

AUTOMORPHISMS OF ORDER 4 OF THE SIMPLY CONNECTED COMPACT LIE GROUP E_6

By

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Using the theory of Kac-Moody Lie algebras, for compact simple Lie algebra \mathfrak{g} , automorphisms ρ of finite order of \mathfrak{g} can be classified and the type of Lie subalgebras \mathfrak{g}^ρ of fixed points are determined [1]. Now for the simply connected compact Lie group E_6 , we realize automorphisms ρ of order 4 and determine the subgroups $(E_6)^\rho$ of fixed points. Among compact exceptional Lie groups, only E_6 has outer automorphisms, so we consider the case of E_6 . As results, the group E_6 has eight inner automorphisms named as $\gamma_1, \gamma_2, \dots, \gamma_5, \sigma_1, \sigma_2, \sigma_3$ and three outer automorphisms named as $\tau\gamma_2', \tau\gamma_3, \tau\sigma_3$, and the subgroups $(E_6)^\rho$ of fixed points are given as follows.

ρ	$(\mathfrak{e}_6)^\rho$	$(E_6)^\rho$
γ_1	$T^1 \oplus A_1 \oplus A_4$	$(Sp(1) \times S(U(1) \times U(5))) / \mathbf{Z}_2$
σ_2	$T^1 \oplus A_1 \oplus A_1 \oplus A_3$	$(Sp(1) \times S(U(2) \times U(4))) / \mathbf{Z}_2$
γ_2	$T^1 \oplus A_1 \oplus A_2 \oplus A_2$	$(Sp(1) \times S(U(3) \times U(3))) / \mathbf{Z}_2$
γ_3	$T^1 \oplus A_5$	$(U(1) \times SU(6)) / \mathbf{Z}_2$
γ_4	$T^2 \oplus A_4$	$(U(1) \times S(U(1) \times U(5))) / \mathbf{Z}_2$
γ_5	$T^2 \oplus A_1 \oplus A_3$	$(U(1) \times S(U(2) \times U(4))) / \mathbf{Z}_2$
σ_1	$T^1 \oplus D_5$	$(U(1) \times Spin(10)) / \mathbf{Z}_4$
σ_3	$T^2 \oplus D_4$	$(U(1) \times (Spin(2) \times Spin(8))) / (\mathbf{Z}_2 \times \mathbf{Z}_4)$
$\tau\gamma_2'$	$A_1 \oplus D_3$	$(Sp(1) \times SO(6)) / \mathbf{Z}_2$
$\tau\gamma_3$	$T^1 \oplus C_3$	$(U(1) \times Sp(3)) / \mathbf{Z}_2$
$\tau\sigma_3$	$A_1 \oplus B_3$	$(SU(2) \times Spin(7)) / \mathbf{Z}_2$

1. Preliminaries

Let $\mathbb{C} = \mathbf{H} \oplus \mathbf{H}e$ (\mathbf{H} is the field of quaternions with the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$) be the Cayley algebra with the multiplication $(m + ae)(n + be) = (mn - \bar{b}a) + (a\bar{n} + bm)e$, the conjugation $\overline{m + ae} = \bar{m} - ae$, the inner product $(x, y) = (\bar{x}y + \bar{y}x)/2$ and the length $|x| = \sqrt{(x, x)}$, and \mathbb{C}^c be its complexification. Let $\mathfrak{S} = \{X \in M(3, \mathbb{C}) \mid X^* = X\}$

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be the exceptional Jordan algebra with the multiplication $X \circ Y = (XY + YX)/2$ and $\mathfrak{J}^{\mathbb{C}}$ be its complexification. \mathfrak{J} and $\mathfrak{J}^{\mathbb{C}}$ have the inner product $(X, Y) = \text{tr}(X \circ Y)$, the Freudenthal multiplication $X \times Y = (2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E)/2$ and the determinant $\det X = (X \times X, X)/3$. ($\mathfrak{J}(3, \mathbf{H}) = \{M \in M(3, \mathbf{H}) \mid M^* = M\}$ and $\mathfrak{J}(3, \mathbf{H})^{\mathbb{C}}$ are also defined). The complex conjugations of $\mathbb{C}^{\mathbb{C}}$, $\mathfrak{J}^{\mathbb{C}}$ are denoted by τ . In $\mathfrak{J}^{\mathbb{C}}$, the positive definite inner product $\langle X, Y \rangle$ is defined by $(\tau X, Y)$. Now

$$E_6 = \{\alpha \in \text{Iso}_{\mathbb{C}}(\mathfrak{J}^{\mathbb{C}}) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$$

$$= \{\alpha \in \text{Iso}_{\mathbb{C}}(\mathfrak{J}^{\mathbb{C}}) \mid \alpha X \times \alpha Y = \tau \alpha \tau (X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$$

is the simply connected compact Lie group of type E_6 [2]. Throughout this paper, we use such notations and theorems in [4] as $E, E_i, F_i(x_i), i=1, 2, 3$ of $\mathfrak{J}, \mathfrak{J}^{\mathbb{C}}$ and Lie subgroups $F_4 = \{\alpha \in E_6 \mid \tau \alpha = \alpha \tau\} = \{\alpha \in E_6 \mid \alpha E = E\}, Spin(9) = \{\alpha \in F_4 \mid \alpha E_1 = E_1\}, Spin(10) = \{\alpha \in E_6 \mid \alpha E_1 = E_1\}$ of E_6 etc..

