TSUKUBA J. MATH.
Vol. 2 (1978). 135—143

PHASE TRANSITIONS IN CLASSICAL HEISENBERG MODELS

By

Koji KURODA

§ 1. Introduction and notations

In this paper we extend the result of Malyshev [2] and show the existence of the phase transitions in Heisenberg model which has more than three phases by using the method of Peierls. Also we discuss the phase transitions in finite spin systems.

In the corollary of therem 1 in § 2 we also show that phase transitions occur in classical Heisenberg antiferromagnet under sufficiently low temperature.

As to the finite spin systems the existence of phase transition can be proved in a similar way to the previous one.

Now we give the definition of Gibbsian random field of the system and introduce some notations used below. We take a compact seperable metric space S as a spin space and denote by μ a non negative finite measure on the topological Borel field in S.

We consider a spin system on a 2-dimensional lattice T with a nearest neighbour potential $U(s_1, s_2)$, where $U(s_1, s_2)$ is a real function on $S\times S$ which is measurable and bounded from below.

For a finite subset $V=\{t_1, \cdots, t_v\}\subset T$, we associate a σ-field $\mathscr{B}_V$ generated by $\{\omega(t);\ t\in V\}$ $(\omega\in\Omega=S^T)$ and let $\mathscr{B}=\mathscr{B}_T$. A probability measure P on $(\Omega, \mathscr{B})$ is called a Gibbsian random field if for each finite set V and for each atom

$$A=\{\omega\in\Omega;\ \omega(t_i)=s_i\ (i=1, \cdots, v)\}\ \text{of}\ \mathscr{B}_V$$

$$P(A|\mathscr{B}_{V^c})=q_{V,\omega}(A)\quad a.e.\ P \tag{1.1}$$

where

$$q_{V,\omega}(A)=\Xi(V, \omega)^{-1}\exp\{-\beta U_V(s_1,\cdots s_v|\omega(t))\}$$

$$U_V(s_1, \cdots, s_v|\omega(t))=\frac{1}{2}\sum_{|t_i-t_j|=1}U(s_i, s_j)+\sum_{i=1}^{V}\sum_{\substack{t\in V^c\\ |t-t_i|=1}}U(s_i, \omega(t))$$

and

Received March 14, 1978. Revised July 5, 1978

$$\Xi(V, \omega)=\int e^{-\beta U_V(s_1,\cdots,s_v|\omega(t))}d\mu(s_1)\cdots d\mu(s_v)$$

A one-to-one transformation h from the measure space $(\Omega_1, \mathcal{F}_1, \mu_1)$ onto the measure space $(\Omega_2, \mathcal{F}_2, \mu_2)$ is called "admissible" if $\mu_1 h^{-1}$ is absolutely continuous with respect to μ_2 and satisfies

$$0<c_1\leq\frac{d(\mu_1\cdot h^{-1})}{d\mu_2}\leq c_2<\infty \quad \text{or} \quad \frac{d(\mu_1\cdot h^{-1})}{d\mu_2}=0$$

$$a.e.\ \mu_2 \text{ for some } c_1, c_2>0.$$

§ 2. Classical Heisenberg model

In this section, we prove the existence of phase transition in the case of the following classical Heisenberg model on square lattice.

We assume that potential function $U(s_1, s_2)$ satisfies the following conditions 1~5;

1) $U(s_1, s_2)$ is a symmetric continuous function on $S\times S$

2) There exist a one-to-one transformation g: $S\rightarrow S$ and n points $s^{[0]}, s^{[1]},\cdots, s^{[n-1]}$ of S such that the following conditions i)~v) are satisfied,

i) $U(s_1, s_2)$ takes its minimum at exactly following n points $(s^{[0]}, s^{[0]})$, $(s^{[1]}, s^{[1]}), \cdots, (s^{[n-1]}, s^{[n-1]})$

ii) $g^n=\text{identity}$

iii) $gs^{[i]}=s^{[i+1]}\ (i=0, 1, \cdots, n-1) \qquad gs^{[n-1]}=s^{[0]}$

iv) $U(gs_1, gs_2)=U(s_1, s_2)$

v) $\mu\cdot g=g$

3) There exist two neighbourhoods O_1 and O_2 of the point $s^{[0]}$ such that following three conditions are satisfied,

i) $O_1\subset O_2$, $\mu(O_1)>0$

ii) The n sets $O_2, gO_2, \cdots, g^{n-1}O_2$ are mutually disjoint.

