sigma\}}+\lambda^{\sigma}$ and set $H=$ **Span Spt** λ^{σ}. Then $\dim H=d-1$ or d because of Span **Spt** $\lambda=\boldsymbol{R}^d$. If $\dim H=d-1$, then by taking Q in (3.1) such that $Q\sigma=\sigma$ and $Q(H)=\{x_1=0\}$ we see that $p_{\lambda_Q}(r\sigma)=p_1(r)p_{d-1}(0, \cdots, 0)$, where p_1 (resp. p_{d-1}) is a one-dimensional density function (resp. $(d-1)$-dimensional density function) corresponing to $\delta_{\{\sigma\}}$ (resp. λ_Q^{σ}). Hence we get $p(r\sigma)=|\det Q|p_1(r)p_{d-1}(0, \cdots, 0)$. If $\dim H=d$, then we can define a d-dimensional density function p_d by λ^{σ}. Thus we have

$$\begin{aligned}(2\pi)^d p(x)&=\int_{\boldsymbol{R}^d}\exp[-i\langle x, z\rangle+\Psi_{\delta_{\{\sigma\}}}(z)+\Psi_{\lambda^{\sigma}}(z)]dz\\ &=\int_{-\infty}^{\infty}dy\ p_1(y)\int_{\boldsymbol{R}^d}\exp[-i\{(x_1-y)z_1+x_2z_2+\cdots+x_dz_d\}+\Psi_{\lambda^{\sigma}}(z)]dz\\ &=(2\pi)^d\int_{-\infty}^{\infty}p_1(y)p_d(x_1-y, x_2, \cdots, x_d)dy.\end{aligned}$$

Therefore (3.3) holds. Here in the second equation we use

$$\exp[\Psi_{\delta_{\{\sigma\}}}(z)]=\int_{-\infty}^{\infty}p_1(y)\exp[i\,yz_1]dy.$$

Now noting that (3.2) does not occur when $0<\alpha<1$ and **Con Spt** $\lambda\neq S^{d-1}$, we see that $p_{d-1}(0, \cdots, 0)>0$ and $p_d(y, 0, \cdots, 0)>0$ if at least $y>0$ by Lemma 1. Hence in the case of (3.2) our claim holds. In the case of (3.3) we have $p(r\sigma)\geqq cp_1(2r)$ for sufficiently large r with a positive constants c. In fact there are a compact set K in $(0, \infty)$ and a positive constant r_0 such that $\varepsilon\equiv\inf_{y\in K}p_d(y, 0, \cdots, 0)>0$ and $\inf_{y\in K}p_1(r-y)\geqq p_1(2r)$ for all $r\geqq r_0$. Thus $p(r\sigma)\geqq\varepsilon|K|\inf_{y\in K}p_1(r-y)\geqq\varepsilon|K|p_1(2r)$ for $r\geqq r_0$. Since $p_1(2r)\sim c'r^{-1-\alpha}$ as $r\to+\infty$, there is a constant $C_1>0$ such that $p(r\sigma)\geqq C_1r^{-1-\alpha}$ for all $r\geqq r_0$. **Q.E.D.**

PROOF OF THEOREM 2 AND THEOREM 3. Let $d=2, 3$ and let μ be a non-degenerate stable distribution on $\boldsymbol{R}^d$ with exponent $0<\alpha<2$. Recall that we are assuming that **Spt** λ is a finite set of S^{d-1}, and we say that λ has mass at $(m+1)$-directions $\sigma_j\in S^{d-1}$, $j=0, 1, 2, \cdots, m$, if λ has mass at σ_j and/or $-\sigma_j$ for each $j=0, 1, 2, \cdots, m$ (of course $\sigma_j\neq\pm\sigma_k$ if $j\neq k$).

Now we begin with the case $d=2$. The proof is divided into three cases.

CASE 1. λ has mass at only two directions σ_0, σ_1 ($\sigma_0\neq\pm\sigma_1$). By (3.1) we may assume that $\sigma_0=(1, 0)$, $\sigma_1=(a, b)$ and with $a\neq 1$, $b>0$ such that $a^2+b^2=1$. Then

$$p(r\sigma)=b^{-1}p_0(rh_0)p_1(rh_1)$$

where, h_j are defined by the decomposition $\sigma=h_0\sigma_0+h_1\sigma_1$, and $p_j(y)$, $y\in\boldsymbol{R}$ are defined by (2.1) with some constants $(c_{j,0}, \beta_{j,0})$ instead of (c_0, β_0), $j=0, 1$. Here one can easily check that $b^{-1}=g(\{0, 1\})$; which is defined in Theorem 3, and that $p_0^{\dagger}(0)=b^{-1}p_1(0)$ and $p_1^{\dagger}(0)=b^{-1}p_0(0)$. Hence our claim immediately follows by using the facts (2.2), (2.4), (2.8) and (2.9). In particular if $1<\alpha<2$ and $\sigma\notin$ **Con Spt** λ, then by (2.5) and (2.7),

$$p(r\sigma)\sim K_1r^{K_2}\exp[-K_3r^{K_4}] \quad \text{as} \quad r\to+\infty \text{ if } 1<\alpha<2, \tag{3.4}$$

$$p(r\sigma)\sim \tilde{K}_1\exp[\tilde{K}_2r-\tilde{K}_3e^{\tilde{K}_4r}] \quad \text{as} \quad r\to+\infty \text{ if } \alpha=1, \tag{3.5}$$

where K_j, $\tilde{K}_j$ are positive constants which are independent of r. For instance, when **Spt** $\lambda=\{\pm\sigma_0, \sigma_1\}$ with $\sigma_0=(1, 0)$ and $\sigma_1=(0, 1)$, let $\sigma=(s, t)$,

if $\sigma\in T(2)$, i.e., $t>0$ and $\sigma\neq\sigma_1$, then $p(r\sigma)\sim cr^{-2(1+\alpha)}$ as $r\to+\infty$;

if $\sigma\in T(1)\cap$**Int** $S(2)$, i.e., $\sigma=\sigma_1$, then $p(r\sigma)\sim cr^{-(1+\alpha)}$ as $r\to+\infty$;

if $\sigma\in T(1)\cap\partial S(2)$, i.e., $\sigma=\pm\sigma_0$, then $p(r\sigma)=0$ $(0<\alpha<1)$, $p(r\sigma)\sim cr^{-(1+\alpha)}(1\leqq\alpha<2)$ as $r\to+\infty$;

if $\sigma\notin S(2)$, i.e., $t<0$, then $p(r\sigma)=0$ for all $r\geqq 0 (0<\alpha<1)$ (3.4) $(1<\alpha<2)$ and (3.5) $(\alpha=1)$ hold.

CASE 2. $\alpha\neq 1$ and λ has mass at only $(m+1)$-directions σ_j, $j=0, 1, 2, \cdots, m (m\geqq 2)$. Then $\Psi(z)$, $z=(z_1, z_2)$, is expressed by

$$\begin{aligned}\Psi(z)&=-\sum_{j=0}^{m}c_{j,0}|\langle\sigma_j, z\rangle|^{\alpha}\left[1-i\beta_{j,0}\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\sigma_j, z\rangle\right]\\&=-\sum_{j=0}^{m}c_j|\langle\sigma_j, z\rangle|^{\alpha}\exp[-i\theta_j\operatorname{sgn}\langle\sigma_j, z\rangle],\end{aligned}$$

where $c_{j,0}>0$, $|\beta_{j,0}|\leqq 1$ and c_j, θ_j are defined by (2.3).