2. Inner automorphisms $\gamma_1, \gamma_2, \dots, \gamma_8$ of order 4 of E_6

The field \mathbf{H} is embedded in $M(2, \mathbb{C})$ by $k: \mathbf{H} = \mathbb{C} \oplus \mathbb{C}j \rightarrow M(2, \mathbb{C})$ (where $\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$) by $k(a + bj) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}$. This k is naturally extended to \mathbf{R} -linear mappings $k: M(3, \mathbf{H}) \rightarrow M(6, \mathbb{C}), k: \mathbf{H}^3 \rightarrow M(2, 6, \mathbb{C})$. Moreover these k are extended to \mathbb{C} - \mathbb{C} -linear isomorphisms $k: M(3, \mathbf{H})^{\mathbb{C}} \rightarrow M(6, \mathbb{C}), k: (\mathbf{H}^3)^{\mathbb{C}} \rightarrow M(2, 6, \mathbb{C})$,

$$k(M_1 + iM_2) = k(M_1) + ik(M_2), \quad M_i \in M(3, \mathbf{H}),$$

$$k(\mathbf{a}_1 + i\mathbf{a}_2) = k(\mathbf{a}_1) + ik(\mathbf{a}_2) \quad \mathbf{a}_i \in \mathbf{H}^3.$$

Finally we define the \mathbb{C} -vector space $\mathfrak{E}(6, \mathbb{C})$ by $\{S \in M(6, \mathbb{C}) \mid {}^t S = -S\}$ and the \mathbb{C} - \mathbb{C} -linear isomorphism $k_J: \mathfrak{J}(3, \mathbf{H})^{\mathbb{C}} \rightarrow \mathfrak{E}(6, \mathbb{C})$ by

$$k_J(M_1 + iM_2) = k(M_1)J + ik(M_2)J, \quad M_i \in \mathfrak{J}(3, \mathbf{H})$$

where $J = \text{diag}(J, J, J), J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In $\mathfrak{J}(3, \mathbf{H})^{\mathbb{C}} \oplus (\mathbf{H}^3)^{\mathbb{C}}$, we define the Freudenthal multiplication [2] as

$$(M + \mathbf{a}) \times (N + \mathbf{b}) = \left(M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a}) \right) - \frac{1}{2}(\mathbf{a}N + \mathbf{b}M).$$

Then $\mathfrak{J}^{\mathbb{C}}$ is isomorphic to $\mathfrak{J}(3, \mathbf{H})^{\mathbb{C}} \oplus (\mathbf{H}^3)^{\mathbb{C}}$ by the correspondence

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3)$$

(where $x_i = m_i + a_i e$, $m_i, a_i \in \mathbf{H}^C$) as Freudenthal algebra [4]. Hereafter we identify \mathfrak{S}^C and $\mathfrak{S}(3, \mathbf{H})^C \oplus (\mathbf{H}^3)^C$. We define an involutive C -linear mapping $\gamma: \mathfrak{S}^C \rightarrow \mathfrak{S}^C$ by

$$\gamma(M + \mathbf{a}) = M - \mathbf{a}, \quad M + \mathbf{a} \in \mathfrak{S}(3, \mathbf{H})^C \oplus (\mathbf{H}^3)^C = \mathfrak{S}^C.$$

Then $\gamma \in E_6$ and $\gamma^2 = 1$.

PROPOSITION 2.1. $(E_6)^\gamma \cong (Sp(1) \times SU(6)) / \mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

PROOF. Let $Sp(1) = \{p \in \mathbf{H} \mid \bar{p}p = 1\}$ and $SU(6) = \{A \in M(6, \mathbf{C}) \mid A^*A = E, \det A = 1\}$. Now the mapping $\phi: Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$,

$$\phi(p, A)(M + \mathbf{a}) = k_J^{-1}(A k_J(M)^t A) + p k^{-1}(k(\mathbf{a})A^*), \quad M + \mathbf{a} \in \mathfrak{S}^C$$

induces the required isomorphism. The details of proof are in [2] or [4].

REMARK. $\phi: Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ satisfies $\gamma = \phi(-1, E)$ and $\tau\phi(p, A)\tau = \phi(p, -J\bar{A}J)$.

Using $\phi: Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ of Proposition 2.1, we define

$$\begin{aligned} \gamma_1 &= \phi(1, iI_1), & I_1 &= \text{diag}(-1, 1, 1, 1, 1, 1), \\ \gamma_2 &= \phi(1, iI_3), & I_3 &= \text{diag}(-1, -1, -1, 1, 1, 1), \\ \gamma_3 &= \phi(i, E), \\ \gamma_4 &= \phi(\mathbf{e}, \mathbf{e}\Gamma_1), & \mathbf{e} &= (1+i)/\sqrt{2}, \Gamma_1 = \text{diag}(i, 1, 1, 1, 1, 1), \\ \gamma_5 &= \phi(i, I_2), & I_2 &= \text{diag}(-1, -1, 1, 1, 1, 1). \end{aligned}$$

Then $\gamma_i \in E_6$ and the order of γ_i is 4, for $i = 1, 2, \dots, 5$.

THEOREM 2.2. (1) $(E_6)^{\gamma_1} \cong (Sp(1) \times S(U(1) \times U(5))) / \mathbf{Z}_2$,

(2) $(E_6)^{\gamma_2} \cong (Sp(1) \times S(U(3) \times U(3))) / \mathbf{Z}_2$,

(3) $(E_6)^{\gamma_3} \cong (U(1) \times SU(6)) / \mathbf{Z}_2$,

(4) $(E_6)^{\gamma_5} \cong (U(1) \times S(U(2) \times U(4))) / \mathbf{Z}_2$

where $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$ in any case.

PROOF. (1) Since $\gamma_1^2 = \gamma$, we have $(E_6)^{\gamma_1} \subset (E_6)^\gamma$. Hence, for $\alpha \in (E_6)^{\gamma_1}$ there exist $p \in Sp(1)$, $A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). From the

condition $\gamma_1\alpha = \alpha\gamma_1$, we have $\phi(p, \mathbf{i}I_1A) = \phi(p, \mathbf{i}AI_1)$, that is, $I_1A = AI_1$, therefore $A \in S(U(1) \times U(5))$. Thus we have the required isomorphism.

(2), (3), (4) are proved to be similar to (1).

