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ON GENERALIZED CONVOLUTION RINGS OF ARITHMETIC FUNCTIONS

By

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The set of all functions defined in the semi-group of the natural numbers whose values are in a commutative ring R, becomes an associative ring when addition and multiplication are defined by the functional addition and the γ-convolution $(f*g)(n)=\sum_{rs=n} f(r)g(s)\gamma(r, s)$.

In this paper, we will give some characterization of these rings. Finally, we will study on the extension of the base ring.

Let N be the multiplicative semi-group of the natural numbers and let K be a field. In [4] the ring of arithmetic functions is defined as the set of all functions $f: N\to K$ whose addition and multiplication are defined by the functional addition and the convolution $(f*g)(n)=\sum_{rs=n} f(r)g(s)$ for every $n\in N$. In [3], the generalized convolution ring is studied. The concept of convolution is generalized by a weighting kernel $\gamma: N\times N\to K$ and the multiplication is defined by $(f*g)(n) =\sum_{rs=n} f(r)g(s)\gamma(r, s)$.

All weighting kernels γ are characterized by the requirement that the set of all arithmetic functions still remains as an associative integral ring.

We consider here a commutative unitary ring R and we define the generalized convolution ring of arithmetic functions over R, in an analogous way. If $\gamma(r, s) =1$ for every r, s in N, the ring of arithmetic functions, which is denoted by $\mathcal{F}(N, R)$, is naturally isomorphic to the formal power series ring with infinitely many indeterminates $R[[X_1, \cdots, X_n, \cdots]]$ via the the application Γ which is defined as follow. If $\{p_1, \cdots, p_n \cdots\}$ is the sequence of all prime numbers, we put $\Gamma(X_i)=f_{p_i}$ where $f_{p_i}(n)=\delta_{p_i, n}$ (δ is the Kronecker's delta).

In section 1, we characterize these rings as solution of an universal problem. Section 2 is devoted to characterize them in an intrinsic way. In section 3 we concern with the extension of the base ring.

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tions arithmétiques", that the author did in collaboration with Carlos Meritano [2]. Several results in sections 1 and 3 of this paper are obtained in that paper. The author wishes to thank him, who kindly permites to restate them here.

1. Introduction.

In what follows we denote by N the multiplicative semi-group of the natural integers and by R a commutative unitary ring. Let $\gamma: N\times N\to R$ an application such that, for every r, s, t in N

$$\gamma(r, s)\gamma(rs, t)=\gamma(r, st)\gamma(s, t) \qquad [1.1]$$

and

$$\gamma(r, 1)=\gamma(1, r)=1 \qquad [1.2]$$

We denote by $\mathcal{F}_\gamma(N, R)$ the ring whose elements are the functions $a: N\to R$ with the ordinary addition and multiplication defined by $(a \underset{\gamma}{*} b)(n)=\sum_{rs=n} a(r)b(s)\gamma(r, s)$, for every a, b in $\mathcal{F}_\gamma(N, R)$. If $\gamma(r, s)=1$ for all r, s in N, we denote by $\mathcal{F}_1(N, R)$ or simply by $\mathcal{F}(N, R)$. Actually, we have an R-algebra with the identity 1 defined by $1(n)=\delta_{1,n}$, for every $n\in N$ (see [3]). We say that $\mathcal{F}_\gamma(N, R)$ is the γ-convolution ring over R.

Let $A=\mathcal{F}_\gamma(N, R)$ and $\phi: N\times A\to R$ the application defined by $\phi(n, a)=a(n)$, for every $a\in A$ and $n\in N$. Then it is easy to see that the followings are verified

(I) For every $n\in N$, the application $\Phi_n: A\to R$ defined by $\Phi_n(a)=\phi(n, a)$, for all $a\in A$, is R-linear.

(II) $\phi(n, ab)=\sum_{rs=n}\phi(r, a)\phi(s, b)\gamma(r, s)$, for every a, b in A and n in N.

(III) For every sequence $(a_n)_{n\in N}\in R^N$ there is a unique $a\in A$ such that $\phi(n, a)=a_n$, for all $n\in N$.

Let now A be an arbitrary R-algebra and $\phi: N\times A\to R$ an application such that (I) and (II) are satisfied. Then it is easy to see that the conditions (III′) $\phi(1, 1)=1$ and (III″) $\phi(n, 1)=\delta_{1,n}$ for every $n\in N$, are equivalent conditions. Further, if ϕ satisfies (III) then it satisfies (III′) too.

In the rest of this section, we consider the pairs (A, ϕ) where A is an R-algebra and $\phi: N\times A\to R$.

PROPOSITION 1.1. *There is a pair (A, ϕ) such that* (I), (II) *and* (III′) *are satisfied and for every pair (A', ϕ') which satisfies* (I), (II) *and* (III′) *there is a unique homomorphism of R-algebra $f: A'\to A$ such that, the following diagram is commutative*

$$\begin{tikzcd} N\times A \arrow[dr, "\phi"] & \\ & R \\ N\times A' \arrow[uu, "1\times f"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

Further, such (A, ϕ) is unique up to isomorphisms.

PROOF. It is sufficient to choose any pair (A, ϕ) such that (I), (II) and (III) are satisfied (i.e., $A=\mathcal{F}_{\gamma}(N, R)$ and ϕ defined as before). If (A', ϕ') satisfied (I), (II) and (III′) and $a'\in A'$, we can define $f(a')\in A'$ by $\phi(n, f(a'))=\phi'(n, a')$ for every $n\in N$.