In order to prove Theorem 2 and Theorem 3 in Case 2 we first consider the special case, however we show that the general case is reduced to this special one (see Second step).

First step. Set $\sigma=\sigma_0=(1, 0)$ and let $\sigma_j=(s_j, t_j)$, $j=0, 1, 2, \cdots, m$, where $s_j=\cos\varphi_j$ and $t_j=\sin\varphi_j$ with $0=\varphi_0<\varphi_1<\cdots<\varphi_m=\pi/2$. Note that if λ has no mass at $\sigma=(1, 0)$, then λ has mass at $-\sigma=(-1, 0)$ by our definition of directions, and $\beta_{0,0}=-1$.

We define the following α-stable densities:

(i) For $y, z\in\boldsymbol{R}$, $p_0(y)$ (resp. $p_0^{\dagger}(y)$) is a one-dimensional density with a log-characteristic function $\Psi_0(z)=-c_0|z|^{\alpha}\exp[-i\theta_0 \operatorname{sgn} z]$ (resp. $\Psi_0^{\perp}(z)=\Psi(0, z)$)

(ii) For $x, z\in\boldsymbol{R}^2$ and $j\neq k$, $p_{j,k}(x)$ is a two-dimensional density with a log-characteristic function $\Psi_{j,k}(z)=-\sum_{r=j,k} c_r|\langle\sigma_r, z\rangle|^{\alpha}\exp[-i\theta \operatorname{sgn}\langle\sigma_r, z\rangle]$.

PROPOSITION. *Let* $r\geqq 0$.

a) *If* $\sigma\in\mathbf{Spt}\ \lambda$ *and* $p_0^{\perp}(0)>0$, *then*

$$p(r\sigma)\sim p_0(r)p_0^{\perp}(0) \quad \text{as} \quad r\to+\infty; \tag{3.6}$$

b) *If* $\sigma\notin\mathbf{Spt}\ \lambda$ *and* $\sigma\in\mathbf{Con\ Spt}\ \lambda$, *then*

$$p(r\sigma)\sim\sum_{1\leqq j<k\leqq m} p_{j,k}(r\sigma) \quad \text{as} \quad r\to+\infty; \tag{3.7}$$

c) *If* $1\leqq\alpha<2$ *and* $\sigma\notin\mathbf{Con\ Spt}\ \lambda$, *then* $p(r\sigma)$ *is rapidly decreasing as* $r\to+\infty$;

d) *If* $0<\alpha<1$ *and* $\sigma\notin\mathbf{Int}$ $(\mathbf{Con\ Spt}\ \lambda)$, *then* $p(r\sigma)=0$.

Note that (b), (c) and (d) also hold in the case that λ has no mass at $\{\pm\sigma\}$ (in this case $c_{0,0}=c_0=0$ in $\Psi(z)$) and that, by (2.9)

$$\operatorname{Re}\int_0^{\infty}\exp\Psi(0, z_2)dz_2=\pi p_0^{\perp}(0)=\bar{c}^{-1/\alpha}\Gamma(\alpha^{-1}+1)\cos\left(\frac{\pi}{2\alpha}L(\alpha)\bar{\beta}\right),$$

where $(\bar{c}, \bar{\beta})$ is (c, β) in (2.3) which is given by using $(\bar{c}_0, \bar{\beta}_0)=(\sum_{j=1}^m c_{j,0}t_j^{\alpha}, \sum_{j=1}^m c_{j,0}\beta_{j,0}t_j^{\alpha}/\bar{c}_0)$ instead of (c_0, β_0) in (2.3). Hence by (2.4) and (2.8) $p_0^{\perp}(0)=0$ if and only if $0<\alpha<1$ and $\beta_{1,0}=\beta_{2,0}=\cdots=\beta_{m,0}=\pm 1$ (i.e., $\sigma\notin\mathbf{Int}$ $(\mathbf{Con\ Spt}\ \lambda)$).

From this proposition we can easily deduce Theorem 2 and Theorem 3 in Case 2 by using the one-dimensional results.

To prove Proposition we need some lemmas. The following lemma is obtained by elementary analysis.

LEMMA 2. *Set* $a_j=t_j/s_j=\tan\phi_j(a_0=0,\ a_m=\infty)$. *Then*

$$\begin{aligned} p(r\sigma)&=(2\pi)^{-2}\int_{R^2}\exp[-irz_1+\Psi(z)]dz \\ &\approx r^{-1}\pi^{-2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}r^{-n\alpha}c_0^n\sin n\eta_0\int_0^{\infty}du\ e^{-u}u^{n\alpha}\operatorname{Re}\int_{u/(ra_1)}^{\infty}\exp\Psi\left(-i\frac{u}{r}, \nu\right)d\nu \\ &\quad +r^{-2}\pi^{-2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}r^{-n\alpha}c_0^n\sin n\eta_0\int_0^{\infty}du\ e^{-u}u^{n\alpha+1} \\ &\qquad \int_0^{\pi/2}d\phi e^{i\phi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}r^{-n\alpha}\operatorname{Im}\left[\sum_{j=1}^m c_ju^{\alpha}(s_j+ie^{i\phi}t_j/a_1)^{\alpha}e^{-i\eta_j}\right]^n \\ &\quad +r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^{\infty}d\nu\int_{a_j\nu}^{a_{j+1}\nu}due^{-u} \end{aligned} \tag{3.8}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}r^{-n\alpha}\,\mathrm{Im}[c_0u^{\alpha}e^{-i\eta_0}+c_1s_1^{\alpha}(u-a_1\nu)^{\alpha}e^{-i\eta_1}+\cdots$$
$$+c_js_j^{\alpha}(u-a_j\nu)^{\alpha}e^{-i\eta_j}]^n$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}r^{-n\alpha}\,\mathrm{Im}[c_{j+1}s_{j+1}^{\alpha}(a_{j+1}\nu-u)^{\alpha}e^{-i\hat{\eta}_{j+1}}+\cdots+c_m\nu^{\alpha}e^{-i\hat{\eta}_m}]^n$$

as $r\to+\infty$, where $\eta_j=\eta(\theta_j)$, $\hat{\eta}_j=\eta(-\theta_j)$ are defined by (2.3). This expansion holds in equal provided $0<\alpha<1$, and if $1<\alpha<2$, then it holds in the sense of (1.2).

