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DIRECT SUM OF τ-INJECTIVE MODULES

By

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Throughout this paper R is a ring with identity and every R-module is unital. Let τ be a hereditary torsion theory with respect to R (see Golan [3]). A submodule N of a right R-module M is said to be τ-dense in M, if M/N is a τ-torsion module. We shall denote the set of all τ-dense right ideals of R by $\mathcal{L}_\tau$. A right R-module M is called τ-injective, if for every right R-module L and its τ-dense submodule N every R-homomorphism $N\rightarrow M$ is extended to $L\rightarrow M$. Let us denote by $E(L)$ the injective hull of the right R-module L. Then, $E_\tau(L)=\{x\in E(L) \mid \text{there exists } I\in\mathcal{L}_\tau \text{ such that } xI\subset L\}$ is said to be τ-injective hull of L. If N is a submodule of L, $E_\tau(N)$ is contained in $E_\tau(L)$.

By a result of Matlis [5] and Papp, R is right Noetherian, if and only if every injective right R-module is a direct sum of (injective) indecomposable submodules. It is to be noted that this result was generalized to injective τ-torsion free right R-modules by Teply [7]. Let τ_G be the Goldie torsion theory with respect to R. Clearly, injective indecomposable right R-modules coincide with those modules each of which is a τ_G-injective hull of its every non-zero submodule. Furthermore, if R is right Noetherian, the ring of quotient of R with respect to τ_G is semi-simple Artinian (cf. Kutami and Oshiro [4]) and hence τ_G is a perfect torsion theory (see [3], [6]). Now, concerning the above result of Matlis and Papp we shall study in this paper a right R-module M such that $M=\bigoplus_{i\in I} M_i$, where each M_i is a τ-injective hull of its every non-zero submodule. In the following such a module M will be said to be τ-completely decomposable.

Now, at first we shall prove the next

THEOREM 1. *Let τ be a hereditary torsion theory with respect to R. Then, $\mathcal{L}_\tau$ satisfies the ascending chain condition, if and only if, every τ-injective τ-torsion R-module is τ-completely decomposable.*

A ring R is called right semi-Artinian, if every non-zero right R-module has a non-zero socle. Then, we shall prove the following

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THEOREM 2. *The following are equivalent, if τ is a perfect torsion theory.*

(i) *$\mathcal{L}_\tau$ satisfies the ascending chain condition and the ring of quotient R_τ of R with respect to τ is right semi-Artinian.*

(ii) *Every τ-injective right R-module is an essential extension of a τ-injective τ-completely decomposable R-module.*

Let τ_G be a Goldie torsion theory. Then every τ_G-torsion free direct summand of a τ_G-completely decomposable module is quasi-injective (see [4]). In Theorem 3, we shall see that this result remains true, even if τ is an arbitrary hereditary torsion theory.

LEMMA 1. *Let M be a right R-module such that M is a τ-injective hull of its every non-zero submodule. Then* $\mathrm{End}(M_R)$ *is a local ring.*

PROOF. Let $f \in \mathrm{End}(M_R)$ be a non-zero element. Since M is uniform and $\mathrm{Ker}\, f \cap \mathrm{Ker}(1-f)=0$, f or $1-f$ is a monomorphism and hence a unit.

LEMMA 2. *Let M be a submodule of a right R-module K such that $M=\bigoplus_{a\in A} M_a$ and $K=\bigoplus_{b\in B} K_b$, which are τ-complete decompositions. If the cardinal number $|A|$ is at most countable, there exists a subset C of B such that $M\cong \bigoplus_{c\in C} K_c$.*

PROOF. Let $a\in A$. Put $A_a=\{d\in A \mid M_d\cong M_a\}$ and $B_a=\{b\in B \mid K_b\cong M_a\}$. It sufficies to show $|A_a|\leqq|B_a|$. Let $d_1, d_2, \cdots, d_n$ be distinct elements of A_a. Assume x_i be a non-zero element of M_{d_i}, $i=1, 2, \cdots, n$. Then, $\bigoplus_{i=1}^{n} M_{d_i}=E_\tau(\bigoplus_{i=1}^{n} x_i R)$, which is contained in a direct sum of finite number of K_b, $b\in B$. Let $b_1, b_2, \cdots, b_t$ be elements of B such that there exists a monomorphism $f: \bigoplus_{i=1}^{n} M_{d_i} \to \bigoplus_{j=J}^{t} K_{b_j}$ and for every p $(p=1, 2, \cdots, t)$ there is no monomorphism from $\bigoplus_{i=1}^{n} M_{d_i}$ to $K_b \oplus K_{b_2}\oplus \cdots \oplus K_{b_{p-1}}\oplus K_{b_{p+1}}\oplus \cdots \oplus K_{b_t}$. Let π_{b_p} be the projection $K\to K_{b_p}$. Then, for every p $(p=1, \cdots, t)$ we can pick up $y_p\in \bigoplus_{i=1}^{n} M_{d_i}$ so that $\pi_{b_p} f(y_p)\neq 0$ and $\pi_{b_s} f(y_p)=0$, $s\neq p$. Put $N=y_1R\oplus \cdots \oplus y_tR$. Since $f(N)=f(y_1)R\oplus \cdots \oplus f(y_t)R\subset K_{b_1}\oplus\cdots\oplus K_{b_t}$, we have $E_\tau(\bigoplus_{i=1}^{t} f(y_i)R)=K_{b_1}\oplus \cdots \oplus K_{b_t}$. Therefore, $\bigoplus_{i=1}^{n} M_{d_i}=\bigoplus_{j=1}^{t} K_{b_j}$. Then, by Lemma 1 and a theorem of Azumaya [1] every K_{b_j} is isomorphic to M_a. Since $t=n$, the consequence is immediate when $|A|$ is finite. Assume A_a is infinite. Then, the above argument implies B_a is an infinite set. Hence $|A_a|\leqq|B_a|$.