REMARK 1.2. If follows from the proof that there is a unique pair (A, ϕ) such that (I), (II) and (III) are verified, i.e., the γ-convolution ring over R.

REMARK 1.3. We denote by $\mathcal{C}(R)$ the category whose objects are the pairs (A, ϕ) such that (I), (II) and (III′) are verified and the morphisms are the R-algebra homomorphisms $f: A\to A'$ such that the diagram in proposition 1.1 is a commutative diagram. By proposition 1.1, this category has the γ-convolution ring over R as a final object.

REMARK 1.4. Let now A be a ring (which is not necessarily an R-algebra "a priori") and $\phi: N\times A\to R$ an application which satisfied similar conditions but in (I) we change the condition "Φ_n is R-linear" for (I′): "Φ_n is additive". Then it is easy to obtain that similar results as in proposition 1.1. In particular, the object (A, ϕ) that satisfied (I′), (II) and (III) are unique and it is the same $\mathcal{F}_{\gamma}(N, R)$ ("a posteriori" A is an R-algebra and Φ_n is R-linear).

Let now (A, ϕ) be a pair which satisfies (I), (II) and (III). We define $\psi: R^N\to A$ by $\psi((r_n)_{n\in N})=a$ if $\phi(n, a)=r_n$ for every $n\in N$. It is easy to see that

(I_1) ψ is R-linear mapping

(II_1) $\psi((a_n)_{n\in N}(b_n)_{n\in N})=\psi((\sum_{rs=n} a_r b_s \gamma(r, s))_{n\in N})$ for all $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ in R^N.

(III_1) ψ is a bijective mapping.

Conversely, if (A, ψ) is an R-algebra A and an application $\psi: R^N\to A$ which satisfies conditions (I_1), (II_1) and (III_1), we can define $\phi: N\times A\to R$ in a similar way, and (A, ϕ) satisfies (I), (II) and (III).

We consider also (III_1') $\psi((\delta_{1,n})_{n\in N})=1$.

Then, we have trivially

PROPOSITION 1.5. *There is a pair* (A, ϕ) *such that* (I_1), (II_1) *and* (III_1) *are satisfied and for every pair* (A', ϕ') *which satisfies* (I_1), (II_1) *and* (III'_1), *there is a unique homomorphism of R-algebras* $f_1: A \to A'$ *such that the following diagram is commutative*

$$\begin{tikzcd} & R^N \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ A \arrow[rr, "f_1"] & & A' \end{tikzcd}$$

Further, (A, ϕ) *is unique up to isomorphism.*

REMARK 1.6. It is clear that the object which is a solution of the former universal problem is the unique object that satisfies (I_1), (II_1) and (III_1) and it corresponds to the solution of the problem in proposition 1.1.

Let now R and R' be commutative rings and $h: R \to R'$ a ring homomorphism. If $\gamma: \boldsymbol{N} \times \boldsymbol{N} \to R$ satisfies [1.1] and [1.2] and if for every pair $(r, s) \in \boldsymbol{N} \times \boldsymbol{N}$ we put $\gamma'(r, s) = h(\gamma(r, s))$, then $\gamma': \boldsymbol{N} \times \boldsymbol{N} \to R'$ satisfies [1.1] and [1.2]. We denote by (A, ϕ) (resp. (B, μ)) the γ-convolution ring over R (resp. γ'-convolution ring over R'). We have, from proposition 1.1,

COROLLARY 1.7. *There is a unique ring homomorphism* $h^*: A \to B$ *such that the following diagram is commutative*

$$\begin{tikzcd} \boldsymbol{N} \times A \arrow[r, "\phi"] \arrow[d, "1 \times f"] & R \arrow[d, "h"] \\ \boldsymbol{N} \times B \arrow[r, "\mu"] & R' \end{tikzcd}$$

PROOF. Since $(A, h \circ \phi)$ satisfies (I′), (II) and (III′) it is enough to apply remark 1.4.

2. Characterization of Convolution Rings.

Let R be a commutative unitary ring, $\gamma: \boldsymbol{N} \times \boldsymbol{N} \to R$ a map which satisfies [1.1] and [1.2] and (A, ϕ) the γ-convolution ring over R. We put

$$A_1 = \{a \in A \mid \phi(n, a) = 0 \ \forall n \in \boldsymbol{N},\ n \neq 1\}$$

It is clear that A_1 is a commutative unitary subring of A which is isomorphic to R via $f: a \mapsto \phi(1, a)$. Further $\gamma' = f^{-1} \circ \gamma: \boldsymbol{N} \times \boldsymbol{N} \to A_1$ satisfies [1.1] and

[1.2] and (A, ϕ) is the γ'-convolution ring over A_1. Then, in the following we consider $R=A_1\subset A$ and for every $a\in A_1$, $\phi(1, a)=a$. We also put

$$I=\{a\in A \mid \phi(1, a)=0\}.$$

Then it is easy to see that I is a two sided ideal of A such that $A=A_1\oplus I$ (as A_1-modules).