PROOF. For simplicity we only prove the case that $m=3$, $\sigma=\sigma_0=(1,0)$, $\sigma_1=(s_1,t_1)$, $\sigma_2=(s_2,t_2)$ and $\sigma_3=(0,1)$. That is, for $\tilde{c}_j=c_js_j^{\alpha}\,(j=1,2)$,

$$\Psi(z)=-c_0|z_1|^{\alpha}\exp[-i\theta_0\,\mathrm{sgn}\,z_1]-\tilde{c}_1|z_1+a_1z_2|^{\alpha}\exp[-i\theta_1\,\mathrm{sgn}(z_1+a_1z_2)]$$
$$-\tilde{c}_2|z_1+a_2z_2|^{\alpha}\exp[-i\theta_2\,\mathrm{sgn}(z_1+a_2z_2)]-c_3|z_2|^{\alpha}\exp[-i\theta_3\,\mathrm{sgn}\,z_2].$$

Then

$$p(r\sigma)=\frac{\mathrm{Re}}{2\pi^2}\int_0^{\infty}dz_2\int_0^{\infty}dz_1\exp[-irz_1-c_0z_1^{\alpha}e^{-i\theta_0}]$$
$$(\exp[-\tilde{c}_1(z_1+a_1z_2)^{\alpha}e^{-i\theta_1}-\tilde{c}_2(z_1+a_2z_2)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}]$$
$$+\exp[-\tilde{c}_1|z_1-a_1z_2|^{\alpha}e^{-i\theta_1\mathrm{sgn}(z_1-a_1z_2)}$$
$$-\tilde{c}_2|z_1-a_2z_2|^{\alpha}e^{-i\theta_2\mathrm{sgn}(z_1-a_2z_2)}-c_3z_2^{\alpha}e^{i\theta_3}]).$$

By changing variable rz_1 to u we have

$$2\pi^2rp(r\sigma)=\mathrm{Re}\int_0^{\infty}dz_2\Big\{\int_0^{ra_1z_2}du\,\exp[-iu-c_0r^{-\alpha}u^{\alpha}e^{-i\theta_0}]$$
$$\Big(\exp\Big[-\tilde{c}_1\Big(a_1z_2+\frac{u}{r}\Big)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\Big(a_2z_2+\frac{u}{r}\Big)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\Big]$$
$$+\exp\Big[-\tilde{c}_1\Big(a_1z_2-\frac{u}{r}\Big)^{\alpha}e^{i\theta_1}-\tilde{c}_2\Big(a_2z_2-\frac{u}{r}\Big)^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\Big]\Big)$$
$$+\int_{ra_1z_2}^{ra_2z_2}du\,\exp[-iu-c_0r^{-\alpha}u^{\alpha}e^{-i\theta_0}]$$
$$\Big(\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}+a_1z_2\Big)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\Big(a_2z_2+\frac{u}{r}\Big)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\Big]$$
$$+\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}-a_1z_2\Big)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\Big(a_2z_2-\frac{u}{r}\Big)^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\Big]\Big)$$
$$+\int_{ra_2z_2}^{\infty}du\,\exp[-iu-c_0r^{-\alpha}u^{\alpha}e^{-i\theta_0}]$$

$$\left(\exp\left[-\tilde{c}_1\left(\frac{u}{r}+a_1z_2\right)^\alpha e^{-i\theta_1}-\tilde{c}_2\left(\frac{u}{r}+a_2z_2\right)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{-i\theta}\right]\right.$$

$$\left.+\exp\left[-\tilde{c}_1\left(\frac{u}{r}a_1z_2\right)^\alpha e^{-i\theta_1}-\tilde{c}_2\left(\frac{u}{r}-a_2z_2\right)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{i\theta_3}\right]\right)\Big\}.$$

First assume $0<\alpha<1$. Rotate the contour of integration with respect to du through an angle $-\pi/2$. Then

$$(3.9)\quad 2\pi^2 rp(r\sigma)=\mathrm{Re}\Big[-i\int_0^\infty dz_2\Big\{\int_0^{ra_1z_2}du\ e^{-u}\exp[-c_0r^{-\alpha}u^\alpha e^{-i\eta_0}]$$

$$\left(\exp\left[-\tilde{c}_1\left(a_1z_2-i\frac{u}{r}\right)^\alpha e^{-i\theta_1}-\tilde{c}_2\left(a_2z_2-i\frac{u}{r}\right)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{-i\theta_3}\right]\right.$$

$$\left.+\exp\left[-\tilde{c}_1\left(a_1z_2+i\frac{u}{r}\right)^\alpha e^{i\theta_1}-\tilde{c}_2\left(a_2z_2+i\frac{u}{r}\right)^\alpha e^{i\theta_2}-c_3z_2^\alpha e^{i\theta_3}\right]\right)$$

$$+\int_{r_1a_{z_2}}^{ra_2z_2}du\ \exp[-u-c_0r^{-\alpha}u^\alpha e^{-i\eta_0}]$$

$$\left(\exp\left[-\tilde{c}_1\left(\frac{u}{r}+ia_1z_2\right)^\alpha e^{-i\eta_1}-\tilde{c}_2\left(a_2z_2-i\frac{u}{r}\right)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{-i\theta_3}\right]\right.$$

$$\left.+\exp\left[-\tilde{c}_1\left(\frac{u}{r}-ia_1z_2\right)^\alpha e^{-i\eta_1}-\tilde{c}_2\left(a_2z_2+i\frac{u}{r}\right)^\alpha e^{i\theta_2}-c_3z_2^\alpha e^{i\theta_3}\right]\right)$$

$$+\int_{ra_2z_2}^\infty du\ \exp[-u-c_0r^{-\alpha}u^\alpha e^{-i\eta_0}]$$

$$\left(\exp\left[-\tilde{c}_1\left(\frac{u}{r}+ia_1z_2\right)^\alpha e^{-i\eta_1}-\tilde{c}_2\left(\frac{u}{r}+ia_2z_2\right)^\alpha e^{-i\eta_2}-c_3z_2^\alpha e^{-i\theta_3}\right]\right.$$

$$\left.+\exp\left[-\tilde{c}_1\left(\frac{u}{r}-ia_1z_2\right)^\alpha e^{-i\eta_1}-\tilde{c}_2\left(\frac{u}{r}-ia_2z_2\right)^\alpha e^{-i\eta_2}-c_3z_2^\alpha e^{i\theta_3}\right]\right)\Big\}$$

$$+i\int_0^\infty dz_2ra_1z_2\int_0^{\pi/2}d\varphi e^{-i\varphi}$$

$$\exp[-ra_1z_2e^{i(\pi/2-\varphi)}-c_0a_1^\alpha z_2^\alpha e^{-i(\theta_0+\alpha\varphi)}-\tilde{c}_2z_2^\alpha(a_2-a_1e^{-i\varphi})^\alpha e^{i\theta_2}-c_3z_2^\alpha e^{i\theta_3}]$$

$$\{\exp[-\tilde{c}_1a_1^\alpha z_2^\alpha(1-e^{-i\varphi})^\alpha e^{i\theta_1}]-\exp[-\tilde{c}_1a_1^\alpha z_2^\alpha(e^{-i\varphi}-1)^\alpha e^{-i\theta_1}]\}$$

$$+i\int_0^\infty dz_2ra_2z_2\int_0^{\pi/2}d\varphi e^{-i\varphi}$$

$$\exp[-ra_2z_2e^{i(\pi/2-\varphi)}-c_0a_2^\alpha z_2^\alpha e^{-i(\theta_0+\alpha\varphi)}-\tilde{c}_1z_2^\alpha(a_2e^{-i\varphi}-a_1)^\alpha e^{-i\theta_1}-c_3z_2^\alpha e^{i\theta_3}]$$

$$\{\exp[-\tilde{c}_2a_2^\alpha z_2^\alpha(1-e^{-i\varphi})^\alpha e^{i\theta_2}]-\exp[-\tilde{c}_2a_2^\alpha z_2^\alpha(e^{-i\varphi}-1)^\alpha e^{-i\theta_2}]\}\Big].$$

