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ANOTHER PROOF OF THE STRONG COMPLETENESS OF THE INTUITIONISTIC FUZZY LOGIC

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Takeuti and Titani [3] introduced the system, which we shall call TT, for the intuitionistic fuzzy logic, and proved the following theorem:

STRONG COMPLETENESS THEOREM (Takeuti and Titani [**3**, Theorem 1.3]). *Suppose that the language of* TT *is countable. If a sequent* $\Sigma\Rightarrow\Delta$ *is valid then it is provable in* TT, *where* Σ *may be infinite.*

The purpose of this note is to give another proof of the above theorem.

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§**1.** Recall, first, that the axioms and inference rules of TT are those of the intuitionistic logic (Gentzen's LJ) together with the following ones:

EXTRA AXIOM SCHEMATA FOR TT.

1. $\Rightarrow(A\rightarrow B)\vee((A\rightarrow B)\rightarrow B)$;
2. $(A\rightarrow B)\rightarrow B\Rightarrow(B\rightarrow A)\vee B$;
3. $(A\wedge B)\rightarrow C\Rightarrow(A\rightarrow C)\vee(B\rightarrow C)$;
4. $A\rightarrow(B\vee C)\Rightarrow(A\rightarrow B)\vee(A\rightarrow C)$;
5. $\forall x(C\vee A(x))\Rightarrow C\vee\forall xA(x)$, where x does not occur in C;
6. $\forall xA(x)\rightarrow C\Rightarrow\exists x(A(x)\rightarrow D)\vee(D\rightarrow C)$, where x does not occur in D.

EXTRA INFERENCE RULE FOR TT.

$$\frac{\Gamma\Rightarrow A\vee(C\rightarrow p)\vee(p\rightarrow B)}{\Gamma\Rightarrow A\vee(C\rightarrow B)},$$

where p is any propositional variable not occurring in the lower sequent.

We call that system TT$^-$ which is obtained from TT by deleting Extra Inference Rule for TT.

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Fifteen years before, Horn had introduced another system for the logic, which we shall call H, and had shown the weak completeness (Horn [1, Theorem 3.8]): *A formula is valid iff it is provable in* H. The system H has also been characterized by means of Kripke models in Ono [2, Theorem 3.3]. Recall that the axioms and inference rules of H are those of LJ together with the following axioms:

EA 1. $\forall x(C \vee A(x)) \Rightarrow C \vee \forall x A(x)$, where x does not occur in C;
EA 2. $\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)$.

Then we claim the following theorem:

THEOREM. *Suppose that the language concerned is countable. The following properties* (a)-(d) *of a sequent* $\Sigma \Rightarrow \Delta$ *are equivalent, where* Σ *may be infinite*:

(a) $\Sigma \Rightarrow \Delta$ *is valid.*
(b) $\Sigma \Rightarrow \Delta$ *is provable in* H.
(c) $\Sigma \Rightarrow \Delta$ *is provable in* TT^-.
(d) $\Sigma \Rightarrow \Delta$ *is provable in* TT.

The proof of (a)⇒(b) is postponed until § 2. Since **EA 1** is identical with Extra Axiom Schema 5 for TT and **EA 2** follows from Extra Axiom Schemata 1 and 2 by the intuitionistic logic, (b) implies (c). Clearly (c) implies (d), while the proof of (d)⇒(a) is routine. Thus, the proofs of (a)⇒(b)⇒(c)⇒(d) form another proof of Strong Completeness Theorem stated in the introduction.

The author confesses that he does not know any syntactical proof of (d)⇒(c).

§ 2. We shall prove that (a) implies (b). In this section, provability means that in H.

To show the contraposition we assume that $\Sigma \Rightarrow \Delta$ is unprovable, and we shall construct a model $\langle \mathcal{A}, [\![\ ]\!] \rangle$ in which $\Sigma \Rightarrow \Delta$ is not valid. As in [3] we further assume, for simplicity, that there exist infinitely many individual free variables which do not occur in $\Sigma \Rightarrow \Delta$, and that Δ consists of one formula A.

Let $\mathcal{A}$ and $\mathcal{F}$ be the sets of all terms and all formulas, respectively.

PROPOSITION 1. *There exists a set* $\mathcal{Q}$ *of formulas which satisfies the following conditions* (1)-(3):

(1) $\Sigma \subseteq \mathcal{Q}$ *and* $A \notin \mathcal{Q}$.
(2) *If* $\vdash \mathcal{Q} \Rightarrow B_1 \vee \cdots \vee B_\nu$, *then* $B_i \in \mathcal{Q}$ *for some* i.
(3) *If* $B(t) \in \mathcal{Q}$ *for every* t *in* $\mathcal{A}$, *then* $\forall x B(x) \in \mathcal{Q}$.