We define $\Gamma: A\to I$ by $\Gamma(a)=a'$ if and only if a' satisfies $\phi(1, a')=0$ and $\phi(n, a')=\phi(n-1, a)$ for every $n>1$ in N. We can easily see that Γ is an A_1-isomorphism. Further, if $I_n=\Gamma^n(A)$ we can obtain by induction that

$$I_n=\{a\in A \mid \phi(p, a)=0 \text{ for all } p\leqq n\} \qquad [2.1]$$

Then, using (II) we have that I_n is a two sided ideal too and we have

Lemma 2.1. *Let A be the γ-convolution ring over the commutative subring A_1 of A. Then*

(i) *A_1 is a direct summand of A as an A_1-module and its complement is a two sided ideal I of A.*

(ii) *There is an A_1-isomorphism $\Gamma: A\to I$ such that for every $n\in N$, $I_n=\Gamma^n(A)$ is a two sided ideal of A and for every pair $(r, s)\in N\times N$, the following diagram is commutative.*

$$\begin{array}{ccc} A_1\times A_1 & \xrightarrow{\Gamma^{r-1}\times\Gamma^{s-1}} & A\times A \\ \downarrow & & \downarrow \\ A_1 & \xrightarrow{\Gamma_{(r,s)}} & A \end{array}$$

where the vertical maps are the multiplications and $\Gamma_{(r,s)}(a)=\Gamma^{rs-1}(a)\gamma(r, s)$.

(iii) *The ring A with the topology defined by $(I_n)_{n\in N}$ becomes a Hausdorff and complete topological ring.*

Proof. Let a_1 and b_1 be in A_1 and we put $a=\Gamma^{r-1}(a_1)$, $b=\Gamma^{s-1}(b_1)$, $c=\Gamma_{(r,s)}(a_1b_1)=\Gamma^{rs-1}(a_1b_1\gamma(r, s))$ and $c'=ab$. Then it is clear that $\phi(n, a)=\phi(1, a_1)\delta_{n,r}$, $\phi(n, b)=\phi(1, b_1)\delta_{n,s}$. Then, since

$$\begin{aligned}\phi(n, c')&=\sum_{uv=n}\phi(u, a)\phi(v, b)\gamma(u, v)\\&=\phi(1, a_1)\phi(1, b_1)\gamma(r, s)\delta_{n,rs}=\phi(1, a_1b_1\gamma(r, s))\delta_{n,rs}\\&=\phi(n, c), \qquad \text{for every } n\in N.\end{aligned}$$

We have $c=c'$ and (ii) is proved.

Finally, $\bigcap_{n\in N} I_n=0$ is clear by [2.1]. Let $(x_n)_{n\in N}$ be a Cauchy sequence in A. Then, for every integer m there exists $n_m\in N$ such that for $p\geqq n_m$ and $q\geqq n_m$, $x_p-x_q\in I_m$, i.e. $\phi(s, x_p)=\phi(s, x_q)$ for all $s\leqq m$. We denote by $x\in A$ the unique element such that $\phi(m, x)=\phi(m, x_p)$ for arbitrary $p\geqq n_m$. It is clear that $x=\lim_{n\to\infty} x_n$, which completes the proof.

Given a ring A (resp. a topological ring A) we shall determine conditions for the existence of an unitary commutative subring A_1 of A and weighting kernel $\gamma: N\times N\to A_1$ such that A is isomorphic to $\mathcal{F}_\gamma(N, A_1)$ as a ring (resp. as a topological ring). If this is the case, we say that A is a convolution ring (resp. a convolution ring as a topological ring).

COROLLARY 2.2. *If the ring A is a convolution ring, then there is a unitary commutative subring A_1 of A and a function $\gamma: N\times N\to A_1$ which satisfies* [1.1] *and* [1.2] *such that the conditions* (i), (ii) *and* (iii) *in lemma* 2.1 *are verified. Moreover, if A is a convolution ring as a topological ring, $(I_n)_{n\in N}$ is a base of the topology of A (via the isomorphism $A\cong\mathcal{F}_\gamma(N, A_1)$).*

PROOF. If follows trivially from lemma 2.1.

To obtain the converse we need some lemmas.

LEMMA 2.3. *Let A be a ring and let A_1 be a unitary commutative subring of A satisfying the conditions* (i) *in lemma* 2.1 *and*

(ii′) *there is an A_1-module isomorphism $\Gamma: A\to I$.*

Then, for every $n\in N$, $A\cong\bigoplus_{i=1}^{n} A_i\oplus I_n$ (internal direct sum), where $A_r=\Gamma^{r-1}(A_1)$ and $I_r=\Gamma^{r-1}(I)$ for every $r\in N$.

PROOF. If $n=1$, the result is clear by condition (i). It is not difficult to complete the proof by induction argument.

COROLLARY 2.4. *With the same assumptions and notations in the above lemma, for every $a\in A$ and for every $n\in N$, there exist unique elements $a_i\in A_i$, $i=1, 2, \cdots, n$, $\alpha_n\in I_n$, such that $a=\sum_{i=1}^{n} a_i+\alpha_n$. Further, these elements does not depend of n.*

PROOF. It is enough to prove that if $a=\sum_{i=1}^{n} a_i+\alpha_n=\sum_{j=1}^{m} b_j+\beta_m$, where $a_i\in A_i$, $(i=1, 2, \cdots n)$, $b_j\in A_j$, $(j=1, \cdots, m)$, $\alpha_n\in I_n$ and $\beta_m\in I_m$, then $a_i=b_i$ for all $i\leqq \min\{n, m\}$. But this follows from $I_r\subset I_s$ and $A_r\subset I_s$ for every $r>s$.