In the last two terms change ra_1z_2 and ra_2z_2 to u, $\pi/2-\varphi$ to ϕ respectively and rotate the contour of the integration with respect to du through an angle $-\phi$. Moreover, in the second and third terms exchange the order of integration with respect to dz_2 and du and change rz_2 to ν. Then $2\pi^2 rp(r\sigma)$ is equal to

$$\int_0^\infty du \int_{u/(ra_1)}^\infty dz e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]\}$$

$$\mathrm{Re}\left\{\exp\left[-\tilde{c}_1\left(a_1 z_2 - i\frac{u}{r}\right)^\alpha e^{-i\theta_1} - \tilde{c}_2\left(a_2 z_2 - i\frac{u}{r}\right)^\alpha e^{-i\theta_2} - c_3 z_2^\alpha e^{-i\theta_3}\right]\right\}$$

$$+\frac{\mathrm{Im}}{r}\int_0^\infty du\ e^{-u}\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]$$

$$\left\{\int_{u/a_2}^{u/a_1} d\nu(\exp[-\tilde{c}_1 r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(a_2\nu - iu)^\alpha e^{-i\theta_2} - c_3 r^{-\alpha}\nu^\alpha e^{-i\theta_3}]\right.$$

$$+\exp[-\tilde{c}_1 r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(a_2\nu+iu)^\alpha e^{i\theta_2} - c_3 r^{-\alpha}\nu^\alpha e^{i\theta_3}])$$

$$+\int_0^{u/a_2} d\nu(\exp[-\tilde{c}_1 r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(u+ia_2\nu)^\alpha e^{-i\eta_2} - c_3 r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$\left.+\exp[-\tilde{c}_1 r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(u-ia_2\nu)^\alpha e^{-i\eta_2} - c_3 r^{-\alpha}\nu^\alpha e^{i\theta_3}])\right\}$$

$$+\frac{\mathrm{Re}}{r}\int_0^\infty du\ e^{-u}\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]\frac{u}{a_1}\int_0^{\pi/2} d\phi$$

$$\exp[-i\phi - \tilde{c}_2 r^{-\alpha}u^\alpha(1-ie^{-i\phi}a_2/a_1)^\alpha e^{i\eta_2} - c_3 r^{-\alpha}a_1^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}]$$

$$\{\exp[-\tilde{c}_1 r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{i\eta_1}] - \exp[-\tilde{c}_1 r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{-i\eta_1}]\}$$

$$+\frac{\mathrm{Re}}{r}\int_0^\infty du\ e^{-u}\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]\frac{u}{a_2}\int_0^{\pi/2} d\phi$$

$$\exp[-i\phi - \tilde{c}_1 r^{-\alpha}u^\alpha(1-ie^{-i\phi}a_1/a_2)^\alpha e^{i\eta_1} - c_3 r^{-\alpha}a_2^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}]$$

$$\{\exp[-\tilde{c}_2 r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{i\eta_2}] - \exp[-\tilde{c}_2 r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{-i\eta_2}]\}.$$

Moreover in the second term we see that

$$\int_{u/a_2}^{u/a_1} d\nu(\exp[\tilde{c}_1 r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(a_2\nu - iu)^\alpha e^{-i\theta_2} - c_3 r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$+\exp[-\tilde{c}_1 r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(a_2\nu+iu)^\alpha e^{i\theta_2} - c_3 r^{-\alpha}\nu^\alpha e^{i\theta_3}])$$

$$+\int_0^{u/a_2} d\nu(\exp[-\tilde{c}_1 r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(u+ia_2\nu)^\alpha e^{-i\eta_2} - c_3 r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$+\exp[-\tilde{c}_1 r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(u-ia_2\nu)^\alpha e^{-i\eta_2} - c_3 r^{-\alpha}\nu^\alpha e^{i\theta_3}])$$

$$=i\int_{u/a_2}^{u/a_1} d\nu(\exp[-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(a_2\nu-u)^\alpha e^{i\hat{\eta}_2} - c_3 r^{-\alpha}\nu^\alpha e^{i\hat{\eta}_3}]$$

$$-\exp[-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(a_2\nu-u)^\alpha e^{-i\hat{\eta}_2} - c_3 r^{-\alpha}\nu^\alpha e^{-i\hat{\eta}_3}])$$

$$+i\int_0^{u/a_2} d\nu(\exp[-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1} - \tilde{c}_2 r^{-\alpha}(u-a_2\nu)^\alpha e^{-i\eta_2} - c_3 r^{-\alpha}\nu^\alpha e^{i\hat{\eta}_3}]$$

$$-\exp[-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^{\alpha}e^{-i\eta_1}-\tilde{c}_2 r^{-\alpha}(u-a_2\nu)^{\alpha}e^{-i\eta_2}-c_3 r^{-\alpha}\nu^{\alpha}e^{-i\hat{\eta}_3}])$$

$$-i\frac{u}{a_1}\int_0^{\pi/2}d\phi(\exp[i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1+ie^{i\phi})e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1+ie^{i\phi}a_2/a_1)^{\alpha}e^{-i\eta_2}-c_3 r^{-\alpha}a_1^{-\alpha}u^{\alpha}e^{-i(\theta_3-\alpha\phi)}]$$

$$-\exp[-i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1-ie^{-i\phi})^{\alpha}e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1-ie^{i\phi}a_2/a_1)^{\alpha}e^{i\eta_2}-c_3 r^{-\alpha}a_1^{-\alpha}u^{\alpha}e^{i(\theta_3-\alpha\phi)}])$$

$$+i\frac{u}{a_2}\int_0^{\pi/2}d\phi(\exp[i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1+ie^{i\phi}a_1/a_2)^{\alpha}e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1+ie^{i\phi})^{\alpha}e^{-i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^{\alpha}e^{-i(\theta_3-\alpha\phi)}]$$

$$-\exp[-i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1-ie^{i\phi}a_1/a_2)^{\alpha}e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1-ie^{i\phi})^{\alpha}e^{i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^{\alpha}e^{i(\theta_3-\alpha\phi)}])$$

$$-i\frac{u}{a_2}\int_0^{\pi/2}d\phi(\exp[i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1+ie^{i\phi}a_1/a_2)^{\alpha}e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1+ie^{i\phi})^{\alpha}e^{-i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^{\alpha}e^{-i(\theta_3-\alpha\phi)}]$$

$$-\exp[-i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1-ie^{i\phi}a_1/a_2)^{\alpha}e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1-ie^{i\phi})^{\alpha}e^{-i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^{\alpha}e^{i(\theta_3-\alpha\phi)}]),$$

where we rotate the contours through angles $\pm\pi/2$. Substitute this equation for the above one, then we get

$$p(r\sigma)=\frac{1}{\pi^2 r}\int_0^{\infty}du\ e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^{\alpha}e^{-i\eta_0}]\}$$

$$\mathrm{Re}\int_{u/(ra_1)}^{\infty}dz_2\left\{\exp\left[-\tilde{c}_1\left(a_1 z_2-i\frac{u}{r}\right)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\left(a_2 z_2-i\frac{u}{r}\right)^{\alpha}e^{-i\theta_2}-c_3 z_2^{\alpha}e^{-i\theta_3}\right]\right\}$$