PROOF. Let $\mathcal{F}=\{F_n \mid n=1, 2, \cdots\}$. We define a pair $\mathcal{G}_n$, $\mathcal{H}_n$ of subsets of $\mathcal{F}$ inductively as follows.

Let $\mathcal{G}_1=\Sigma$ and $\mathcal{H}_1=\Delta=\{A\}$.

Assume that $\mathcal{G}_n$ and $\mathcal{H}_n$ have been defined already. CASE 1: $\vdash \mathcal{G}_n \Rightarrow \bigvee \mathcal{H}_n \vee F_n$. Set $\mathcal{G}_{n+1}=\mathcal{G}_n \cup \{F_n\}$ and $\mathcal{H}_{n+1}=\mathcal{H}_n$. CASE 2: Otherwise. Set $\mathcal{G}_{n+1}=\mathcal{G}_n$, and $\mathcal{H}_{n+1}=\mathcal{H}_n \cup \{F_n, B(a)\}$ or $\mathcal{H}_{n+1}=\mathcal{H}_n \cup \{F_n\}$ according as F_n has the form $\forall x B(x)$ or not, where a is any individual free variable which does not occur in $\mathcal{G}_n \cup \mathcal{H}_n \cup \{F_n\}$.

In view of **EA 1** we see $\nvdash \mathcal{G}_n \Rightarrow \bigvee \mathcal{H}_n$ by induction on n, and $\bigcup_{n=1}^{\infty} \mathcal{H}_n = \mathcal{F} - \bigcup_{n=1}^{\infty} \mathcal{G}_n$. So $\mathcal{G}=\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is the required set. □

We assume hereafter that a set $\mathcal{G}$ of formulas satisfying (1)-(3) is given.

Now we define the relation $\leqq^{\circ}$ and $\equiv$ on $\mathcal{F}$ by

$$B \leqq^{\circ} C \Leftrightarrow B \to C \in \mathcal{G} \quad \text{and} \quad B \equiv C \Leftrightarrow B \leqq^{\circ} C \ \& \ C \leqq^{\circ} B.$$

Then $\leqq^{\circ}$ is reflexive and transitive; since for every B, C and D, $\vdash \Rightarrow B \to B$ and $\vdash B \to C, C \to D \Rightarrow B \to D$, so $B \to B \in \mathcal{G}$ and if $B \to C \in \mathcal{G}$ and $C \to D \in \mathcal{G}$ then $B \to D \in \mathcal{G}$. Hence $\equiv$ is an equivalence relation on $\mathcal{F}$. For every B in $\mathcal{F}$ we let $|B|$ be the equivalence class under $\equiv$ to which B belongs, and $\mathcal{F}/\equiv$ the set of all equivalence classes. Next we define the relation $\leqq$ on $\mathcal{F}/\equiv$ by

$$|B| \leqq |C| \Leftrightarrow B \leqq^{\circ} C \Leftrightarrow B \to C \in \mathcal{G}.$$

This is an unambiguous definition, and $\langle \mathcal{F}/\equiv, \leqq \rangle$ forms an ordered structure.

PROPOSITION 2. *$\langle \mathcal{F}/\equiv, \leqq \rangle$ is a countable linearly ordered structure with the distinct maximal element $|A \to A|$ and the minimal element $|\neg(A \to A)|$.*

PROOF. Since $\mathcal{F}$ is countably infinite, $\mathcal{F}/\equiv$ is countable. For every B and C, $\vdash \Rightarrow (B \to C) \vee (C \to B)$ by **EA 2**, and so either $B \to C \in \mathcal{G}$ or $C \to B \in \mathcal{G}$; hence $\leqq$ is linear. For every B, $\vdash \Rightarrow B \to (A \to A)$ and $\vdash \Rightarrow \neg(A \to A) \to B$, and so $B \to (A \to A) \in \mathcal{G}$ and $\neg(A \to A) \to B \in \mathcal{G}$; hence $|A \to A|$ and $|\neg(A \to A)|$ are the maximal and the minimal elements, respectively. Since $\vdash (A \to A) \to \neg(A \to A) \Rightarrow A$ and since $A \notin \mathcal{G}$, $(A \to A) \to \neg(A \to A) \notin \mathcal{G}$; so $|A \to A| \neq |\neg(A \to A)|$. □

We abbreviate $|A \to A|$ and $|\neg(A \to A)|$ by **1** and **0**, respectively.

PROPOSITION 3. *The following properties hold in $\langle \mathcal{F}/\equiv, \leqq \rangle$:*

1°) $|B \wedge C| = \min(|B|, |C|)$.

2°) $|B \vee C| = \max(|B|, |C|)$.

3°) $|B \to C| = \mathbf{1}$ *if* $|B| \leqq |C|$; $|B \to C| = |C|$ *otherwise.*

4°) $|\neg B|=\mathbf{1}$ *if* $|B|=\mathbf{0}$; $|\neg B|=\mathbf{0}$ *otherwise.*

5°) $|\exists x B(x)|=\sup\{|B(t)| \mid t\in \mathcal{A}\}$.

