COMPACT 2-TRANSNORMAL HYPERSURFACE IN A KAEHLER MANIFOLD OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

By

Fumiko OHTSUKA

§ 0. Introduction

The theory of "transnormality" has arisen from attempts to generalize a concept of curves with constant width in a Euclidian plane to an analogue in a Riemannian manifold. S.A. Robertson [4, 5] has accomplished this in the case where the ambient space is Euclidian. Subsequently, J. Bolton [1] has extended this to the case of a hypersurface in a Riemannian manifold.

Let M be an n-dimensional connected complete Riemannian manifold isometrically imbedded into an $(n+1)$-dimensional connected complete Riemannian manifold $\bar{M}$. For each $x \in M$ there exists, up to parametrization, a unique geodesic τ_x of $\bar{M}$ which intersects M orthogonally at x. M is called a *transnormal hypersurface* of $\bar{M}$ if, for each pair $x, y \in M$, the relation $\tau_x \ni y$ implies that $\tau_x = \tau_y$. We difine an equivalent relation $\sim$ for points on a transnormal hypersurface M by writing $x \sim y$ to mean $y \in \tau_x$, with respect to which the quotient space $\hat{M} = M/\sim$ with the quotient topology is considered. M is called an *r-transnormal* hypersurface if the natural projection ϕ of M onto $\hat{M}$ is an r-fold covering map.

Recently, S. Nishikawa [2] has studied some global properties of transnormal hypersurfaces of a complete Riemannian manifold. He gave also in [3] differential geometric structures of a compact 2-transnormal hypersurface of a simply connected complete Riemannian manifold of constant curvature.

In this paper, we shall investigate 2-transnormal hypersurfaces in a Kaehler manifold of constant holomorphic sectional curvature and prove that these hypersurfaces are geodesic hyperspheres if principal curvatures are bounded from below (or above) by certain constants which depend only upon the diameters of the hypersurfaces and the holomorphic sectional curvatures of the ambient Kaehler manifolds. (Theorem 3.3 and Theorem 4.1)

The auther would like to express her hearty gratitude to Prof. Shun-ichi Tachibana for his kind advice and suggestion.

§1. Preliminaries

In this section, relevant concepts and formulas used throughout the paper are prepared in brief.

Let $\overline{M}$ be an $(n+1)$-dimensional connected complete Riemannian manifold. We denote by $\overline{M}_x$ the tangent space of $\overline{M}$ at $x \in \overline{M}$, and by $<,>$ the inner product on the tangent space. Let M be an r-transnormal hypersurface of $\overline{M}$ and assume that there exists a point $p \in M$ satisfying the condition $C(p) \cap M = \emptyset$, where $C(p)$ denotes the cut locus of p in $\overline{M}$. The function Λ_p on M is defined by

$$\Lambda_p(x) = d_{\overline{M}}(p,\ x)^2 \qquad x \in M,$$

where $d_{\overline{M}}$ denotes the distance function in $\overline{M}$. By definition, $\Lambda_p(x)$ means the square of the length of the unique minimizing geodesic segment $\gamma(p,\ x)$ of $\overline{M}$ from p to x. It is known in [2] that Λ_p is a Morse function and a point $x \in M$, $x \neq p$ is a critical point of Λ_p if and only if $\gamma(p,\ x)$ is perpendicular to M at x and then at p owing to the transnormality of M.

A geodesic segment γ is said to be *normal* if it is parametrized by arc length. Let $x \in M$, $x \neq p$ be a critical point of Λ_p and $\gamma = \gamma(p,\ x)$ the minimizing normal geodesic segment from p to x. The length of the geodesic segment γ is denoted by $L(\gamma)$. By the second variation formula of the arc length of γ, the Hessian H of Λ_p at x, which is a symmetric bilinear form on M_x, is given by

$$(1.1) \qquad H(X,\ Y) = 2L < S_{\gamma'(L)} X(L) + X'(L),\ Y(L) > X,\ Y \in M_x,$$

where $L = L(\gamma)$, and S denotes the second fundamental tensor of M and $X(t)$, $Y(t)$ are the Jacobi fields along γ with $X(0) = 0$, $X(L) = X$ and $Y(0) = 0$, $Y(L) = Y$ respectively.