LEMMA 2.5. *Let A be a ring, A_1 an unitary commutative subring of A and*

$\gamma: N\times N\to A_1$ a map such that [1.1] and [1.2] are verified. Let us suppose that the conditions (i) *and* (ii) *in lemma* 2.1 *are verified. Then, there is an application $\phi: N\times A\to A_1$ such that the conditions* (I), (II) *and* (III′) *in section* 1 *are verified.*

PROOF. Let $a\in A$ and $r\in N$. We denote by $a_r\in A_r$ the elements which are determined in corollary 2.4. Then, we put $\phi(n, a)=\Gamma^{1-r}(a_r)\in A_1$. It is clear that (I) and (III′) are verified. Finally, if $a=\sum_{i=1}^{n} a_i+\alpha_n$, $b=\sum_{j=1}^{m} b_j+\beta_m$ are the representations in corollary 2.4 we have

$$\begin{aligned} a_i b_j &= \Gamma^{i-1}(\phi(i, a))\Gamma^{j-1}(\phi(j, b)) \\ &= \Gamma^{ij-1}(\phi(i, a)\phi(j, b)\gamma(i, j))\in A_{ij} \end{aligned}$$

Then, it follows that

$$ab=\sum_{r=1}^{n} c_r+\delta\,, \qquad \text{where } c_r=\sum_{uv=r} a_u b_v\in A_r$$

and $\delta\in I_n$. Thus, we have

$$\begin{aligned} \phi(r, ab) &= \Gamma^{1-r}\Big(\sum_{uv=r} a_u b_v\Big)=\sum_{uv=r} \Gamma^{1-uv}(a_u b_v) \\ &= \sum_{uv=r} \phi(u, a)\phi(v, b)\gamma(u, v)\,. \end{aligned}$$

REMARK 2.6. The object (A, ϕ) which is defined in lemma 2.5, is an object in the category $\mathcal{C}(A_1)$ (Remark 1.3).

PROPOSITION 2.7. *Let A be a ring, A_1 an unitary commutative subring of A and $\gamma: N\times N\to A$ such that [1.1] and [1.2] are verified. If the conditions* (i), (ii) *in lemma* 1.1 *and*

(iii′) *The ring A becomes a Hausdorff topological space with respect to the topology defined by $(I_n)_{n\in N}$,*

are verified, the completion $\hat{A}$ of A is isomorphic to $\mathcal{F}_\gamma(N, A_1)$ as a topological ring.

PROOF. Using corollary 2.4 and the definition of ϕ we can easily see that

$$I_n=\{a\in A \mid \phi(r, a)=0, \ \forall r\leqq n\}.$$

Let $x\in\hat{A}$ and let $(x_n)_{n\in N}$ be a sequence in A such that $\lim_{n\to\infty} x_n=x$. Since $(x_n)_{n\in N}$ is a Cauchy sequence, for every m there exists n_m such that for every $p\geqq n_m$, $\phi(r, x_p-x_{n_m})=0$, for $r<m$. Therefore, the sequence $(\phi(r, x_p))_{p\in N}$ is a stationary sequence in A_1. Thus, we can define $\hat{\phi}: N\times\hat{A}\to A_1$ by $\hat{\phi}(r, x)=\lim_{n\to\infty}\phi(r, x_n)$. It is easy to see that this definition does not depend on $(x_n)_{n\in N}$

and $\hat{\phi}/N\times A=\phi$. Then, the conditions (I) and (II) in section 1 are trivially true. Let $(a_n)_{n\in N}\in A_1^N$; we put $b_i=\Gamma^{i-1}(a_i)\in A_i$ and $x_n=\sum_{i=1}^{n} b_i\in A$. We can see that $(x_n)_{n\in N}$ is a Cauchy sequence in A. If $x=\lim_{n\to\infty} x_n\in\hat{A}$, for every $r\in N$, $\hat{\phi}(r, x)=\phi(r, x_p)=a_r$ (takinp p adequately). Finally, if $\hat{\phi}(r, x)=\hat{\phi}(r, y)$ for all $r\in N$, $\hat{\phi}(r, x-y)=0$. We write $x-y=\lim_{n\to\infty} t_n$, where $(t_n)_{n\in N}\in A^N$ and using $\bigcap_{n\geqq 1} I_n=(0)$, it follows that $x-y=0$. Then, condition (III) in section 1 is verified and remark 1.2 shows that $\hat{A}\cong\mathcal{F}_\gamma(N, A_1)$. Since this isomorphism is a canonical isomorphism and $\hat{I}_n=\{a\in\hat{A}\mid\hat{\phi}(r, a)=0, \forall r\leqq n\}$ define the topology in $\hat{A}$, it is clear that it is a topological isomorphism.

Putting together all the pieces we have

THEOREM 2.8. *A ring A is a convolution ring if and only if there exist a unitary commutative subring A_1 of A and a map $\gamma: N\times N\to A$ which satisfies* [1.1] *and* [1.2] *such that the conditions* (i), (ii) *and* (iii) *in lemma* 2.1 *are verified. Furthermore, A is a convolution ring as a topological ring if and only if all above conditions are verified and $(I_n)_{n\in N}$ is a base of the topology of A (via the isomorphism $A\cong\mathcal{F}_\gamma(N, A_1)$).*

PROOF. The corollary 2.2 shows the "only if" part. The converse is a consequence of proposition 2.7. In fact, if $(I_n)_{n\in N}$ define the complete topological ring A, then $A=\hat{A}\cong\mathcal{F}_\gamma(N, A_1)$. The last part is true if the topology given in A coincide with the topology which is defined by $(I_n)_{n\in N}$. Then, the proof is complete.