$$+\frac{1}{\pi^2 r^2}\int_0^{\infty}du\left\{\int_{u/a_2}^{u/a_1}d\nu\ e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^{\alpha}e^{-i\eta_0}-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^{\alpha}e^{-i\eta_1}]\}\right.$$

$$\mathrm{Im}\{\exp[-\tilde{c}_2 r^{-\alpha}(a_2\nu-u)^{\alpha}e^{-i\hat{\eta}_2}-c_3 r^{-\alpha}\nu^{\alpha}e^{-i\hat{\eta}_3}]\}$$

$$+\int_0^{u/a_2}d\nu e^{-u}\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^{\alpha}e^{-i\eta_0}-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^{\alpha}e^{-i\eta_1}$$

$$\left.-\tilde{c}_2 r^{-\alpha}(u-a_2\nu)^{\alpha}e^{-i\eta_2}]\}\,\mathrm{Im}\{\exp[-c_3 r^{-\alpha}\nu^{\alpha}e^{-i\hat{\eta}_3}]\}\right\}$$

$$+\frac{1}{\pi^2 r^2}\int_0^{\infty}du\ e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^{\alpha}e^{-i\eta_0}]\}\frac{u}{a_1}$$

$$\int_0^{\pi/2}d\phi\ \exp[i\phi-\tilde{c}_1 r^{-\alpha}u^{\alpha}(1+ie^{i\phi})^{\alpha}e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^{\alpha}(1+ie^{i\phi}a_2/a_1)^{\alpha}e^{-i\eta_2}-c_3 r^{-\alpha}a_1^{-\alpha}u^{\alpha}e^{-i(\theta_3-\alpha\phi)}].$$

This implies (3.8). Next let $1<\alpha<2$. In this case it is impossible to proceed in the same way as above, because the integral in (3.9) may diverge. However in a similar way to the one-dimensional case (cf. [3] Th. 2.4.2), if we choose suitable angles in the rotation of the contours of integration and use Taylor's formula: For $x>0$, $y\in\boldsymbol{R}$

$$\exp[-x+iy]=\sum_{n=0}^{N}\frac{(-x+iy)^n}{n!}+\varepsilon\frac{(-x+iy)^{N+1}}{(N+1)!},\quad \varepsilon\in\boldsymbol{C},\ |\varepsilon|\leqq 1,$$

then we will obtain the same asymptotic expansion (3.8). In fact, first we see that

$$\begin{aligned}2\pi^2 rp(r\sigma)=&\mathrm{Re}\int_0^\infty dz_2\int_0^{ra_1z_2}du\exp[-iu-c_0r^{-\alpha}u^\alpha e^{-\theta_0}]^n/n!\\ &\left(\exp\left[-\tilde{c}_1\left(a_1z_2+\frac{u}{r}\right)^\alpha e^{-i\theta_1}-\tilde{c}_2\left(a_2z_2+\frac{u}{r}\right)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{-i\theta_3}\right]\right.\\ &\left.+\exp\left[-\tilde{c}_1\left(a_1z_2-\frac{u}{r}\right)^\alpha e^{i\theta_1}-\tilde{c}_2\left(a_2z_2-\frac{u}{r}\right)^\alpha e^{i\theta_2}-c_3z_2^\alpha e^{i\theta_3}\right]\right)\\ &+\frac{\mathrm{Re}}{r}\int_0^\infty d\nu\left\{\int_{a_1\nu}^{a_2\nu}du\exp[-iu-c_0r^{-\alpha}u^\alpha e^{-i\theta_0}]\right.\\ &\left(\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u+a_1\nu)^\alpha e^{-i\theta_1}]^n/n!\sum_{n=0}^{N}[-\tilde{c}_2r^{-\alpha}(a_2\nu+u)^\alpha e^{-i\theta_2}]^n/n!\right.\\ &\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]^n/n!+\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\theta_1}]^n/n!\\ &\sum_{n=0}^{N}[-\tilde{c}_2r^{-\alpha}(a_2\nu-u)^\alpha e^{i\theta_2}]^n/n!\\ &\left.\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}]^n/n!\right)+\int_{a_2\nu}^{\infty}du\exp[-iu-c_0r^{-\alpha}u^\alpha e^{-i\theta_0}]\\ &\left(\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u+a_1\nu)^\alpha e^{-i\theta_1}]^n/n!\sum_{n=0}^{N}[-\tilde{c}_2r^{\alpha}(u+a_2\nu)^\alpha e^{-i\theta_2}]^n/n!\right.\\ &\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]^n/n!+\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\theta_1}]^n/n!\\ &\left.\left.\sum_{n=0}^{N}[-\tilde{c}_2r^{-\alpha}(u-a_2\nu)^\alpha e^{-i\theta_2}]^n/n!\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}]^n/n!\right)\right\}\\ &+O(r^{-1-(N+1)\alpha}).\end{aligned}$$

In each term we rotate the contour of integration with respect to du through an angle $\gamma=\pi[(\alpha-2)\beta_0-1]/(2\alpha)$, then $\exp[-iu]$ is to $\exp[-ue^{i(\pi/2+\gamma)}]$ and $\exp[-c_0r^{-\alpha}u^\alpha e^{-i\theta_0}]$ is to $\exp[ic_0r^{-\alpha}u^\alpha]=\sum_{n=0}^{N}[ic_0r^{-\alpha}u^\alpha]^n/n!+\varepsilon[ic_0r^{-\alpha}u^\alpha]^{N+1}/(N+1)!$ with $\varepsilon\in\boldsymbol{C}$, $|\varepsilon|\leqq 2$. Note that $-\pi<\gamma<0$ and $|\pi/2+\gamma|<\pi/2$. Moreover we rotate the contour through an angle $-\pi/2-\gamma$. Then we have the expansion which is similar to (3.9). Then by the same way to the case of $0<\alpha<1$ we can easily obtain (3.8). Q.E.D.

Thus if $\sigma\in\mathbf{Spt}\,\lambda$ and $p_0^{\perp}(0)>0$, then

$$\mathrm{Re}\int_{u/r}^{\infty}\exp\Psi\left(-i\frac{u}{r},\,\nu\right)d\nu\longrightarrow\pi p_0^{\perp}(0)\qquad\text{as}\quad r\to+\infty,$$

and

$$p(r\sigma)\sim r^{-1-\alpha}\pi^{-1}c_0\sin\eta_0\Gamma(\alpha+1)p_0^{\perp}(0)\qquad\text{as}\quad r\to+\infty.$$

Therefore we have (a) in Proposition:

If $\sigma\notin\mathbf{Spt}\,\lambda$ then $\beta_{0,0}=-1$, i.e., $\eta_0=0$ or π (see §2), thus the first and second terms of (3.8) vanish. Hence by change of variables $u-a_1\nu$ to u' we have the following:

LEMMA 3. *Set $b_j=a_j-a_1(b_1=0,\ b_m=\infty)$. Then for $\sigma\notin\mathbf{Spt}\,\lambda$,*

$$(3.10)\quad p(r\sigma)\approx r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^{\infty}d\nu\int_{b_j\nu}^{b_{j+1}\nu}du\ e^{-u-a_1\nu}$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}r^{-n\alpha}\,\mathrm{Im}\,[c_1s_1^{\alpha}u^{\alpha}e^{-i\eta_1}+c_2s_2^{\alpha}(u-b_2\nu)^{\alpha}e^{-i\eta_2}+\cdots$$
$$+c_js_j^{\alpha}(u-b_j\nu)^{\alpha}e^{-i\eta_j}]^n$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}r^{-n\alpha}\,\mathrm{Im}\,[c_{j+1}s_{j+1}^{\alpha}(b_{j+1}\nu-u)^{\alpha}e^{-i\hat{\eta}_{j+1}}+\cdots+c_m\nu^{\alpha}e^{-i\hat{\eta}_m}]^n\,.$$

as $r\to+\infty$.