6°) $|\forall x B(x)|=\inf\{|B(t)| \mid t\in \mathcal{A}\}$.

7°) $|B|=\mathbf{1}\Leftrightarrow B\in \mathcal{G}$.

PROOF. 1°) From $\vdash\Rightarrow B\wedge C\rightarrow B$, $\vdash\Rightarrow B\wedge C\rightarrow C$ and $\vdash D\rightarrow B, D\rightarrow C\Rightarrow D\rightarrow B\wedge C$ for every D, it follows $|B\wedge C|=\inf(|B|, |C|)$, from which 1°) follows since $\leqq$ is linear.

2°) is proved similarly to 1°).

3°) From $\vdash\Rightarrow(B\rightarrow C)\wedge B\rightarrow C$ and $\vdash D\wedge B\rightarrow C\Rightarrow D\rightarrow(B\rightarrow C)$ for every D, it follows $|B\rightarrow C|=\max\{|D| \mid |D\wedge B|\leqq|C|\}$. Hence in view of 1°), follows 3°) since $\leqq$ is linear.

4°) From $\vdash\Rightarrow\neg B\wedge B\rightarrow\neg(A\rightarrow A)$ and $\vdash D\wedge B\rightarrow\neg(A\rightarrow A)\Rightarrow D\rightarrow\neg B$ for every D, it follows $|\neg B|=\max\{|D| \mid |D\wedge B|=\mathbf{0}\}$. Hence in view of 1°), follows 4°) since $\leqq$ is linear.

5°) Since $\vdash\Rightarrow B(t)\rightarrow\exists x B(x)$, $|B(t)|\leqq|\exists x B(x)|$ for every t in $\mathcal{A}$. On the other hand, for every D,

$$
\begin{aligned}
& |B(t)|\leqq|D| \quad \text{for every } t \text{ in } \mathcal{A} \\
\Leftrightarrow\ & B(t)\rightarrow D\in\mathcal{G} \quad \text{for every } t \text{ in } \mathcal{A} \\
\Rightarrow\ & \forall x(B(x)\rightarrow D)\in\mathcal{G} \qquad\qquad \text{since (3)} \\
\Rightarrow\ & \exists x B(x)\rightarrow D\in\mathcal{G} \quad \text{since } \vdash\forall x(B(x)\rightarrow D)\Rightarrow\exists x B(x)\rightarrow D \\
\Leftrightarrow\ & |\exists x B(x)|\leqq|D|.
\end{aligned}
$$

Hence 5°) follows.

6°) is proved similarly to 5°).

7°) Since $\vdash(A\rightarrow A)\rightarrow B\Rightarrow B$ and $\vdash B\Rightarrow(A\rightarrow A)\rightarrow B$,

$$|B|=\mathbf{1}\Leftrightarrow|A\rightarrow A|\leqq|B|\Leftrightarrow(A\rightarrow A)\rightarrow B\in\mathcal{G}\Leftrightarrow B\in\mathcal{G}. \qquad \square$$

PROPOSITION 4 (Horn [**1**, Lemma 3.7]). *If* $\langle L, \leqq\rangle$ *is a countable linearly ordered structure with the distinct maximal and minimal elements, then there exists a monomorphism on* $\langle L, \leqq\rangle$ *to* $\langle[0, 1]\cap\boldsymbol{Q}, \leqq\rangle$ *which preserves the maximal and the minimal elements as well as all existing supremums and infimums in* $\langle L, \leqq\rangle$. *Hence there exists such a monomorphism on* $\langle L, \leqq\rangle$ *to* $\langle[0, 1], \leqq\rangle$. □

By Propositions 2 and 4, there exists a monomorphism h on $\langle\mathcal{F}/\equiv, \leqq\rangle$ into $\langle[0, 1], \leqq\rangle$ which preserves the maximal and the minimal elements as well as all existing supremums and infimums in $\langle\mathcal{F}/\equiv, \leqq\rangle$. Put $[\![B]\!]=h(|B|)$ for every B in $\mathcal{F}$, and we obtain a model $\langle\mathcal{A}, [\![\ ]\!]\rangle$ by Proposition 3. Note that for every B,

$$[\![B]\!]=1 \Leftrightarrow |B|=1 \Leftrightarrow B \in \mathcal{G} .$$

In this model,

$$B \in \Sigma \Rightarrow B \in \mathcal{G} \Leftrightarrow [\![B]\!]=1 ,$$

while $A \notin \mathcal{G}$ so $[\![A]\!] \neq 1$; so $\Sigma \Rightarrow \Delta$ is not valid.

Thus we have found, on the assumption that $\Sigma \Rightarrow \Delta$ is unprovable, a model $\langle \mathcal{A}, [\![\ ]\!] \rangle$ in which it is not valid. Q. E. D.

References

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