THEOREM 2.3. $(E_6)^{\gamma_4} \cong (U(1) \times S(U(1) \times U(5))) / \mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

PROOF. Since the operation of γ_4 on $\mathfrak{Z}^c = \mathfrak{Z}(3, \mathbf{H})^c \oplus (\mathbf{H}^3)^c$ is given by

$$\begin{aligned} & \gamma_4 \left(\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3) \right) \\ &= \begin{pmatrix} -\xi_1 & i\varepsilon m_3 & i\varepsilon \bar{m}_2 \\ i\varepsilon \bar{m}_3 \bar{\varepsilon} & i\xi_2 & im_1 \\ i\varepsilon m_2 \bar{\varepsilon} & i\bar{m}_1 & i\xi_3 \end{pmatrix} + (-i\varepsilon a_1 \bar{\varepsilon}, -i\varepsilon a_2, -i\varepsilon a_3) \end{aligned}$$

(where $\varepsilon = (1+i)/\sqrt{2}$), the eigen C -vector spaces $(\mathfrak{Z}^c)_\nu$, $\nu = 1, -1, i, -i$ with respect to γ_4 are

$$\begin{aligned} (\mathfrak{Z}^c)_1 &= \{M + \mathbf{a} \in \mathfrak{Z}(3, \mathbf{H})^c \oplus (\mathbf{H}^3)^c \mid \gamma_4(M + \mathbf{a}) = M + \mathbf{a}\} \\ &= \{(a_1(\mathbf{i}-i), (\mathbf{i}+i)a_2, (\mathbf{i}+i)a_3) \mid a_1 \in \mathbf{C}\mathbf{j}, a_2, a_3 \in \mathbf{H}\}, \\ (\mathfrak{Z}^c)_{-1} &= \{M + \mathbf{a} \in \mathfrak{Z}(3, \mathbf{H})^c \oplus (\mathbf{H}^3)^c \mid \gamma_4(M + \mathbf{a}) = -M - \mathbf{a}\} \\ &= \left\{ \begin{pmatrix} \xi_1 & (\mathbf{i}+i)m_3 & \overline{m_2(\mathbf{i}-i)} \\ \overline{(\mathbf{i}+i)m_3} & 0 & 0 \\ m_2(\mathbf{i}-i) & 0 & 0 \end{pmatrix} + (a_1(\mathbf{i}+i), 0, 0) \right\}_{\substack{\xi_1 \in \mathbf{C}, \\ a_1 \in \mathbf{C}\mathbf{j}, m_2, m_3 \in \mathbf{H}}}, \\ (\mathfrak{Z}^c)_i &= \{M + \mathbf{a} \in \mathfrak{Z}(3, \mathbf{H})^c \oplus (\mathbf{H}^3)^c \mid \gamma_4(M + \mathbf{a}) = i(M + \mathbf{a})\} \\ &= \left\{ \begin{pmatrix} 0 & (\mathbf{i}-i)m_3 & \overline{m_2(\mathbf{i}+i)} \\ \overline{(\mathbf{i}-i)m_3} & \xi_2 & m_1 \\ m_2(\mathbf{i}+i) & \bar{m}_1 & \xi_3 \end{pmatrix} \right\}_{\substack{\xi_2, \xi_3 \in \mathbf{C}, \\ m_1 \in \mathbf{H}^c, m_2, m_3 \in \mathbf{H}}}, \\ (\mathfrak{Z}^c)_{-i} &= \{M + \mathbf{a} \in \mathfrak{Z}(3, \mathbf{H})^c \oplus (\mathbf{H}^3)^c \mid \gamma_4(M + \mathbf{a}) = -i(M + \mathbf{a})\} \\ &= \{(a_1, (\mathbf{i}-i)a_2, (\mathbf{i}-i)a_3) \mid a_1 \in \mathbf{C}^c, a_2, a_3 \in \mathbf{H}\} \end{aligned}$$

where $\mathbf{C}\mathbf{j} = \{sj + tk \mid s, t \in \mathbf{R}\}$. These spaces are invariant under the group $(E_6)^{\gamma_4}$. We shall show that $(\mathbf{H}^3)^c$ is invariant under $(E_6)^{\gamma_4}$. From the forms of $(\mathfrak{Z}^c)_\nu$, it is sufficient to show that $\alpha\mathbf{a} \in (\mathbf{H}^3)^c$ for $\alpha \in (E_6)^{\gamma_4}$ and $\mathbf{a} = (a(\mathbf{i}+i), 0, 0) = F_1((a(\mathbf{i}+i))e)$ ($\mathbf{a} \in \mathbf{C}\mathbf{j}$). Now, in fact,

$$\begin{aligned} \alpha F_1((a(i+i)e)) &= 4\alpha((F_3((i-i)\bar{a}) \times F_1(1)) \times F_3(e)) \\ &= 4(\alpha F_3((i-i)\bar{a}) \times \alpha F_1(1)) \times \tau \alpha F_3(e) \\ &\in 4(\mathfrak{Z}(3, \mathbf{H})^c \times \mathfrak{Z}(3, \mathbf{H})^c) \times (\mathbf{H}^3)^c \subset \mathfrak{Z}(3, \mathbf{H})^c \times (\mathbf{H}^3)^c \subset (\mathbf{H}^3)^c. \end{aligned}$$

Thus we see that $(\mathbf{H}^3)^c$ is invariant under $(E_6)^{\gamma^4}$, hence $\mathfrak{Z}(3, \mathbf{H})^c = ((\mathbf{H}^3)^c)^\perp = \{X \in \mathfrak{Z}^c \mid \langle X, Y \rangle = 0 \text{ for all } Y \in (\mathbf{H}^3)^c\}$ is also invariant under $(E_6)^{\gamma^4}$. Consequently, $\alpha \in (E_6)^{\gamma^4}$ commutes with γ , that is, $(E_6)^{\gamma^4} \subset (E_6)^\gamma$. Hence, for $\alpha \in (E_6)^{\gamma^4}$, there exist $p \in Sp(1)$, $A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). From the condition $\gamma_4 \alpha = \alpha \gamma_4$, we have $\phi(\mathfrak{e}p, \mathfrak{e}A) = \phi(p\mathfrak{e}, A\mathfrak{e})$, that is, $\mathfrak{e}p = p\mathfrak{e}$, $\mathfrak{e}A = A\mathfrak{e}$ (or $\mathfrak{e}p = -p\mathfrak{e}$, $\mathfrak{e}A = -A\mathfrak{e}$ (which is impossible)), therefore $p \in U(1)$, $A \in S(U(1) \times U(5))$. Thus we have the required isomorphism.

3. Inner automorphisms $\sigma_1, \sigma_2, \sigma_3$ of order 4 of E_6

Let $U(1) = \{\theta \in C \mid (\tau\theta)\theta = 1\}$ (where $C = \mathbf{R}^c$) and we define an embedding $\phi : U(1) \rightarrow E_6$ by

$$\phi(\theta) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \theta^4 \xi_1 & \theta x_3 & \theta \bar{x}_2 \\ \theta \bar{x}_3 & \theta^{-2} \xi_2 & \theta^{-2} x_1 \\ \theta x_2 & \theta^{-2} \bar{x}_1 & \theta^{-2} \xi_3 \end{pmatrix}$$

and put $\sigma = \phi(-1) \in E_6$.