REMARK 2.9. In theorem 2.8, it is not necessary to assume that γ satisfies [1.1]. If we define the ring $\mathcal{F}_\gamma(N, R)$ even as a non-associative ring, the associativity of A and $A\cong\mathcal{F}_\gamma(N, A_1)$ carry [1.1] as a consequence (see [3]). Further, A is a commutative ring if and only if γ is symmetric. A has not zero divisors if and only if A_1 is an integer domain and $\gamma(r, s)\neq 0$ for every r, s in N. Finally, if A has not zero divisors, the condition [1.2] is also not necessary, because it is implied by [1.1].

Using the same notations in theorem 2.8 we have

LEMMA 2.10. *If A is the γ-convolution ring over A_1, A is commutative and $\gamma(N\times N)\subset U(A_1)$ (the set of all unit elements in A_1), then there exists $\omega: N\to U(A_1)$ such that for every pair $(r, s)\in N\times N$,*

$$\gamma(r, s)=\frac{\omega(rs)}{\omega(r)\omega(s)}$$

PROOF. Since γ is symmetric, the proof of theorem 3.1 in [3] applies unaltered here.

LEMMA 2.11. *Let A be a commutative γ-convolution ring over A_1 and we assume that there exists a function $\omega: \boldsymbol{N}\to U(A_1)$ such that*

$$\gamma(r, s)=\frac{\omega(rs)}{\omega(r)\omega(s)},$$

for every pair $(r, s)\in \boldsymbol{N}\times\boldsymbol{N}$. Then, there exists a canonical isomorphism $A\cong \mathcal{F}_1(\boldsymbol{N}, A_1)$.

PROOF. We denote by $\phi: \boldsymbol{N}\times A\to A_1$ and by $\phi_1: \boldsymbol{N}\times\mathcal{F}_1(\boldsymbol{N}, A_1)\to A_1$ the canonical applications respectively. If $x\in\mathcal{F}_1(\boldsymbol{N}, A_1)$, we put $\psi(x)\in A$ the element which is defined by $\phi(n, \psi(x))=\omega(n)\phi_1(n, x)$, for all $n\in\boldsymbol{N}$. It is not difficult to check that ψ is an isomorphism.

COROLLARY 2.12. *Let A be a ring which has not zero divisors and we suppose that A is the γ-convolution ring over a field A_1. Then A is commutative if and only if $A\cong\mathcal{F}_1(\boldsymbol{N}, A_1)$.*

PROOF. The "if" part is trivial. Conversely, if A is a commutative domain, $\gamma(r, s)\neq 0$ for every $(r, s)\in\boldsymbol{N}\times\boldsymbol{N}$. Then $\gamma(\boldsymbol{N}\times\boldsymbol{N})\subset A_1-\{0\}=U(A_1)$ and we can apply lemmas 2.10 and 2.11.

REMARK 2.13. In Lemma 2.11 and Corollary 2.12, if A is a topological ring and is the γ-convolution ring over A_1 as a topological ring, the isomorphism is a topological isomorphism, since it is canonical. It is clear that a topological ring A can be a convolution ring but not a convolution ring as a topological ring.

If A is the γ-convolution ring over A_1, this subring is not, in general, uniquely determine. In fact, if K is a field we have

$$\begin{aligned}\mathcal{F}_1(\boldsymbol{N}, K[[X]])&\cong K[[X]][[X_1, \cdots, X_n, \cdots]]\\ &\cong K[[X, X_1, \cdots, X_n, \cdots]]\cong\mathcal{F}_1(\boldsymbol{N}, K).\end{aligned}$$

We can obtain some sufficient condition for A_1 to be uniquely determine when A is a γ-convolution ring as a topological ring.

PROPOSITION 2.14. *Let A be a topological ring which has not non-zero nilpotent elements. If A is a γ-convolution ring over A_1 as a topologycal ring, then A_1 is obtained uniquely by $A_1\cong A/I$, where $I=\{a\in A \mid \lim_{n\to\infty} a^n=0\}$ (I is the ideal in theorem 2.8).*

PROOF. We can put $A=\mathcal{F}_{\gamma}(N, A_1)$ and $I=\{a\in A \mid \phi(1, a)=0\}$. If $x\in I$, $x^2\in I$, $\phi(1, x^2)=0$ and $\phi(2, x^2)=2\phi(1, x)\phi(2, x)\gamma(1, 2)=0$. Then $x^2\in I_2$. By induction we obtain $x^n\in I_n$ for every $n\in N$ and then $\lim_{n\to\infty} x^n=0$. Conversely, let $x=a+\alpha$, where $a\in A_1$ and $\alpha\in I$, such that $\lim_{n\to\infty} x^n=0$. Since $x^n=a^n+\beta$, $\beta\in I$, for every $n\in N$, we have $a^n=0$ for some n and then $a=0$. Thus, $x=\alpha\in I$, which completes the proof.