iii) For some $\varepsilon>0$

$$U(s_1, s_2)<U(s_1', s_2')-\varepsilon$$

for any $(s_1, s_2)\in(O_1\times O_1)\cup(gO_1\times gO_1)\cup\cdots\cup(g^{n-1}O_1\times g^{n-1}O_1)$
and for any $(s_1', s_2')\notin(O_2\times O_2)\cup\cdots\cup(g^{n-1}O_2\times g^{n-1}O_2)$

4) If $\mu(O_2-O_1)>0$, then there exists an admissible transformation χ: $O_2-O_1 \longrightarrow A_1\subset O_1$ such that the following two conditions i) and ii) are satisfied, where A_1 is a open subset of O_1

i) $U(s_1, \chi s_2)<U(s_1, s_2)-\varepsilon$
for all $s_1\in O_1$, and for all $s_2\in O_2-O_1$

ii) $U(\chi s_1, \chi s_2) < U(s_1, s_2) - \varepsilon$
for all $s_1, s_2 \in O_2 - O_1$

5) There exists a finite system of mutually disjoint subsets $\{F_i\}_{i=1}^k$ of S such that

$$\sum_i F_i + \sum_i gF_i + \cdots + \sum_i g^{n-1}F_1 = S - (O_2 \cup \cdots \cup g^{n-1}O_2)$$

and $(\sum_i F_i)$, $(\sum_i gF_i)$, $\cdots$, $(\sum_i g^{n-1}F_i)$ are mutally disjoint. Also there exist admissible transformations f_i: $F_i \longrightarrow B_i$ (open subset of O_1)

THEOREM 1

If $U(s_1, s_2)$ satisfies above five conditions 1~5, then there exist at least n distinct limiting Gibbs distributions for sufficiently large β.

(PROOF)

We take a square V centered at origin and take a following boundary condition ω_0 such that

$$\omega_0(t) \in O_1 \qquad t \in V^c$$

For a given configulation ω, we call a site t regular with respect to ω if $\omega(t) \in O_1$. We also call a site A-site if it is not a regular site.

If we can prove

(2-1) $$q_{V, \omega_0}(\text{origin } O \in O_1) \geqslant \frac{1}{2} \qquad \text{(independently of } V)$$

then from the symmetry of g we get

(2-2) $$q_{V, g^k\omega_0}(\text{origin } O \in g^k O_1) \geqslant \frac{1}{2} \qquad (k=1, \cdots, n-1),$$

and so we get the distinct n limiting Gibbs distributions, where $g^k\omega_0$ is a following boundary condition

$$g^k\omega_0(t) \in g^k O_1 \qquad t \in V^c$$

Consequently we have only to show (2-1) for proving Th 1. We take a configuration ω such that origin O is A-site, and let R be a one-connected component of A-sites which cotains O. We also let $\bar{R}$=the region enclosed by the outer boundary of R, and $R_0 = T - \bar{R}$.

Next we define a subset $R(B)$ of $\bar{R}$ inductively as follows. Namely a subset $R(B)$ consist of following sites of $\bar{R}$;

I) the sites which interact with R_0

II) the sites which interact with at least one element of the above set and which has at least active bond, where O_1-O_1 bonds, gO_1-gO_1 bonds, $\cdots$, and $g^{n-1}O_1-g^{n-1}O_1$ bonds are called static bonds and other bonds are called active bonds.

III) repeat the step II), and so on.

Put $R_{int}=\bar{R}-R(B)$. From the definition of $R(B)$, the components of R_{int} can be classified into following n classes.

class 1 $C(g^0)$; The component whose boundary sites are reqular.

class k $C(g^{k-1})$; $(k=2,\cdots,n)$

The component whose boundary sites are transformed into regular sites by the transformation g^{k-1}.

Let W be the $\sqrt{2}$-connected subset of T, and

$$\boldsymbol{B}=\{\,\omega;\ R(B)(\omega)=W\,\}$$

where we say that the set W is $\sqrt{2}$-connected if for arbitrary two sites x, y of W there exists a chain $\{x_0, x_1, \cdots, x_{n-1}, x_n\}$ such that $x_0=x$, $x_1, \cdots, x_{n-1}, \in W$, $x_n=y$ and $|x_i-x_{i+1}|\leq\sqrt{2}$ $(i=0,1,\cdots,n-1)$.