The group $Spin(10)$ is defined by $(E_6)_{E_1} = \{\alpha \in E_6 \mid \alpha E_1 = E_1\}$ which is the covering group of $SO(10) = SO(V^{10})$ where $V^{10} = \{X \in \mathfrak{Z}^c \mid 2E_1 \times X = -\tau X\} = \{\xi E_2 - \tau \xi E_3 + F_1(x) \mid \xi \in C, x \in \mathfrak{G}\}$. Note that $Spin(10)$ leaves invariant $\{X \in \mathfrak{Z}^c \mid E_1 \times X = 0\} = \{F_2(x) + F_3(y) \mid x, y \in \mathfrak{G}^c\}$.

PROPOSITION 3.1. $(E_6)^\sigma \cong (U(1) \times Spin(10)) / \mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

PROOF. The mapping $\varphi : U(1) \times Spin(10) \rightarrow (E_6)^\sigma$,

$$\varphi(\theta, \beta) = \phi(\theta)\beta$$

induces the required isomorphism. The details of proof are in [2] or [4].

Using $\phi : U(1) \rightarrow E_6$ or $\phi : Sp(1) \times SU(6) \rightarrow E_6$ of Proposition 2.1, we define

$$\sigma_1 = \phi(i) = \phi(-1, \Gamma_4), \quad \Gamma_4 = \text{diag}(1, 1, i, i, i, i),$$

$$\sigma_2 = \gamma \sigma_1 = \phi(1, \Gamma_4).$$

Then $\sigma_i \in E_6$ and $\sigma_i^2 = \sigma$, $\sigma_i^4 = 1$ for $i = 1, 2$.

THEOREM 3.2. $(E_6)^{\sigma_1} \cong (U(1) \times Spin(10)) / \mathbf{Z}_4$, $\mathbf{Z}_4 = \langle (i, \phi(-i)) \rangle$.

PROOF is clear from Proposition 3.1, because $\sigma_1 = \phi(i)$ commutes with any elements of $U(1)$ and $Spin(10)$.

THEOREM 3.3. $(E_6)^{\sigma_2} \cong (Sp(1) \times S(U(2) \times U(4))) / \mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

PROOF. Since $\sigma_2^2 = \sigma$, we have $(E_6)^{\sigma_2} \subset (E_6)^\sigma$. Hence, for $\alpha \in (E_6)^{\sigma_2}$, there exist $\theta \in U(1)$, $\beta \in Spin(10)$ such that $\alpha = \phi(\theta)\beta$ (Proposition 3.1). In particular, α commutes with $\sigma_1 = \phi(i)$. Therefore, from the condition $\sigma_2\alpha = \alpha\sigma_2$, that is, $\gamma\sigma_1\alpha = \alpha\gamma\sigma_1$, we have $\gamma\alpha = \alpha\gamma$, namely $\alpha \in (E_6)^\gamma$. Hence there exist $p \in Sp(1)$, $A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). Moreover from the condition $\sigma_1\alpha = \alpha\sigma_1$, we have $\phi(p, \Gamma_4 A) = \phi(p, A\Gamma_4)$, that is, $\Gamma_4 A = A\Gamma_4$, therefore $A \in S(U(2) \times U(4))$. Thus we have the required isomorphism.

REMARK. The group $(E_6)^{\sigma_2}$ has also the following expression

$$(E_6)^{\sigma_2} \cong (U(1) \times Sp(1) \times (SU(2) \times SU(4))) / (\mathbf{Z}_2 \times \mathbf{Z}_4)$$

where $\mathbf{Z}_2 = \langle (1, -1, -I_2) \rangle$, $\mathbf{Z}_4 = \langle (-i, 1, -\Gamma_4) \rangle$. In fact, for $\alpha = \phi(p, A) \in (E_6)^{\sigma_2}$, $p \in Sp(1)$, $A = (P, Q) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(2) \times U(4))$, the condition that α belongs to the group $((E_6)^{\sigma_2})_{E_1} \subset Spin(10)$, that is, $\phi(p, (P, Q))E_1 = E_1$, is $p \in Sp(1)$, $P \in SU(2)$, $Q \in SU(4)$. From this we have easily the required isomorphism.

The field \mathbf{C} of complex numbers is embedded in \mathfrak{G} as $\mathbf{C} = \{x + ye \mid x, y \in \mathbf{R}\}$ and put $\mathbf{C}^+ = \{t \in \mathfrak{G} \mid (t, \mathbf{C}) = 0\}$. Let $Spin(2) = \{a \in \mathbf{C} \mid \bar{a}a = 1\}$ ($\cong U(1)$) and we define an embedding $D : Spin(2) \rightarrow E_6$ by

$$D_a \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_3 \bar{a} & \bar{x}_2 a \\ a \bar{x}_3 & \xi_2 & a x_1 a \\ \bar{a} x_2 & \overline{a x_1 a} & \xi_3 \end{pmatrix}.$$

Put $\sigma_3 = D_{-e}$. Then $\sigma_3 \in E_6$ and $\sigma_3^2 = \sigma$, $\sigma_3^4 = 1$.

The group $Spin(8)$ is defined by

$$\begin{aligned} (E_6)_{E_1, F_1(\sigma)} &= \{ \alpha \in E_6 \mid \alpha E_1 = E_1, \alpha F_1(s) = F_1(s) \text{ for all } s \in \mathbf{C}^+ \} \\ &= \{ \alpha \in Spin(10) \mid \alpha F_1(1) = F_1(1), \alpha F_1(e) = F_1(e) \} \end{aligned}$$

which is the covering group of $SO(8) = SO(V^8)$ where $V^8 = (V^{10})_{\sigma_3} = \{ \xi E_2 - \tau \xi E_3 + F_1(t) \mid \xi \in \mathbf{C}, t \in \mathbf{C}^+ \}$.

LEMMA 3.3. D_a ($a \in Spin(2)$) and $\beta \in Spin(8)$ commute with each other.