THEOREM 2.15. *Let A be a topological ring. The following conditions are equivalent:*

(a) *A is topologically isomorphic to a 1-convolution ring over a field.*

(b) *A is a commutative domain and is topologically isomorphic to a convolution ring over a field.*

(c) *A is a commutative domain, and is topologically isomorphic to a convolution ring and $I=\{a\in A \mid \lim_{n\to\infty} a^n=0\}$ is a maximal ideal.*

(d) *There exist a field $K\subset A$ $(1_K=1_A)$ such that*

(1) *The complement of K in A is a two sided ideal I of A.*

(2) *There exist a K-isomorphism $\Gamma\colon A\to I$ such that for every n, $I_n=\Gamma^n(A)$ is a two sided ideal of A and for every r, s in N, the following diagram is commutative if the vertical applications are the multiplications:*

$$\begin{tikzcd} K\times K \arrow[r, "\Gamma^{r-1}\times\Gamma^{s-1}"] \arrow[d] & A\times A \arrow[d] \\ K \arrow[r, "\Gamma^{rs-1}"] & A \end{tikzcd}$$

(3) *A is a Hausdorff and complete topological ring and $(I_n)_{n\in N}$ is a base of its topology.*

Finally, the field mentioned in each assertion is the same field K and it is uniquely determine by $K\cong A/I$.

PROOF. (a)→(b) it is clear from Remark 2.9. (b)→(c) it follows by Proposition 2.14. (c)→(a): let $A\cong\mathcal{F}_{\gamma}(N, A_1)$. Using Proposition 2.14 and Corollary 2.12 we have that $A_1\cong A/I$ is a field and $A\cong\mathcal{F}_1(N, A_1)$. (a)↔(d) is a particular case of Theorem 2.8.

REMARK 2.16. The above theorem gives a characterization of the formal power series ring with infinitely many indeterminates over a field when we define a convenient topology.

3. Extension of the Base Ring.

Let $h: R\rightarrow T$ a ring homomorphism of commutative rings, $\gamma: N\times N\rightarrow R$ a function which satisfies [1.1] and [1.2] and $\gamma': N\times N\rightarrow T$ the extension of γ defined by $\gamma'(r, s)=h(\gamma(r, s))$. We denote $\gamma'=\gamma$, by $\phi_R: N\times\mathcal{F}_\gamma(N, R)\rightarrow R$ and $\phi_T: N\times\mathcal{F}_\gamma(N, T)\rightarrow T$ the canonical applications. We define $\phi': N\times(\mathcal{F}_\gamma(N, R)\otimes_R T)\rightarrow T$ by $\phi'(n, x\otimes t)=\phi_R(n, x)t$, for every $n\in N$, $x\in\mathcal{F}_\gamma(N, R)$ and $t\in T$. It is easy to check that ϕ' satisfies (I), (II) and (III') in section 1. Thus, there is a unique homomorphisms of T-algebras $\Phi: \mathcal{F}_\gamma(N, R)\otimes_R T\rightarrow\mathcal{F}_\gamma(N, T)$ such that the following diagram is commutative

$$\begin{tikzcd}
N\times(\mathcal{F}_\gamma(N, R)\otimes_R T) \arrow[d, "1\times\phi"] \arrow[dr, "\phi'"] & \\
N\times\mathcal{F}_\gamma(N, T) \arrow[r, "\phi_T"'] & T
\end{tikzcd}$$

On the other hand, by Corollary 1.7, there is a unique ring homomorphism $h^*: \mathcal{F}_\gamma(N, R)\rightarrow\mathcal{F}_\gamma(N, T)$ such that the following diagram is commutative

$$\begin{tikzcd}
N\times\mathcal{F}_\gamma(N, R) \arrow[r, "\phi_R"] \arrow[d, "1\times h^*"] & R \arrow[d, "h"] \\
N\times\mathcal{F}_\gamma(N, T) \arrow[r, "\phi_T"] & T
\end{tikzcd}$$

Let $\mu: \mathcal{F}_\gamma(N, T)\otimes_R T\rightarrow\mathcal{F}_\gamma(N, T)$ the multiplication ($\mu(y\otimes t)=yt$). Then it is easy to see that $\Phi=\mu\circ(h^*\otimes 1)$ and for every $\sum_{i=1}^{r} x_i\otimes t_i\in\mathcal{F}_\gamma(N, R)\otimes_R T$, $\Phi(\sum_{i=1}^{n} x_i\otimes t_i)$ is defined by

$$\phi_T(n, \Phi(\sum_{i=1}^{r} x_i\otimes t_i))=\sum_{i=1}^{r}\phi_R(n, x_i)t_i,$$

for all $n\in N$.

We ask about conditions for Φ to be an isomorphism. Since Φ is a homomorphism of R-module, this fact does not depend on γ. We omit it in the following.

We say that an R-module M is countably generated if we can find a sequence

$(x_n)_{n\in N}\in M^N$ such that $M=\sum_{i\in N} Rx_i$.

LEMMA 3.1. *Φ is surjective if and only if for every countably generated R-submodule M of T there is a finitely generated R-submodule P of T such that $M\subset P$.*

PROOF. Let Φ be surjective and $(t_n)_{n\in N}\in T^N$. We define $y\in\mathcal{F}(N, T)$ by $\phi_T(n, y)=t_n$, for all $n\in N$. Then, there is

$$x=\sum_{i=1}^{p} c_i\otimes e_i\in\mathcal{F}(N, R)\otimes_R T$$

such that $\Phi(x)=y$. Using the definition of Φ it is easy to check that $t_n\in\sum_{i=1}^{p} Re_i\subset T$, for every $n\in N$.