PROOF OF THEOREM 1.

Assume $\mathcal{L}_\tau$ satisfies the ascending chain condition. Let M be a τ-injective

τ-torsion R-module. If $0\neq x\in M$, then $xR\cong R/I$ for some $I\in\mathcal{L}_\tau$. Therefore, R/I is a Noetherian right R-module and hence contains a uniform submodule U. Since $E_\tau(U)$ is contained in M, M have a τ-injective submodule which is a τ-injective hull of its every non-zero submodule. Let $\{M_j\}_{j\in J}$ be a maximal independent set of submodules of M such that each M_j is a τ-injective hull of its every non-zero submodule, where J is an index set. Let $y\in M$ be a non-zero element. As is shown above $E_\tau(yR)$ contains a submodule which is a τ-injective hull of its every non-zero submodule and hence $\bigoplus_{j\in J} M_j\cap yR\neq 0$. It follows $\bigoplus_{j\in J} M_j$ is an essential submodule of M. On the other hand, since $\mathcal{L}_\tau$ satisfies the ascending chain condition, by [3, p. 128, Proposition 14.2] $\bigoplus_{j\in J} M_j$ is τ-injective. Since M is a τ-injective τ-torsion module, $\bigoplus_{j\in J} M_j$ is a direct summand of M. Thus we have $M=\bigoplus_{j\in J} M_j$.

Conversely, assume every τ-injective τ-torsion R-module is τ-completely decomposable. Let $\{P_i\}_{i\in I}$ be a class of (non-isomorphic) representatives of all τ-injective τ-torsion uniform R-modules. Then, each P_i is the τ-injective hull of its every non-zero submodule. Let us denote $P_i^{(N)}$ the direct sum of countably copies of P_i. Since $E_\tau(\bigoplus_{i\in I} P_i^{(N)})$ is a τ-torsion module, it has a τ-complete decomposition such that $E_\tau(\bigoplus_{i\in I} P_i^{(N)})=\bigoplus_{j\in J} Q_j$, where J is an index set. From Lemma 2 $\bigoplus_{i\in I} P_i^{(N)}$ is isomorphic to a direct summand of $\bigoplus_{j\in J} Q_j$. Hence we have $(\bigoplus_{i\in I} P_i)^{(N)}$ is τ-injective. Let $K_1\subset K_2\subset K_3\subset\cdots$ be a strictly ascending chain of right ideals in $\mathcal{L}_\tau$. Then, R/K_j $(j=1, 2, \cdots)$ is a submodule of a τ-torsion τ-completely decompossable R-module $E_\tau(R/K_j)$. It follows K_j is an annihilator right ideal of a subset of $\bigoplus_{i\in I} P_i$. Therefore, we can choose $a_1, a_2, a_3, \cdots\in\bigoplus_{i\in I} P_i$ such that $a_jK_j=0$ and $a_jK_{j+1}\neq 0$ $(j=1, 2, 3, \cdots)$. Put $K=\bigcup_{j=1} K_j$. Clearly, the map $f: K\to(\bigoplus_{i\in I} P_i)^{(N)}$ by $f(x)=(a_1x, a_2x, \cdots)$, $x\in K$, is an R-homomorphism. Since $K\in\mathcal{L}_\tau$, f is extended to $R\to(\bigoplus_{i\in I} P_i)^{(N)}$. However this is a contradiction, since for every integer $n>0$ there exists $x\in K$ such that $a_nx\neq 0$. This complete the proof.

In the following let us denote $T_\tau(M)$ the τ-torsion submodule of M.

PROOF OF THEOREM 2.

(i) $\Rightarrow$ (ii). Let M be a τ-injective R-module. Since $E_\tau(T_\tau(M))$ is contained in M, it is equal to $T_\tau(M)$. Then, $T_\tau(M)$ is τ-completely decomposable by Theorem 1. We may assume $T_\tau(M)$ is not an essential submodule of M. Let N be a closed submodule of M such that $N\cap T_\tau(M)=0$ and $N\oplus T_\tau(M)$ is essential in M. Since N has no essential extension in M, N is τ-injective. Then, N becomes a right R_τ-module, which has an essential socle $S=\bigoplus_{h\in H} S_h$, where S_h is a simple right R_τ-module. Let L be a non-zero R-submodule of S_h. Then, $S_h=LR_\tau$ and hence L is a τ-dense submodule of S_h. On the other hand, since τ is a perfect torsion theory, S is τ-injective τ-completely decomposable and so is $S\oplus T_\tau(M)$.