This lemma also holds in the case that λ has mass at neither σ nor $-\sigma$, because $c_0=0$ in (3.8).

Thus if $\sigma\notin\mathbf{Spt}\,\lambda$ and $\sigma\in\mathbf{Int}\,S(2)$, then

$$p(r\sigma)\sim r^{-2(1+\alpha)}\pi^{-2}\Gamma(\alpha+1)^2\sum_{1\leq j<k\leq m}g_{j,k}c_j|h_{j,k}|^{-1-\alpha}\sin\eta_jc_k|h_{k,j}|^{-1-\alpha}\sin\hat{\eta}_k$$
$$\sim\sum_{1\leq j<k\leq m}g_{j,k}p_j(rh_{j,k})p_k(rh_{k,j})\qquad\text{as}\quad r\to+\infty$$
$$=\sum_{1\leq j<k\leq m}p_{j,k}(r\sigma),$$

where $g_{j,k}=(s_jt_k-s_kt_j)^{-1}>0$ for $j<k$, $h_{j,k}$ and $h_{k,j}$ are defined by $\sigma=h_{j,k}\sigma_j+h_{k,j}\sigma_k$ (i.e., $h_{j,k}=t_k/(s_jt_k-s_kt_j)$). Thus, we get (b) in Proposition.

Moreover if $1<\alpha<2$ and $\sigma\notin S(2)$, then $\beta_{1,0}=\cdots=\beta_{m,0}=\pm1$ (i.e., $\hat{\eta}_1=\cdots=\hat{\eta}_m=\pi$ or $\eta_1=\cdots=\eta_m=\pi$). Hence every term of (3.10) vanish. We have (c) in Proposition.

Finally (d) is followed by Lemma 2.

Second step. Suppose that λ has mass at only $(m+1)$-directions σ_j, $j=0,1,2,\cdots,m$. We may assume that $\sigma=(1,0)$ and $0\leqq\varphi_0<\varphi_1<\varphi_2<\cdots<\varphi_{m-1}<\varphi_m<$

$\pi(\varphi_j=\arg\sigma_j)$. If λ has no mass at $\{\pm\sigma\}$, then by taking $c_0=0$ in $\Psi(z)$ and seting $\sigma_0=\sigma$ we may include σ as a member of directions σ_j, $j=0, 1, \cdots, m$. Moreover in (3.1) let Q be a linear transformation such that $Q\sigma_0=\sigma_0$ and $Q\sigma_m=(0, 1)$, then $0=\tilde{\varphi}_0<\tilde{\varphi}_1<\cdots<\tilde{\varphi}_m=\pi/2$ where $\tilde{\varphi}_j=\arg Q\sigma_j$. Thus by $Qr\sigma=r\sigma$ we have $p_\lambda(r\sigma)=|\det Q|p_{\lambda_Q}(r\sigma)$ and λ_Q has mass at only $(m+1)$-directions $Q\sigma_j$, $j=0, 1, \cdots, m$. Therefore the general case is reduced to the special case of First step.

The proof of Theorem 2 and Theorem 3 in Case 2 is complete.

CASE 3. $\alpha=1$ and λ has mass at $(m+1)$-directions $\sigma_0, \sigma_1, \cdots, \sigma_m$ $(m\geqq 2)$. We may also take $\{\sigma_j, j=0, 1, 2, \cdots, m\}$ as in First step of Case 2. Then for $z=(z_1, z_2)\in \boldsymbol{R}^2$

$$\Psi(z)=-\sum_{j=0}^{m} c_j\left\{|\langle\sigma_j, z\rangle|+i\frac{2}{\pi}\beta_j\langle\sigma_j, z\rangle\log|\langle\sigma_j, z\rangle|\right\},$$

where $c_j>0$, $|\beta_j|\leqq 1$, $j=0, 1, 2, \cdots, m$ are constants.

The following lemma is corresponding to Lemma 2 and Lemma 3.

LEMMA 4. *Let* $r\geqq 0$ *and* $\sigma=\sigma_0=(1, 0)$.

(i) *Then for* $a_j=\tan\phi_j$

$$p(r\sigma)=(2\pi)^{-2}\int_{\boldsymbol{R}^2}\exp[-irz_1+\Psi(z)]dz$$

$$\approx r^{-1}\pi^{-2}\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}c_0^n\int_0^\infty e^{-u}u^n\,\mathrm{Im}\left[i(1+\beta_0)-\frac{2\beta_0}{\pi}\log\frac{u}{r}\right]^n du$$

$$\mathrm{Re}\int_{u/r}^{\infty}\exp\Psi\left(-i\frac{u}{r}, \nu\right)d\nu$$

$$+r^2\pi^{-2}\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}c_0^n\int_0^\infty e^{-u}u^{n+1}\,\mathrm{Im}\left[i(1+\beta_0)-\frac{2}{\pi}\beta_0\log\frac{u}{r}\right]^n du$$

$$+r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^\infty d\nu\int_{a_j\nu}^{a_{j+1}\nu}du\, e^{-u}\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}\mathrm{Im}\left[c_0u\left\{i(1+\beta_0)-\frac{2}{\pi}\beta_0\log\frac{u}{r}\right\}\right.$$

$$+c_1s_1(u-a_1\nu)\left\{i(1+\beta_1)-\frac{2}{\pi}\beta_1\log[s_1(u-a_1\nu)/r]\right\}+\cdots$$

$$\left.+c_js_j(u-a_j\nu)\left\{i(1+\beta_j)-\frac{2}{\pi}\beta_j\log[s_j(u-a_j\nu)/r]\right\}\right]^n$$

$$\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}\mathrm{Im}\left[c_{j+1}s_{j+1}(a_{j+1}\nu-u)\left\{i(1-\beta_{j+1})+\frac{2}{\pi}\beta_{j+1}\log[s_{j+1}(a_{j+1}\nu-u)/r]\right\}\right.$$

$$+\cdots+c_m\nu\Big\{i(1-\beta_m)+\frac{2}{\pi}\beta_m\log\frac{\nu}{r}\Big\}\Big]^n$$

as $r\to+\infty$.