We now define the transformation G: $\boldsymbol{B}\longrightarrow\Omega$ which erase the set $R(B)$ as follows.

$$Gs(t)=\begin{cases} s(t) & t\in R_0 \\ hs(t) & t\in R(B) \\ s(t) & t\in C(g^0) \\ gs(t) & t\in C(g) \\ g^2s(t) & t\in C(g^2) \\ \cdots\cdots\cdots\cdots & \\ g^{n-1}s(t) & t\in C(g^{n-1}) \end{cases}$$

$$hs=\begin{cases} \chi s & s\in O_2-O_1 \\ f_is & s\in F_i\ (i=1,2,\cdots,k) \\ g^{n-1}s & s\in gO_1 \\ \chi g^{n-1}s & S\in g(O_2-O_1) \\ f_ig^{n-1}s & s\in gF_i\ (i=1,2,\cdots,k) \\ \cdots\cdots\cdots\cdots & \\ gs & s\in g^{n-1}O_1 \\ \chi gs & s\in g^{n-1}(O_2-O_1) \\ f_igs & s\in g^{n-1}F_i\ (i=1,2,\cdots,k) \end{cases}$$

Next we consider the energy variation of the system by the transformation G.

From the property 2-iv) of g the energy in R_{int} and the mutual energy between R_{int} and $R(B)$ are conserved. So, we have only to consider the energy variation in $R(B)$ and the mutual energy variation between $R(B)$ and R_0.

When two spins s_1 and s_2 are connected through a active bond, this bond is transformed into static bond by the transformation G.

By the property 3-iii) of potential function the next inequality is satisfied,

$$U(Gs_1, Gs_2)<U(s_1, s_2)-\varepsilon \tag{2-2}$$

From the definition of $R(B)$, the number of active bonds must be greater than the half of the number of sites in $R(B)$.

Consequently setting the number of elements of $R(B)$ to be m, the following relation is satisfied

$$U_V(Gs|\omega_0)<U_V(s|\omega_0)-\frac{1}{2}\varepsilon m \tag{2-4}$$

Next we decmpose $\boldsymbol{B}$ into several subsets such that the restriction of G on each subset becomes one-to-one.

We consider a partition of S

$$\boldsymbol{P}=\{O_1, O_2-O_1, F_1, F_2, \cdots, F_k, gO_1, \cdots, g(O_2-O_1), \cdots, g^{n-1}O_1, \cdots, g^{n-1}F_k\}$$

and we say ω_1 and $\omega_2 \epsilon \boldsymbol{B}$ belong to the subset L if and only if the following condition is satisfied,

(*) both $\omega_1(t)$ and $\omega_2(t)$ belong to a common element of $\boldsymbol{P}$ for all $t \epsilon R(B)$.

By this definition, the restriction of the transformation G to each L_i becomes one-to-one. When the number of elements of W is m, the number of subsets of this decomposition is at most $(n(k+2))^m$.

Then we get

$$\int_B e^{-\beta U_V(\omega)}d\mu(\omega)=\sum_i\int_{Li} e^{-\beta U_V(\omega)}d\mu(\omega)<e^{-\frac{1}{2}\beta\varepsilon m}\sum_i\int_{Li} e^{-\beta U_V(G\omega)}d\mu(\omega)$$

$$<e^{-\frac{1}{2}\beta\varepsilon m}(n(k+2)h)^m\int_\Omega e^{-\beta U_V(\omega)}d\mu(\omega)$$

where

$$h=\max\left\{\frac{d\mu\cdot\chi^{-1}}{d\mu}\ \frac{d\mu\cdot f_i^{-1}}{d\mu}\ (i=1,\cdots,k)\right\}$$

Hence,

$$P[\boldsymbol{B}]=\frac{\int_B e^{-\beta U_V(\omega)}d\mu(\omega)}{\int_\Omega e^{-\beta U_V(\omega)}d\mu(\omega)}<(n(k+2)^{-\frac{1}{2}\varepsilon\beta})^m \tag{2-5}$$