PROOF.

$$\begin{aligned} \beta D_a F_1(z) &= \beta F_1(aza) = \beta F_1(a^2s+t) \quad (z=s+t \in \mathcal{C}^c \oplus (\mathcal{C}^+)^c = \mathfrak{G}^c) \\ &= F_1(a^2s) + \beta F_1(t) = F_1(a^2s) + (\xi_2 E_2 + \xi_3 E_3 + F_1(t')) \quad (\xi_i \in \mathcal{C}, t' \in (\mathcal{C}^+)^c) \\ &= D_a(F_1(s) + \xi_2 E_2 + \xi_3 E_3 + F_1(t')) = D_a(F_1(s) + \beta F_1(t)) \\ &= D_a \beta F_1(s+t) = D_a \beta F_1(z). \end{aligned}$$

$$\begin{aligned} \beta D_a F_2(z) &= \beta F_2(\bar{a}z) = 4\beta((F_1(1) \times F_2(z)) \times F_1(a)) = 4(\beta F_1(1) \times \beta F_2(z)) \times \tau \beta \tau F_1(a) \\ &= 4(F_1(1) \times (F_2(x) + F_3(y))) \times F_1(a) \quad (\text{for some } x, y \in \mathfrak{G}^c) \\ &= F_2(\bar{a}x) + F_3(y\bar{a}) = D_a(F_2(x) + F_3(y)) = D_a \beta F_2(z). \end{aligned}$$

Similarly $\beta D_a F_3(z) = D_a \beta F_3(z)$. Clearly $D_a \beta = \beta D_a$ on E_1 .

$$D_a \beta E_2 = D_a(\xi_2 E_2 + \xi_3 E_3 + F_1(t)) = \xi_2 E_2 + \xi_3 E_3 + F_1(t) = \beta E_2 = \beta D_a E_2$$

(for some $\xi_i \in \mathcal{C}, t \in (\mathcal{C}^+)^c$). Similarly $D_a \beta E_3 = \beta D_a E_3$. Thus we have $D_a \beta = \beta D_a$.

LEMMA 3.4. Let $\beta \in (\text{Spin}(10))^{\sigma_3}$. Then we can put $\beta F_1(1) = F_1(s), \beta F_1(e) = F_1(es), s \in \mathcal{C}, |s| = 1$.

PROOF. Since the group $(\text{Spin}(10))^{\sigma_3}$ acts on $\{F_1(s) | s \in \mathcal{C}\} = \{X \in \mathfrak{F}^c | \sigma_3 X = -X, 2E_1 \times X = -\tau X\}$, we can put

$$\beta F_1(1) = F_1(s), \quad \beta F_1(e) = F_1(s'), \quad s, s' \in \mathcal{C}, |s| = |s'| = 1.$$

Operate $\tau \beta \tau$ on the relation $F_1(1) \times F_1(e) = -(1, e)E_1 = 0$, then $0 = \tau \beta \tau(F_1(1) \times F_1(e)) = \beta F_1(1) \times \beta F_1(e) = F_1(s) \times F_1(s') = -(s, s')E_1$, hence $(s, s') = 0$. Together with $|s| = |s'| = 1$, we have $s' = es$ or $s' = -es$. The latter case is impossible. In fact, choose $s \in \mathcal{C}$ such that $a^2 = \bar{s}$ and put $\delta = D_a \beta$, then $\delta F_1(1) = F_1(1), \delta F_1(e) = -F_1(e)$. Then

$$\begin{aligned} \delta F_2(e) &= \delta \sigma_3 F_2(1) = \sigma_3 \delta F_2(1) = \sigma_3(F_2(x) + F_3(y)) \quad (\text{for some } x, y \in \mathfrak{G}^c) \\ &= F_2(ex) + F_3(ye), \\ \delta F_3(1) &= 2\delta(F_1(1) \times F_2(1)) = 2\tau \delta \tau F_1(1) \times \tau \delta \tau F_2(1) \\ &= 2F_1(1) \times (F_2(\tau x) + F_3(\tau y)) = F_3(\tau \bar{x}) + F_2(\tau \bar{y}). \end{aligned}$$

Therefore we have

$$\begin{aligned} F_1(e) &= \delta F_1(-e) = 2\delta(F_2(e) \times F_3(1)) = 2\tau \delta \tau F_2(e) \times \tau \delta \tau F_3(1) \\ &= 2(F_2(e(\tau x)) + F_3((\tau y)e)) \times (F_3(\bar{x}) + F_2(\bar{y})) \\ &= F_1(-x((\tau \bar{x})e) - (e(\tau \bar{y}))y) + *E_2 + *E_3. \end{aligned}$$

Compare the coefficients of e , then we have $1 = -|x_1|^2 - |x_2|^2 - |y_1|^2 - |y_2|^2$ (where $x = x_1 + ix_2, y = y_1 + iy_2, x_i, y_i \in \mathbb{C}$), which is a contradiction. Thus Lemma 3.4 is proved.

THEOREM 3.5. $(E_6)^{\sigma_3} \cong (U(1) \times Spin(2) \times Spin(8)) / (\mathbf{Z}_2 \times \mathbf{Z}_4),$

$$\begin{aligned} \mathbf{Z}_2 \times \mathbf{Z}_4 = & \langle (1, -1, \sigma) \rangle \times \langle (i, e, \phi(i)D_e) \rangle \\ = & \{ (1, 1, 1), (-1, 1, \sigma), (i, e, \phi(i)D_e), (-i, e, \phi(-i)D_e), \\ & (1, -1, \sigma), (-1, -1, 1), (i, -e, \phi(-i)D_e), (-i, -e, \phi(i)D_e) \}. \end{aligned}$$

PROOF. We define a mapping $\phi: U(1) \times Spin(2) \times Spin(8) \rightarrow (E_6)^{\sigma_3}$ by

$$\phi(\theta, a, \delta) = \phi(\theta)D_a\delta.$$

Obviously ϕ is well-defined. Since $\phi(\theta)$ ($\theta \in U(1)$), D_a ($a \in Spin(2)$) and $\delta \in Spin(8)$ commute with one another (Lemma 3.3), ϕ is a homomorphism. We shall show that ϕ is onto. (Although it suffices to show $\dim((e_6)^{\sigma_3}) = 30$, we will give a direct proof). Since $\sigma_3^2 = \sigma$, we have $(E_6)^{\sigma_3} \subset (E_6)^\sigma$. Hence, for $\alpha \in (E_6)^{\sigma_3}$, there exist $\theta \in U(1), \beta \in Spin(10)$ such that $\alpha = \phi(\theta)\beta$ (Proposition 3.1). From $\sigma_3\alpha = \alpha\sigma_3$, we have $\beta \in (Spin(10))^{\sigma_3}$. Hence we can put $\beta F_1(1) = F_1(s), \beta F_1(e) = F_1(es), s \in \mathbf{C}, |s| = 1$ (Lemma 3.4). Choose $a \in \mathbf{C}$ such that $a^2 = s$ and put $\delta = D_a^{-1}\beta$, then $\delta F_1(1) = F_1(1), \delta F_1(e) = F_1(e)$, that is, $\delta \in Spin(8)$. Hence we have a presentation such that

$$\alpha = \phi(\theta)D_a\delta, \quad \theta \in U(1), a \in Spin(2), \delta \in Spin(8).$$