(ii) *If* $\sigma\notin\mathbf{Spt}\,\lambda$, *set* $b_j=a_j-a_1$, *then*

$$p(r\sigma)\approx r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^\infty d\nu\int_{b_j\nu}^{b_{j+1}\nu}du\; e^{-u-a_1\nu}$$

$$\sum_{n=1}^\infty\frac{r^{-n}}{n!}\mathrm{Im}\Big[c_1s_1u\Big\{i(1+\beta_1)-\frac{2}{\pi}\beta_1\log[s_1(u-b_1)/r]\Big\}+\cdots$$

$$+c_js_j(u-b_j\nu)\Big\{i(1+\beta_j)-\frac{2}{\pi}\beta_j\log[s_j(u-b_j\nu)/r]\Big\}\Big]^n$$

$$\sum_{n=1}^\infty\frac{r^{-n}}{n!}\mathrm{Im}\Big[c_{j+1}s_{j+1}(b_{j+1}\nu-u)\Big\{i(1-\beta_{j+1})+\frac{2}{\pi}\beta_{j+1}\log[s_{j+1}(b_{j+1}\nu-u)/r]\Big\}$$

$$+\cdots+c_m\nu\Big\{i(1-\beta_m)+\frac{2}{\pi}\beta_m\log\frac{\nu}{r}\Big\}\Big]^n$$

as $r\to+\infty$.

From this lemma we obtain Theorem 2 and Theorem 3 by the same way as in case of $1<\alpha<2$.

Next we proceed the proof of Theorem 2 in case of $d=3$.

(1) First we see that

$$(3.11)\quad (2\pi)^3p(x)=\int_{R^3}\exp[-i\langle x, z\rangle+\Psi(z)]dz$$

$$=2\,\mathrm{Re}\int_{R_+^3}dz\,\{\exp[-i(x_1z_1+x_2z_2+x_3z_3)+\Psi(z_1, z_2, z_3)]$$

$$+\exp[-i(x_1z_1-x_2z_2+x_3z_3)+\Psi(z^1, -z_2, z_3)]$$

$$+\exp[-i(x_1z_1+x_2z_2-x_3z_3)+\Psi(z_1, z_2, -z_3)]$$

$$+\exp[-i(x_1z_1-x_2z_2-x_3z_3)+\Psi(z_1, -z_2, -z_3)]\}.$$

(2) We divide the integral domain in order to omit the notation "sgn" in $\Psi(z)$.

(3) We change variables z_1, z_2, z_3 appropriately according to σ.

Then we deduce that Theorem 2 and Theorem 3 hold. We will describe the outline of the proof in some details. Here we only consider the case that λ has mass at $(m+1)$-directions $\sigma_0, \sigma_1, \cdots \sigma_m\,(m\geqq 3)$ but that $0<\alpha<1$ and $\sigma\notin$ **Int** $S(3)$, because it is evident in the others.

a) If $\sigma\in T(1)$, i.e., $\lambda(\{\sigma\})>0$, we may take $\sigma=\sigma_0=(1, 0, 0)$ and change z_1 to $-iu/r$, then we have $p(r\sigma)\sim p_0(r)p_0^+(0)\sim cr^{-1-\alpha}(c>0)$ as $r\to+\infty$.

EXAMPLE 1. Let $m=5$, $\sigma=\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$, $\sigma_2=(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\sigma_3=(0, 1/\sqrt{2}, 1/\sqrt{2})$, $\sigma_4=(1/\sqrt{2}, 0, 1/\sqrt{2})$ and $\sigma_5=(0, 0, 1)$. In (3.11) we divide the integral domain as follows:

$$\int_{R_+^3} dz=\int_0^\infty dz_3\Big\{\int_0^{z_3/2} dz_2\Big(\int_0^{z_2} dz_2\int_{z_2}^{z_3-z_2}+\int_{z_3-z_2}^{z_3}+\int_{z_3}^{z_3+z_2}+\int_{z_3+z_2}^{\infty} dz_1\Big) \tag{3.12}$$

$$+\int_{z_3/2}^{z_3} dz_2\Big(\int_0^{z_3-z_2}+\int_{z_3-z_2}^{z_2}+\int_{z_2}^{z_3}+\int_{z_2}^{z_3+z_2}+\int_{z_3+z_2}^{\infty} dz_1\Big)$$

$$+\int_{z_3}^{2z_3} dz_2\Big(\int_0^{z_2-z_3}+\int_{z_2-z_3}^{z_3}+\int_{z_3}^{z_2}+\int_{z_2}^{z_2+z_3}+\int_{z_2+z_3}^{\infty} dz_1\Big)$$

$$+\int_{2z_3}^{\infty} dz_2\Big(\int_0^{z_3}+\int_{z_3}^{z_2-z_3}+\int_{z_2-z_3}^{z_2}+\int_{z_2}^{z_2+z_3}+\int_{z_2+z_3}^{\infty} dz_1\Big)\Big\}$$

and change z_1 to $-iu/r$, then we can see that the sum of terms in (3.11) corresponding to the first integral with respect to dz_1 of each term in (3.12) decreases like $p_0(r)p_0^{\perp}(0)\sim cr^{-1-\alpha}(c>0)$ as $r\to+\infty$. Moreover, the remaining terms are $o(r^{-1-\alpha})$ as $r\to+\infty$.

b) If $\sigma\in T(2)$, then the following two cases arise.

(i) There exists only one plane H which is spanned by some elements $\sigma_0, \sigma_1, \cdots, \sigma_k (k\geqq 1)$ of **Spt** λ and contains σ. In this case we may assume that H is x_1x_2-plane, $\sigma=(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$ and $\sigma_2, \cdots, \sigma_k\in\{\theta_3=0\}\backslash\{\theta_1\geqq 0, \theta_2\geqq 0, \theta_3=0\}$ in S^2. Set $r'=r/\sqrt{2}$. We divide the integral domain as mentioned in (2) and change $(z_1, \pm z_2)$ to $-i(u_1/r', \pm u_2/r')$ in order to $\exp[-ir'(z_1\pm z_2)]$ become $\exp[-u_1-u_2]$ in (3.11). Then we have an asymptotic behaviour $p(r\sigma)\sim r^{-2(1+\alpha)}$ as $r\to+\infty$.

EXAMPLE 2. Let $m=3$, $k=1$, $\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$, $\sigma_2=(0, 1/\sqrt{5}, 2/\sqrt{5})$, $\sigma_3=(0, 0, 1)$ and $\sigma=(1/\sqrt{2}, 1/\sqrt{2}, 0)\in$ **Con** $\{\sigma_0, \sigma_1\}$. In (3.11) we divide the integral as follows:

$$\int_{R_+^3} dz=\int_0^\infty dz_3\int_0^\infty dz_1\Big\{\int_0^{2z_3}+\int_{2z_3}^{\infty} dz_2\Big\}.$$

Change variables z_1 and z_2. Then from the term in (3.11) corresponding to the first integral in the above we have an asymptotic $p_{0,1}(r\sigma(0, 1))p_{0,1}^{\perp}(0)(\sim cr^{-2(1+\alpha)}, c>0)$ as $r\to+\infty$, where $\sigma(0, 1)$ is a restriction of σ to **Span** $\{\sigma_0, \sigma_1\}$. Moreover, from the other we have $o(r^{-2(1+\alpha)})$ as $r\to+\infty$.

(ii) There exist at least two planes H_1, H_2 which are spanned by some elements of **Spt** λ and $H_1\cap H_2$ is a line containing σ. In this case we take $\sigma=(1, 0, 0)$. We change z_1 to $-iu_1/r$ and also z_2 appropriately as seen in the following example. Then we have $p(r\sigma)\sim r^{-2(1+\alpha)}$ as $r\to+\infty$.