From the definition of $R(B)$, it follows $R(B)$ is $\sqrt{2}$ −connected. As the number of $\sqrt{2}$ −connected set which contains the origin is at most $m^2 8^m$, we get the following result.

$$q_{V,\omega_0}(O\in S-O_1)=\sum_{m\geq 1} q_{V,\omega_0}(O\in S-O_1, \sharp(R(B))=m)$$

$$\leq \sum_{m\geq 1} m^2 8^m (n(k+2)he^{-\beta\varepsilon})^m$$

$$\ll \frac{1}{2} \quad \text{for sufficiently large } \beta.$$

COROLLARY

If we impose the following conditions 2)′, 3)′, 4)′ on the potential function $U(s_1, s_2)$ instead of the above conditions 2), 3), 4), then we get the same result as theorem 1.

2-0)′ $n=2m$

2-i)′ $U(s_1, s_2)$ takes its minimum at exactly following n points $(s^{[0]}, s^{[m]})$, $(s^{[1]}, s^{[m+1]})$, …, $(s^{[m-1]}, s^{[2m-1]})$ (antiferromagnet type)

3-iii)′ For some $\varepsilon>0$

$$U(s_1, s_2) < U(s_1', s_2')-\varepsilon$$

for any $(s_1, s_2)\in(O_1\times g^m O_1)\cup(gO_1\times g^{m+1}O_1)\cup\cdots\cup(g^{m-1}O_1\times g^{2m-1}O_1)$ and any $(s_1', s_2')\notin(O_2\times g^m O_2)\cup\cdots\cup(g^{m-1}O_2\times g^{2m-1}O_2)$

4-i)′ $U(s_1, \chi s_2)<U(s_1, s_2)-\varepsilon$

for any $s_1\in g^m O_1$, and Jor any $s_2\in O_2-O_1$

4-ii)′ $U(\chi s_1, g^m\chi g^m s_2)<U(s_1, s_2)-\varepsilon$

(The remainder conditions of 2)′, 3)′, 4)′ are the same as the above conditions 2), 3), 4).)

(PROOF)

We introduce the following potential function $\tilde{U}(s_1, s_2)$ given by

$$\tilde{U}(s_1, s_2)=U(s_1, g^m s_2) \quad s_1\text{: even site} \quad s_2\text{: odd site}$$

where the site (x, y) is called a even site if $x+y\equiv 0 \pmod 2$ and other sites are called cdd sites.

Clearly $\tilde{U}(s_1, s_2)$ satisfies the above conditions in theorem 1. For proving the corollary, we have only to remark the following relation

$$q_{V,\omega}^{(U)}(A)=q_{V,\omega}^{(\tilde{U})}(A')$$

where

$$A=\{\omega;\ \omega(t_i)=s_i\ (i=1,2,\cdots,v)\}$$

$$A'=\{\omega;\ \omega(t_i)=s_i \text{ if } t_i \text{ is even}$$
$$\omega(t_i)=g^m s_i \text{ if } t_i \text{ is odd } (i=1,2,\cdots,v)\}$$

and ω' is the boundary conditon given by

$$\omega'(t)=\omega(t) \text{ if } t \text{ is even}$$
$$\omega'(t)=g^m\omega(t) \text{ if } t \text{ is odd.}$$

§ 3. Finite spin system

In this section we consider the finite systems on square lattice and triangular lattice.

We let $S=\{1,2,\cdots,n\}$ $(\mu(\{i\})=1\ (i=1,2,\cdots,n))$ and α: $S\longrightarrow S$ be surjective, and consider the system with the interaction potential U defined as follows.

(3-1)
$$\begin{cases} U(1,\alpha(1))=U(2,\alpha(2))=\cdots=U(n,\alpha(n))=-\varepsilon & \varepsilon>0 \\ U(i,j)=0 \quad \text{for all other spins} \end{cases}$$

In the simillar way as in the previous section, we can show that this system has at least n distinct limiting Gibbs distributions for sufficiently large β.

In the rest of this section we discuss a finite spin system on triangular lattice.

We let $S=\{1,2,3\}$ $(\mu(\{i\})=1(i=1,2,3))$ and consider the following potential U.