Then ϕ is onto. $\text{Ker } \phi = \mathbf{Z}_2 \times \mathbf{Z}_4$ is easily obtained. Thus we have the required isomorphism.

4. Outer automorphisms $\tau\gamma_2', \tau\gamma_3$ of order 4 of E_6

Using $\phi: Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ of Proposition 2.1, we define $\gamma_2' = \phi(1, J)$ and consider an automorphism $\tau\gamma_2'$ of $E_6: E_6 \ni \alpha \rightarrow \tau\gamma_2'\alpha\gamma_2'^{-1}\tau \in E_6$. Then $(\tau\gamma_2')^2 = \gamma, (\tau\gamma_2')^4 = 1$.

THEOREM 4.1. $(E_6)^{\tau\gamma_2'} \cong (Sp(1) \times SO(6)) / \mathbf{Z}_2, \mathbf{Z}_2 = \{(1, E), (-1, -E)\}.$

PROOF. Since $(\tau\gamma_2')^2 = \gamma$, we have $(E_6)^{\tau\gamma_2'} \subset (E_6)^\gamma$. Hence, for $\alpha \in (E_6)^{\tau\gamma_2'}$, there exist $p \in Sp(1), A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). Since $\tau\gamma_2'\alpha\gamma_2'^{-1}\tau = \alpha$, we have $\tau\phi(p, -JAJ)\tau = \phi(p, A)$, that is, $\phi(p, \bar{A}) = \phi(p, A)$ (Remark of Proposition 2.1) then $A = \bar{A}$, hence $A \in SO(6)$. Thus we have the required isomorphism.

We use $\gamma_3 = \phi(i, E)$ of Section 2 and consider an automorphism $\tau\gamma_3$ of E_6 : $E_6 \ni \alpha \rightarrow \tau\gamma_3\alpha\gamma_3^{-1}\tau \in E_6$. Then $(\tau\gamma_3)^2 = \gamma$, $(\tau\gamma_3)^4 = 1$.

THEOREM 4.2. $(E_6)^{\tau\gamma_3} \cong (U(1) \times Sp(3))/Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

PROOF. Since $(\tau\gamma_3)^2 = \gamma$, we have $(E_6)^{\tau\gamma_3} \subset (E_6)^\gamma$. Hence, for $\alpha \in (E_6)^{\tau\gamma_3}$, there exist $p \in Sp(1)$, $A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). Since $\tau\gamma_3\alpha\gamma_3^{-1}\tau = \alpha$, we have $\phi(-ipi, -J\bar{A}J) = \phi(p, A)$ (Remark of Proposition 2.1), then

$$p = -ipi, A = -J\bar{A}J \text{ or } p = ipi, A = J\bar{A}J.$$

In the first case $p \in U(1) = \{p \in \mathbf{C} \mid \bar{p}p = 1\}$, $A \in Sp(3) = \{A \in M(6, \mathbf{C}) \mid JA = \bar{A}J, A^*A = E, (\det A = 1)\}$ (where $\mathbf{C} \subset \mathbf{H}$). The latter case is impossible. In fact, consider AI where $I = \text{diag}(1, -1, 1, -1, 1, -1)$, then $AI \in Sp(3)$, hence $\det(AI) = 1$. On the other hand, $\det(AI) = (\det A)(\det I) = 1(-1) = -1$, which contradicts $\det(AI) = 1$. Thus we have the required isomorphism.

5. Outer automorphism $\tau\sigma_3$ of order 4 of E_6

As in Section 3, we embed the field \mathbf{C} of complex numbers in \mathfrak{C} as $\mathbf{C} = \{x + ye \mid x, y \in \mathbf{R}\}$. Let $SU(2) = \{A \in M(2, \mathbf{C}) \mid A^*A = E, \det A = 1\}$ and we define a mapping $\phi : SU(2) \rightarrow E_6$ by

$$\phi(A)X = (\rho_1(A))X(\rho_1(A))^*, \quad X \in \mathfrak{Z}^c$$

where $\rho_1(A) = \Gamma^{-1}A\Gamma$, $\Gamma = \text{diag}(i, 1, i)$, $A' = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Explicitly, for $A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$, $z \in \mathfrak{C}^c$,

$$\begin{cases} \phi(A)E_1 = E_1, \\ \phi(A)(E_2 + E_3) = (|a|^2 - |b|^2)(E_2 + E_3) - 2iF_1(a\bar{b}), \\ \phi(A)(E_2 - E_3) = E_2 - E_3, \\ \phi(A)F_1(z) = -2i(z, \bar{b}\bar{a})(E_2 + E_3) + F_1(aza - \bar{b}z\bar{b}), \\ \phi(A)F_2(z) = F_2(\bar{a}z) - iF_3(\bar{z}b), \\ \phi(A)F_3(z) = -iF_2(b\bar{z}) + F_3(z\bar{a}). \end{cases}$$

Note that $D_a = \phi \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$ for $a \in \mathbf{C}$, $|a| = 1$, in particular, $\phi \begin{pmatrix} -e & 0 \\ 0 & e \end{pmatrix} = D_{-e} = \sigma_3$.

LEMMA 5.1. ϕ is well-defined, that is, for $A \in SU(2)$ we have $\phi(A) \in E_6$.