EXAMPLE 3. The setting is the same as in Example 1 except $\sigma_0=(1/\sqrt{2}, 1/\sqrt{2}, 0)\neq\sigma=(1, 0, 0)$, and also divide the integral domain as in it. First in each integral we change z_1 to $-iu_1/r$ then in (3.11) terms vanish which correspond to the first integrals with respect to z_1 in (3.12). In the integral $\int_0^\infty dz_3\int_0^{z_3/2}dz_2\int_{rz_2}^{r(z_3-z_2)}du_1$ we change z_2 to $+iu_2/r$, $-iu_2/r$, $+iu_2/r$ and $-iu_2/r$ according to each term of (3.11). Then we have the asymptotic $p_{0,1}(r\sigma(0, 1))p_{0,1}^{\perp}(0)$ as $r\rightarrow+\infty$. Moreover by the same change of variables we have $o(r^{-2(1+\alpha)})$ as $r\rightarrow+\infty$ from the integrals of

$$\int_0^\infty dz_3\Big\{\int_0^{z_3/2}dz_2\Big(\int_{r(z_3+z_2)}^{rz_3}+\int_{rz_3}^{r(z_3+z_2)}+\int_{r(z_3+z_2)}^{\infty}du_1\Big)$$
$$+\int_{z_3/2}^{z_3}dz_2\Big(\int_{rz_2}^{rz_3}+\int_{rz_3}^{r(z_3+z_2)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)\Big\}.$$

Similarly, in the integral $\int_0^\infty dz_3\int_{2z_3}^\infty dz_2\int_{rz_3}^{r(z_2-z_3)}du_1$ we change z_3 to $+iu_3/r$, $+iu_3/r$, $-iu_3/r$ and $-iu_3/r$ according to each term of (3.11). Then we have the asymptotic $p_{4,5}(r\sigma(4, 5))p_{4,5}^{\perp}(0)$ as $r\rightarrow+\infty$, and by the same change of variables we have $o(r^{-2(1+\alpha)})$ as $r\rightarrow+\infty$ from the integrals of

$$\int_0^\infty dz_3\Big\{\int_{z_3}^{2z_3}dz_2\Big(\int_{rz_3}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)$$
$$+\int_{2z_3}^{\infty}dz_2\Big(\int_{r(z_2-z_3)}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)\Big\}$$
$$=\int_0^\infty dz_2\Big\{\int_{z_2/2}^{z_2}dz_2\Big(\int_{rz_3}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)$$
$$+\int_0^{z_2/2}dz_3\Big(\int_{r(z_2-z_3)}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)\Big\}.$$

Finally we have the asymptotic $p_{2,3}(r\sigma(2, 3))p_{2,3}^{\perp}(0)$ as $r\rightarrow+\infty$ from the remaining terms. In fact, in the integral $\int_0^\infty dz_3\int_{z_3/2}^{z_3}dz_2\int_{rz_3}^{rz_2}du_1$ we change z_2 (resp. z_3) to $-iu_2/r$, $+iu_2/r$, $-iu_2/r$ and $+iu_2/r$ (resp. $+iu_3/r$, $+iu_3/r$, $-iu_3/r$ and $-iu_3/r$) according to each term of (3.11). Moreover change variables (u_1, u_2, u_3) to $(\nu_1+\nu_3, \nu_2, \nu_2+\nu_3)$. Then the sum of the first and 4-th terms vanish and we change ν_2 to $-i\nu_2$ (resp. $+i\nu_2$) in the second term (resp. third term). Similarly in $\int_0^\infty dz_3\int_{z_2}^{2z_3}dz_2\int_{r(z_2-z_3)}^{rz_3}du_1$ change z_2 (resp. z_3) to $+iu_2/r$, $-iu_2/r$, $+iu_2/r$ and $-iu_2/r$ (resp. $-iu_3/r$, $-iu_3/r$, $+iu_3/r$ and $+iu_3/r$) according to each terms of (3.11), and (u_1, u_2, u_3) to $(\nu_1+\nu_2, \nu_2+\nu_3, \nu_3)$. Then the sum of the first and 4-th terms vanish. Hence, we change ν_3 to $+i\nu_3$ (resp. $-i\nu_3$) in the second term (resp. third term). By this way we have $p_{2,3}(r\sigma(2,3))p_{2,3}^{\perp}(0)$ as $r\rightarrow+\infty$. Therefore we see that $p(r\sigma)\sim p_{0,1}(r\sigma(0,1))p_{0,1}^{\perp}(0)+p_{2,3}(r\sigma(2,3))p_{2,3}^{\perp}(0)+p_{4,5}(r\sigma(4,5))$ $p_{4,5}^{\perp}(0)$ $(\sim cr^{-2(1+\alpha)})$ as $r\rightarrow+\infty$.

c) If $\sigma\in T(3)$, it is sufficient to consider the case that $\sigma=(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\sigma_0=(1,0,0)$, $\sigma_1=(0,1,0)$, $\sigma_2=(0,0,1)$ and $\sigma_3, \cdots, \sigma_m\subset S^2\backslash\{\theta_1\geqq 0, \theta_2\geqq 0, \theta_3\geqq 0\}$. Set $r'=r/\sqrt{3}$. We divide the integral domain as mentioned in (2) and change (z_1, z_2, z_3) to $-i(u_1/r', \pm u_2/r', \pm u_3/r')$ in order to $\exp[-ir'(z_1\pm z_2\pm z_3)]$ be to $\exp[-u_1-u_2-u_3]$ in (3.11). For instance, for $\exp[-ir'(z_1-z_2+z_3])$ we change (z_1, z_2, z_3) to $-i(u_1/r', -u_2/r', u_3/r')$. Then we have an asymptoic $p(r\sigma)$ $\sim r^{-3(1+\alpha)}$ as $r\rightarrow+\infty$.

EXAMPLE 4. Let $m=3$, $\sigma_0=(1,0,0)$, $\sigma_1=(0,1,0)$, $\sigma_2=(0,0,1)$, $\sigma_3=(0, -1/\sqrt{2}, 1/\sqrt{2})$ and $\sigma=(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. In (3.11) we divide the integral

$$\int_{R_+^3} dz=\int_0^\infty dz_3\int_0^\infty dz_1\left\{\int_0^{z_3}+\int_{z_3}^\infty dz_2\right\}.$$

Change variables z_1, z_2 and z_3 as above. Then we can easily deduce that $p(r\sigma)\sim p_{0,1,2}(r\sigma)p_{0,1,2}^{\perp}(0)+p_{0,1,3}(r\sigma)p_{0,1,3}^{\perp}(0)+p_{0,2,3}(r\sigma)p_{0,2,3}^{\perp}(0)$ $(\sim cr^{-3(1+\alpha)}, c>0)$ as $r\rightarrow+\infty$.

d) If $\sigma\in S(3)$ and $1\leqq\alpha<2$, then by the same way as in (c) we can see that $p(r\sigma)$ is rapidly decreasing as $r\rightarrow+\infty$.

All of the above change of variables are informal, however we can justify the computations by a similar way to the case of $d=2$.

Then we conclude Theorem 2 and Theorem 3.

REMARK 4. As mentioned in §1, in higher dimensions $(d\geqq 4)$ we belive that our method should work, although the calculations may be more tedious and complicated.

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