(3-2)
$$\begin{cases} U(1,2)=U(2,3)=U(3,1)=-\varepsilon & \varepsilon>0 \\ U(1,1)=U(2,2)=U(3,3)=0 \end{cases}$$

We shall show that this system hs at least six distinct limiting Gibbs distributions for sufficiently large β. We give a coordinate for each site t as Fig. 1, and take the following boundary condition ω_1

$$\omega_1(x,y)=\begin{cases} 1 & \text{if } x+y\equiv 0 \pmod 3 \\ 2 & \text{if } x+y\equiv 1 \pmod 3 \\ 3 & \text{if } x+y\equiv 2 \pmod 3 \end{cases}$$

For given configuration ω, we call a site (x,y)

A_1-site if $x+y\equiv 0 \pmod 3$ and $\omega(x,y)\neq 1$

A_2-site if $x+y\equiv 1 \pmod 3$ and $\omega(x,y)\neq 2$

A_3-site if $x+y\equiv 2 \pmod 3$ and $\omega(x,y)\neq 3$.

We also call a site t A-site if t is A_1-site or A_2-site or A_2-site, and other sites are called regular. In this model, 1-2, 2-3, and 3-1 bonds are called static bonds and other bonds are called active one.

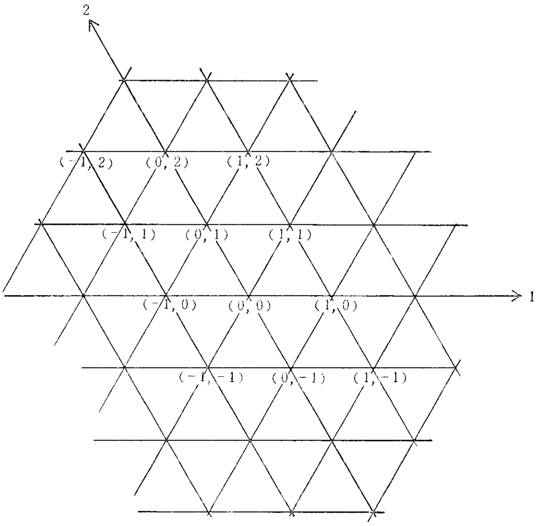

Fig. 1

In the same way as in the previous model we define R, while the definition of $R(B)$ is given by adding the following two steps to the condition I) and II) in the definition of $R(B)$ in section 2.

step III) the sites which have at least one active opposite bond are connected with at least one element of the set determined by step I) and step II), where each bond of 1~5 and 6 is called opposite bond of the site A. (see Fig. 2)

step IV) repeat the step II) and step III), and so on.

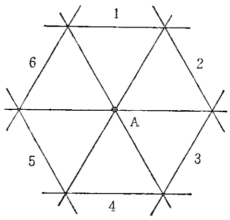

Fig. 2

In this model we can classify the family of connected components of R_{int} into the following six classes.

$$C^i; \text{ class } i: (i=1,\cdots,6)$$

: The component whose boundary sites are transformed into regular sites by the transformation g_i, where transformation $g_1, g_2, \cdots, g_6$ are defined as follows,

$$g_1(s)=s \text{ for all s} \qquad g_2(s)=\begin{cases}1 & s=1\\ 2 & s=3\\ 3 & s=2\end{cases}$$

$$g_3(s)=\begin{cases}1 & s=3\\ 2 & s=2\\ 3 & s=1\end{cases} \qquad g_4(s)=\begin{cases}1 & s=2\\ 2 & s=1\\ 3 & s=3\end{cases}$$

$$g_5(s)=\begin{cases}1 & s=3\\ 2 & s=1\\ 3 & s=2\end{cases} \qquad g_6(s)=\begin{cases}1 & s=2\\ 2 & s=3\\ 3 & s=1\end{cases}$$

In the triangular lattice, we cannot decompose R_{int} in this way unless the new two steps III) and IV) are added.

From this definition of g_i, the energy variation inside R_{int} by the transformation G defined in the same way becomes zero. Furthermore we can make two sites of $R(B)$ correspondence with one active bond, so the remainder part of the proof is accomplished in the same way as in the previous section.

References

[1] Peierls, R.: proc. Cambridge Phil. Soc. **32** (1936)
[2] Malyshev, V.A.: Commun. math. Phys. **40**, 75-82 (1975)
[3] Bortz, Griffith, R.: Commun. math. Phys. **26**, 102-108 (1972)

Institute of Mathematics
The University of Tsukuba
Ibaraki, 300-31, Japan