PROOF. Define $\rho \in E_6$ by $\rho X = \Gamma X \Gamma$, $X \in \mathfrak{Z}^c$. We consider an embedding

$SU(3) \subset F_4 \subset E_6$, in which $SU(3) \subset F_4$ is given by $h : SU(3) \rightarrow F_4$,

$$h(A)(X+M) = AXA^* + MA^*, \quad X+M \in \mathfrak{S}(3, \mathbf{C}^c) \oplus M(3, \mathbf{C}^c) = \mathfrak{S}^c$$

(as for notations see [5]). The $\phi(A)$ is nothing but $\phi(A) = \rho^{-1}h(A')\rho \in E_6$.

We use σ_3 of Section 3 and consider an automorphism $\tau\sigma_3$ of $E_6 : E_6 \ni \alpha \rightarrow \tau\sigma_3\alpha\sigma_3^{-1}\tau \in E_6$. Then $(\tau\sigma_3)^2 = \sigma$, $(\tau\sigma_3)^4 = 1$.

LEMMA 5.2. For $\alpha \in (E_6)^{\tau\sigma_3}$, we have $\alpha E_1 = E_1$. In particular, $(E_6)^{\tau\sigma_3} \subset (E_6)_{E_1} = Spin(10)$.

PROOF. Since $(\tau\sigma_3)^2 = \sigma$, we have $(E_6)^{\tau\sigma_3} \subset (E_6)^\sigma$. Hence, for $\alpha \in (E_6)^{\tau\sigma_3}$, there exists $\xi \in \mathbf{C}$, $(\tau\xi)\xi = 1$ such that $\alpha E_1 = \xi E_1$ (Proposition 3.1). From $\xi E_1 = \alpha E_1 = \alpha\tau\sigma_3 E_1 = \tau\sigma_3 \alpha E_1 = \tau\sigma_3(\xi E_1) = \tau\xi E_1$, we have $\tau\xi = \xi$, that is, $\xi \in \mathbf{R}$, hence $\xi = \pm 1$. The case of $\xi = -1$ is impossible. (Although it follows from the connectedness of $(E_6)^{\tau\sigma_3}$, we will give an elementary proof). Suppose $\xi = -1$. Let $\alpha = \phi(\theta)\beta$, $\theta \in U(1)$, $\beta \in Spin(10)$. Then $-E_1 = \alpha E_1 = \phi(\theta)\beta E_1 = \theta^4 E_1$, hence $\theta^4 = -1$. Now, for $t \in \mathbf{C}^\pm$, we can put $\beta F_1(t) = \eta E_2 - \tau\eta E_3 + F_1(x)$, $\eta \in \mathbf{C}$, $x \in \mathbf{C}$. Then $\phi(\phi)\beta F_1(t) = \pm i\eta E_2 \mp i\tau\eta E_3 \pm iF_1(x)$. Since $\tau\sigma_3\alpha = \alpha\tau\sigma_3$, we have $\eta \in i\mathbf{R}$, $x \in \mathbf{C}$. This shows that for $V = \{F_1(t) | t \in \mathbf{C}^\pm\}$, $\dim V = 6$, $\dim(\alpha V) \leq 3$, which contradicts the regularity of α . Thus Lemma 5.2 is proved.

The group $Spin(7)$ is defined by

$$\begin{aligned} (E_6)_{E_1, E, F_1(s)} &= \{\alpha \in E_6 | \alpha E_1 = E_1, \alpha E = E, \alpha F_1(s) = F_1(s) \text{ for all } s \in \mathbf{C}\} \\ &= \{\alpha \in F_4 | \alpha E_1 = E_1, \alpha F_1(s) = F_1(s) \text{ for all } s \in \mathbf{C}\} \\ &= \{\alpha \in Spin(9) | \alpha F_1(1) = F_1(1), \alpha F_1(e) = F_1(e)\} \end{aligned}$$

which is the covering group of $SO(7) = SO(V^7)$ where $V^7 = \{\xi(E_2 - E_3) + F_1(t) | \xi \in \mathbf{R}, t \in \mathbf{C}^\pm\}$.

LEMMA 5.3. $\phi(A) \in \phi(SU(2))$ and $\beta \in Spin(7)$ commute with each other.

PROOF. For $A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$, $\beta \in Spin(7)$, we shall show

$$\beta\phi(A)X = \phi(A)\beta X, \quad X \in \mathfrak{S}^c. \tag{i}$$

$$\begin{aligned} \beta\phi(A)(E_2 - E_3) &= \beta(E_2 - E_3) = \xi(E_2 - E_3) + F_1(t) \quad (\text{for some } \xi \in \mathbf{C}, t \in (\mathbf{C}^\pm)^c) \\ &= \phi(A)(\xi(E_2 - E_3) + F_1(t)) = \phi(A)\beta(E_2 - E_3), \end{aligned}$$

$$\begin{aligned}\beta\phi(A)(E_2+E_3) &= \beta((|a|^2-|b|^2)(E_2+E_3)-2iF_1(\bar{a}b)) = \phi(A)(E_2+E_3) \\ &= \phi(A)\beta(E_2+E_3).\end{aligned}$$

Thus (i) is true for $X=E_2, E_3$. For $X=E_1$, (i) is trivial.

$$\begin{aligned}\beta\phi(A)F_1(z) &= \beta\phi(A)F_1(s+t) \quad (z=s+t \in \mathbf{C}^c \oplus (\mathbf{C}^+)^c = \mathfrak{C}^c) \\ &= \beta(-2i(s, \bar{b}a)(E_2+E_3) + F_1(a^2s - \bar{b}^2\bar{s}) + F_1(t)) \\ &= -2i(s, \bar{b}a)(E_2+E_3) + F_1(a^2s - \bar{b}^2\bar{s}) + (\xi(E_2-E_3) + F_1(t'))\end{aligned}$$

(for some $\xi \in C, t' \in (\mathbf{C}^+)^c$)

$$= \phi(A)(F_1(s) + \xi(E_2-E_3) + F_1(t')) = \phi(A)\beta F_1(s+t) = \phi(A)\beta F_1(z).$$

$$\begin{aligned}\beta\phi(A)F_2(z) &= \beta(F_2(\bar{a}z) - iF_3(\bar{z}b)) \\ &= 4\beta((F_1(1) \times F_2(z)) \times F_1(a)) - 4i\beta((F_1(1) \times F_3(\bar{z})) \times F_1(\bar{b})) \\ &= 4(\beta F_1(1) \times \beta F_2(z)) \times \beta F_1(a) - 4i(\beta F_1(1) \times \beta F_3(\bar{z})) \times \beta F_1(\bar{b})\end{aligned}$$