(ii) $\Rightarrow$ (i). Let N be a τ-injective τ-torsion R-module. Since every τ-injective submodule of N is a direct summand, N is τ-completely decomposable and $\mathcal{L}_\tau$ satisfies the ascending chain condition. Let K be a right R_τ-module. Since K is a τ-injective R-module it contains an essential τ-completely decomposable R-submodule. In view of [3, p. 186 Corollary] we have that this submodule is a socle of the right R_τ-module K. Hence R_τ is right semi-Artinian.

REMARK. Assume τ_G is the Goldie torsion theory. Put $\mathcal{C}_{\tau_G}=\{$right ideal I of $R\,|\,R/I$ is τ_G-torsion free$\}$. If $\mathcal{C}_{\tau_G}$ and $\mathcal{L}_{\tau_G}$ satisfy the ascending chain condition, then every injective right R-module is a direct sum of indecomposable submodules in view of Theorem 1 and [7, Theorem 1.2] and hence R is right Noetherian. When R is right non-singular, this is a case of Yamagata [9, Theorem 9].

LEMMA 3. *Let M be a τ-torsion free right R-module. Assume $M=\sum_{i\in I}M_i$, where M_i is a τ-injective hull of its every non-zero submodule. Then, there exists a subset J of I such that $M=\bigoplus_{j\in J} M_j$.*

PROOF. Let $\{M_j\}_{j\in J}$ be a maximal independent subset of the class $\{M_i\}_{i\in I}$. For every $i\in I$ $M_i\cap\bigoplus_{j\in J} M_j$ contains a non-zero element x, say. Therefore, there exists a finite subset $\{j_1, \cdots, j_n\}$ of J so that $E_\tau(xR)$ is contained in $\bigoplus_{k=1}^{n} M_{j_k}$. Let $0\neq y\in E_\tau(xR)+M_i$. Put $y=y_1+y_2$, $y_1\in E_\tau(xR)$, $y_2\in M_i$. Then, there exist $L_1, L_2\in \mathcal{L}_\tau$ such that $y_1L_1\subset xR$ and $y_2L_2\subset xR$. So $0\neq y(L_1\cap L_2)\subset xR$. This implies xR is (an essential) τ-dense submodule of $E_\tau(xR)+M_i$. Since M_i is a τ-injective hull of xR, too, we have $E_\tau(E_\tau(xR)+M_i)=E_\tau(xR)=M_i$. Hence M_i is a submodule of $\bigoplus_{j\in J} M_j$.

THEOREM 3. *Let $M=\bigoplus_{i\in I} M_i$ be a τ-complete decomposition. If N is a τ-torsion free direct summand of M, then N is quasi-injective.*

PROOF. Let $0\neq x\in N$. $E_\tau(xR)$ is contained in a sum of finite number of M_i, $i\in I$. Let $\pi: M\to N$ be the projection. Then, the restriction $\pi\,|\,E_\tau(xR)$ is a monomorphism. This implies we may assume $E_\tau(xR)$ is contained in N. Now, $E_\tau(xR)$ is isomorphic to a direct sum of finite number of M_i, $i\in I$, by the same method as in the proof of Lemma 2. Therefore, $N=\sum_{x\in N}E_\tau(xR)$ is τ-completely decomposable by Lemma 3. Let $N=\bigoplus_{h\in H} N_h$ be the τ-complete decomposition and $E(N)$ the injective hull of N. To see that N is quasi-injective, it sufficies to show $f(N)\subset N$ for every $f\in \mathrm{End}(E(N)_R)$. Let f_n be the restriction $f|N_h$. Suppose $f_h\neq 0$. Then f_h is monic (cf. [3, Proposition 18.2]) and hence $\mathrm{Im} f_h$ is a τ-injective hull of its every non-zero submodule. Since $\bigoplus_{h\in H} N_h$ is an essential submodule of a τ-torsion free module $E(N)$, it is easy to check that $\mathrm{Im} f_h \subset \bigoplus_{h\in H} N_h$ from the proof of Lemma 3. This proves the Theorem.

REMARK. In [2] it is proved that if M is an injective right R-module which is a direct sum of indecomposable modules, then so is its direct summand. Now, let M be a right R-module such that $M=\bigoplus_{i\in I} M_i$, where every proper factor module of each M_i is τ-torsion. Assume N is a τ-injective direct summand of M. Then by the same method as in [8, Lemma 2], it is not hard to see that there is a submodule N' of M such that $M=N\oplus N'$ and $N'=\bigoplus_{i\in I} M_i'$, where $M_i'\subset M_i$ $(i\in I)$. Especially, when $\bigoplus_{i\in I} M_i$ is a τ-complete decomposition, there exists a subset J of I such that $N'=\bigoplus_{j\in J} M_j$. Hence N has a τ-complete decomposition, too.

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