Conversely, if $y\in\mathcal{F}(N, T)$, we denote by $P=\sum_{i=1}^{p} Re_i$ a finitely generated R-submodule of T such that $\phi_T(n, y)\in P$ for every $n\in N$. Thus $\phi_T(n, y)=\sum_{i=1}^{p} r_{ni}e_i$, for some $r_{ni}\in R$. Let $c_i\in\mathcal{F}(N, R)$ such that $\phi_R(n, c_i)=r_{ni}$ for all $n\in N$ and $i=1, \cdots, p$. Since

$$\phi_T(n, \Phi(\sum_{i=1}^{p} c_i\otimes e_i))=\phi_T(n, y),$$

for all $n\in N$, Φ is surjective.

The following corollaries are clear.

COROLLARY 3.2. *If T is an R-algebra which is finitely generated as an R-module, Φ is surjective.*

COROLLARY 3.3. *If R is a noetherian ring, Φ is surjective if and only if every countably generated R-submodule of T is finitely generated.*

Let M be an R-module and $\mathcal{F}(N, M)$ denote the set of all functions $f: N\to M$, which is an R-module in a trivial way. As before, we can define $\phi_M: N\times\mathcal{F}(N, M)\to M$ by $\phi_M(n, x)=x(n)$, for every $n\in N$ and $x\in\mathcal{F}(N, M)$. We can see that, as in section 1, $\mathcal{F}(N, M)$ is a solution of the universal problem defined by the following conditions:

(A) For every $n\in N$, $\Phi_n: \mathcal{F}(N, M)\to M$ such that $\Phi_n(x)=\phi_M(n, x)$ is an R-linear mapping.

In fact, $\mathcal{F}(N, M)$ is the unique R-module that satisfies (A) and

(B) For every $(x_n)_{n\in N}\in M^N$, there is a unique $x\in\mathcal{F}(N, M)$ such that $\phi_M(n, x)=x_n$, for each $n\in N$.

Then, as above, we have that there exists a unique application $\rho_M : \mathcal{F}(N, R) \otimes_R M \to \mathcal{F}(N, M)$ such that $\rho_M(a\otimes x)=y$ if and only if

$$\phi_M(n, y)=\phi_R(n, a)x\,, \qquad \text{for all } n\in N \quad (\mathcal{F}(N, R)=\mathcal{F}_\gamma(N, R))\,.$$

Let P be a projective R-module and let L be a free R-module such that $L\cong P\oplus Q$. Since, by Corollary 3 in [1] (pag. A. II. 63), ρ_L is a monomorphism and ρ_P can be can be considered as the restriction of ρ_L on $\mathcal{F}(N, R)\otimes_R P$, ρ_P is a monomorphism. Similarly, if P is finitely generated and projective, L is finitely generated and free, thus ρ_P is an isomorphism. Then, we have

LEMMA 3.4. *If P is a projective R-module (resp. projective and finitely generated), the canonical application ρ_P is a monomorphism (resp. an isomorphism).*

COROLLARY 3.5. *If T is an R-algebra which is projective (resp. projective and finitely generated) as an R-module, Φ is a monomorphism (resp. an isomorphism),*

PROOF. Taking $T=P$. it is clear that $\Phi=\rho_P$. More generally, we can prove,

PROPOSITION 3.6. *Let us assume that for every finitely generated R-submodule M of T there exists a projective R-submodule P of T such that $M\subset P$. Then Φ is a monomorphism.*

PROOF. Let $v=\sum_{i=1}^{p} x_i\otimes t_i\in\mathcal{F}(N, R)\otimes_R T$ such that $\Phi(v)=0$. We denote by P a projective R-submodule of T such that $\sum_{i=1}^{p} Rt_i\subset P$ and by $j: P\to T$ the natural inclusion. We write $u=\sum_{i=1}^{p} x_i\otimes t_i\in\mathcal{F}(N, R)\otimes_R P$ and we have $v=(1\otimes j)(u)$.

Since, for every $n\in N$,

$$0=\phi_T(n, \Phi(v))=\sum_{i=1}^{p}\varphi_R(n, x_i)t_i=j(\sum_{i=1}^{p}\phi_R(n, x_i)t_i)\,,$$

$\phi_P(n, \rho_P(u))=0$ for every $n\in N$. Thus, $\rho_P(u)=0$ and by Lemma 3.4, $u=0$, which completes the proof.

COROLLARY 3.7. *Let us assume that for every countably generated R-submodule M of T there is a finitely generated and projective R-submodule P of T such that $M\subset P$. Then Φ is an isomorphism.*

PROOF. It follows trivially from Lemma 3.1 and Proposition 3.6.

REMARK 3.8. If J is an ideal of R, the above results include the particular case $T=R/J$. In this case, the canonical application $\mathcal{F}(N, R)/J\mathcal{F}(N, R)\to\mathcal{F}(N, R/J)$

is always surjective. If R is a direct sum of two ideals, $R=J\oplus J'$, then, by Corollary 3.5, it is an isomorphism. We shall consider now the case of localization.

Let R be a ring and S a multiplicative system of R. We denote by $h: R\to R_S$ the natural application $h(r)=r/1$ and by $\Phi: (\mathcal{F}(N, R))_S\to\mathcal{F}(N, R_S)$ the application defined above, i. e., for $x\in\mathcal{F}(N, R)$, $s\in S$ and $n\in N$,

$$\phi\left(n, \Phi\left(\frac{x}{s}\right)\right)=\frac{\phi_R(n, x)}{s},$$

where $\phi=\phi_{R_S}$. We consider here the following conditions

(h) For every $(s_n)_{n\in N}\in S^N$, there exist $(r_n)_{n\in N}\in R^N$ and $t\in S$, such that

$$\frac{r_n s_n}{1}=\frac{t}{1} \qquad \text{for every } n\in N, \text{ in } R_S.$$

(H) For every $(s_n)_{n\in N}\in S^N$, there exist $(r_n)_{n\in N}\in R^N$ and $t\in S$ such that

$$r_n s_n=t \qquad \text{for every } n\in N, \text{ in } R.$$

It is clear that (H)$\to$(h) and we shall see that the converse is not true.