(put $\beta F_2(z) = F_2(x) + F_3(y), x, y \in \mathfrak{C}^c$, then $\beta F_3(\bar{z}) = \beta(2F_1(1) \times F_2(z)) = 2F_1(1) \times \beta F_2(z) = 2F_1(1) \times (F_2(x) + F_3(y)) = F_2(\bar{x}) + F_3(\bar{y})$)

$$\begin{aligned}&= 4(F_1(1) \times (F_2(x) + F_3(y))) \times F_1(a) - 4i(F_1(1) \times (F_3(\bar{x}) + F_2(\bar{y}))) \times F_1(\bar{b}) \\ &= F_2(\bar{a}x) + F_3(y\bar{a}) - i(F_3(\bar{x}b) + F_2(b\bar{y})) \\ &= F_2(\bar{a}x) - iF_3(\bar{x}b) - iF_2(b\bar{y}) + F_3(y\bar{a}) \\ &= \phi(A)F_2(x) + \phi(A)F_3(y) = \phi(A)(F_2(x) + F_3(y)) = \phi(A)\beta F_2(z).\end{aligned}$$

Similarly, $\beta\phi(A)F_3(z) = \phi(A)\beta F_3(z)$. Thus (i) is proved.

LEMMA 5.4. $\phi(SU(2))$ and $Spin(7)$ are contained in $(E_6)^{\tau\sigma_3}$.

PROOF. $\phi(SU(2)) \subset (E_6)^{\tau\sigma_3}$ is clear, noting that $\begin{pmatrix} -e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} -e & 0 \\ 0 & e \end{pmatrix}^{-1} = \begin{pmatrix} a & \bar{b} \\ -b & \bar{a} \end{pmatrix}$, $\tau i = -i$ and b, i appear simultaneously in $\phi(A)X$. Next, $\beta \in Spin(7) \subset F_4$ implies $\tau\beta = \beta\tau$ and $\sigma_3\beta = \beta\sigma_3$ (Lemma 5.3). Hence $Spin(7) \subset (E_6)^{\tau\sigma_3}$.

THEOREM 5.5. $(E_6)^{\tau\sigma_3} \cong (SU(2) \times Spin(7))/\mathbf{Z}_2, \mathbf{Z}_2 = \{(E, 1), (-E, \sigma)\}$.

PROOF. We define a mapping $\phi: SU(2) \times Spin(7) \rightarrow (E_6)^{\tau\sigma_3}$ by

$$\phi(A, \beta) = \phi(A)\beta.$$

Then ϕ is well-defined (Lemma 5.4) and is a homomorphism (Lemma 5.3). We

shall show that ϕ is onto. (Although it suffices to show $\dim((e_6)^{\tau\sigma_3})=24$, we will give a direct proof). Let $\alpha \in (E_6)^{\tau\sigma_3}$. Then we can put

$$\alpha(i(E_2+E_3))=i\eta(E_2+E_3)+F_1(s), \quad \eta^2+|s|^2=1, \quad \eta \in \mathbf{R}, s \in \mathbf{C}.$$

In fact, since $\alpha \in (E_6)^{\tau\sigma_3} \subset (E_6)_{E_1} = Spin(10)$ (Lemma 5.2), we can put $\alpha(i(E_2+E_3)) = \xi E_2 - \tau \xi E_3 + F_1(s+t)$, $\xi \in \mathbf{C}$, $s+t \in \mathbf{C} \oplus \mathbf{C}^\perp = \mathfrak{G}$. From $\tau \sigma_3 \alpha = \alpha \tau \sigma_3$, we have $\xi \in i\mathbf{R}$, $t=0$. And $2(\eta^2+|s|^2) = \langle \alpha(i(E_2+E_3)), \alpha(i(E_2+E_3)) \rangle = \langle i(E_2+E_3), i(E_2+E_3) \rangle = 2$. Now put

$$P = \frac{1}{\sqrt{2(1-\eta)}} \begin{pmatrix} \bar{s} & 1-\eta \\ -1+\eta & s \end{pmatrix} \quad (\text{if } \eta=1, \text{ put } P=E).$$

Then $P \in SU(2)$ and $\phi(P)(i\eta(E_2+E_3)+F_1(s))=i(E_2+E_3)$. Hence $\phi(P)\alpha(i(E_2+E_3))=i(E_2+E_3)$, that is, $\delta = \phi(P)\alpha \in (F_4)_{E_1} = Spin(9)$, moreover $((Spin(9))^{\tau\sigma_3} = (Spin(9))^{\sigma_3} \subset (Spin(10))^{\sigma_3}$. Hence we can put

$$\delta F_1(1) = F_1(s_0), \quad \delta F_1(e) = F_1(es_0), \quad s_0 \in \mathbf{C}, |s_0|=1$$

(Lemma 3.4). Choose $a \in \mathbf{C}$ such that $a^2 = \bar{s}_0$ and put $\beta = D_a \delta$, then $\beta(i(E_2+E_3)) = i(E_2+E_3)$, $\beta F_1(1) = F_1(1)$, $\beta F_1(e) = F_1(e)$, that is, $\beta = D_a \phi(P)\alpha \in Spin(7)$. Therefore we have a presentation such that

$$\alpha = \phi(P^{-1}A)\beta, \quad P^{-1}A \in SU(2) \quad (\text{where } A = \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix}), \quad \beta \in Spin(7).$$

Thus ϕ is onto. $\text{Ker } \phi = \mathbf{Z}_2 = \{(E, 1), (-E, \sigma)\}$. In fact, let $\phi(A)\beta=1$, $A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$, $\beta \in Spin(7)$. Then $E = \phi(A)\beta E = \phi(A)E = E_1 + (|a|^2 - |b|^2)(E_2 + E_3)$.

Hence $|a|^2 - |b|^2 = 1$. Together with $|a|^2 + |b|^2 = 1$, we have $b=0$, so $A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$.

From $F_1(1) = \phi(A)\beta F_1(1) = \phi(A)F_1(1) = F_1(a^2)$, we have $a^2=1$, hence $a = \pm 1$, so $A = \pm E$. Then $\beta = \phi(E) = 1$ or $\beta = \phi(-E) = \sigma$. Thus we have the required isomorphism.

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