PROPOSITION 3.9. *The following conditions are equivalent*

(a) *Φ is an epimorphism*

(b) *S verifies* (h)

(c) *For every countably generated R-submodule M of R_S, there is $t\in S$ such that $M\subset R\cdot 1/t$.*

PROOF. (a)$\leftrightarrow$(c) it is a trivial consequence of Lemma 3.1, since every finitely generated R-submodule of R_S are contained in an R-submodule, which is generated by a simple element of the form $1/t$, $t\in S$.

(b)$\to$(c) Given $(y_n/s_n)_{n\in N}\in R_S^N$, we choose $(r_n)_{n\in N}\in R^N$ and $t\in S$ such that $r_n s_n/1=t/1$.

Thus $y_n/s_n=r_n y_n/t\in R\cdot 1/t$, for every $n\in N$.

(c)$\to$(b) Let $(s_n)_{n\in N}\in S^N$; we can find $t\in S$ such that $1/s_n\in R\cdot 1/t$, for every $n\in N$, and (h) follows.

PROPOSITION 3.10. *The following conditions are equivalent*

(a) *Φ is an isomorphism*

(b) *S verifies* (H).

PROOF. (b)$\to$(a). It is enough to prove that Φ is a monomorphism. Let $x\in\mathcal{F}(N, R)$ and $s\in S$ such that $\Phi(x/s)=0$. Then, for every $n\in N$, $\phi_R(n, x)/s=0$ and there is $s_n\in S$ such that $s_n\phi_R(n, x)=0$. We multiply by the element r_n determined by (H) and we have $\phi_R(n, tx)=t\phi_R(n, x)=r_n s_n \phi_R(n, x)=0$ for every $n\in N$. Thus, $tx=0$, i.e., $x/s=0$.

(a)→(b) Given $(s_n)_{n\in N}\in S^N$, since Φ is an epimorphism we can find $(r_n)_{n\in N}\in R^N$ and $t\in S$ such that $s_n r_n/1=t/1$, for every $n\in N$, in R_S. Let $x\in\mathcal{F}(N, R)$ and $y\in\mathcal{F}(N, R)$ such that for every $n\in N$, $\phi_R(n, x)=r_n s_n$ and $\phi_R(n, y)=t$. Since $\phi(n, \Phi(x/1))=r_n s_n/1=t/1=\phi(n, \Phi(y/1))$, $x/1=y/1$ and there is $u\in S$ such that $ux=uy$. It is clear that $ur_n s_n=ut$ for every $n\in N$ and then (H) is verified.

REMARK 3.11. It is clear that every finite multiplicative system verifies (H). Then if R is a finite ring and S is arbitrary, Φ is an isomorphism. On the other hand, if S is countable and it has not zero divisors, the conditions (h) and (H) are equivalent, but they are true only in trivial cases, i.e., if and only if every element of S is an unit in R, $(R_S=R)$. Finally, (h) and (H) are not equivalent in general. In fact, let $R=C[0, 1]$ the ring of all real continuous functions defined in $[0, 1]$ and S the multiplicative system which consists of all of the functions f such that $f(x)=0$ if and only if $x\in[1/j, 1]$ for some $j\geqq 1$, and the constant function 1. For every n, let f_n be a function in S such that $f_n(x)=0$ if and only if $x\in[1/n, 1]$. If there exist some $f\in S$, $g_n\in R$ $(n=1, 2\cdots)$ such that $f=f_n g_n$, we have $f=0$. Then, (H) is not verified. But the condition (h) is verified. In fact, let $f\in S$ and let suppose that $f(x)=0$ if and only if $x\in[1/m, 1]$. We define $g\in R$ as some continuous extension of the function defined for $t\in[0, 1/2m]$ by $g(t)=1/f(t)$, and we denote by $h\in S$ a function such that $h(x)=0$ if and only if $x\in[1/2m, 1]$. Then $h(gf-1)=0$ and we have $gf/1=1/1$ in R_S (this example is due to Ada Maria de Souza Döering and the author thanks to her).

Let now R be a commutative ring and we denote by $R[[X]]$ the ring of the formal power series over R with indeterminates $X=\{X_1, \cdots, X_n, \cdots\}$ (finitely or infinitely many). If $R\to T$ is a ring homomorphism of commutative rings, by $\Phi: R[[X]]\otimes_R T\to T[[X]]$, we denote the canonical application that we define in the begining of this section, or if X is a set of finitely many indeterminates then its restriction. (We can do this restriction without problem since $\Phi(X_i)=X_i$).

We have the following corollaries:

COROLLARY 3.12. *Let us assume that for every contably generated R-submodule M of T, there is a finitely generated projective R-submodule P of T such that $M\subset P$. Then, the canonical application Φ is an isomorphism.*

COROLLARY 3.13. *If S is a multiplicative system in R which satisfies* (h) *(resp.* (H)), *the canonical application* $(R[[X]])_S \to R_S[[X]]$ *is an epimorphism (resp. an isomorphism).*

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