If, in particular, M is compact and 2-transnormal and if $\overline{M}$ is a simply connected complete kaehler manifold of constant holomorphic sectional curvature, then for not only p but also any point $x \in M$ the condition $C(x) \cap M = \emptyset$ is satisfied and there exists exactly one point $\tilde{x} \in M$ such that $d_{\overline{M}}(x,\ \tilde{x})$ equals the diameter of M as a subset of $\overline{M}$. Note that $\gamma(x,\ \tilde{x})$ is perpendicular to M at both of its end points. We call $\tilde{x} \in M$ an *antipodal point* of $x \in M$ and the initial vector $\gamma'(0)$ of $\gamma(x,\ \tilde{x})$ an *inward unit normal vector* at x. Note that for each $x \in M$ the inward unit normalvector at x, say $N(x)$, is determined uniquely. Then $N = N(x)$ gives rise to a C^∞-vector field of $\overline{M}$ on M. Denote by J the complex structure of $\overline{M}$. Then JN forms C^∞-vector field on M. Let $(JN)^*$ denote the dual covariant vector field of JN and define $(1,1)$-tensor field $\tilde{J}$ on M by

$$\tilde{J}(X) = J(X - <X,\ JN> JN)$$

for any vector field X on M. Then $(\tilde{J}, JN, (JN)^*)$ forms an almost contact structure on M. We call $E=JN$ an almost contact structure vector of M.

Assume that at each $x \in M$, an almost contact structure vector $E(x)$ is a principal vector of M. Then M is called C-umbilic if the principal curvature of M at x with principal vector $E(x)$ is constant and if the other principal curvatures are all equal at any $x \in M$.

§2.   Jacobi fields of a Kaehler manifold of constant holomorphic sectional curvature

This section is devoted to determining the Jacobi fields of a Kaehler manifold of constant holomorphic sectional curvature k.

Let $\overline{M}$ be a simply connected complete Kaehler manifold of constant holomorphic sectional curvature k and $\dim_C \overline{M}=m$. We denote by J the complex structure of $\overline{M}$. Let γ be a normal geodesic on $\overline{M}$ and $\{e_i, e_{i*}\}$ $(i=1,\cdots,m)$ an orthonormal basis of $\overline{M}_{\gamma(0)}$ such that $\gamma'(0)=e_1$, $e_{i*}=Je_i$. By the parallel translation of this basis along γ, we obtain 2m parallel vector fields P_i and P_{i*} for $i \in \{1,\cdots,m\}$. Let $\psi(t)$ be a Jacobi field along γ, i.e. $\psi(t)$ satisfies the Jacobi equation;

$$\psi'' + R(\psi, \gamma')\gamma' = 0,$$

where γ' is the tangent vector field of γ, ψ' denotes the covariant derivative of ψ with respect to γ' and R is the Riemannian curvature tensor of $\overline{M}$. Then there exist 2m functions f_i and f_{i*} such that

$$\psi(t) = \sum_{i=1}^m f_i(t)P_i(t) + \sum_{i=1}^m f_{i*}(t)P_{i*}(t),$$

and these functions satisfy the following differential equations;

(2,1) $$f_i''(t) - \sum_{j=1}^m <R(P_1, P_j)P_1, P_i>(t)f_j(t)$$

$$- \sum_{j=1}^m <R(P_1, P_{j*})P_1, P_i>(t)f_{j*}(t)=0$$

for $i \in \{1,\cdots,m, 1^*,\cdots,m^*\}$.

Recall that the components of the sectional curvature of $\overline{M}$ are given as follows;

(2,2) $\begin{cases} \rho_{ii*}=k & (i=1,\cdots,m) \\ \rho_{ij}=\rho_{ij*}=k/4 & (i \neq j) \\ R_{ii*jj*}=-k/2 & (i \neq j) \\ \quad 0 & \text{otherwise} \end{cases}$

where $R_{ijkl}=<R(e_i, e_j)e_k, e_l>$ and $\rho_{ij}=-R_{ijij}$. Hence, substituting (2,2) into (2,1)

and solving these equations, we obtain the form of Jacobi field as follows;

1) in the case $k>0$;

$$\phi(t)=\Sigma f_i(t)P_i(t)$$

where
$$\begin{cases} f_1(t)=a_1t+b_1 \\ f_{1^*}(t)=a_{1^*}\cdot\sin\sqrt{k}t+b_{1^*}\cdot\cos\sqrt{k}t \\ f_i(t)=a_i\cdot\sin(\sqrt{k}t/2)+b_i\cdot\cos(\sqrt{k}t/2) \quad (i\neq1,1^*). \end{cases}$$

2) in the case $k<0$;

$$\phi(t)=\Sigma f_i(t)P_i(t)$$

where
$$\begin{cases} f_1(t)=a_1t+b_1 \\ f_{1^*}(t)=a_{1^*}\cdot\sinh\sqrt{-k}t+b_{1^*}\cdot\cosh\sqrt{-k}t \\ f_i(t)=a_i\cdot\sinh(\sqrt{-k}t/2)+b_i\cdot\cosh(\sqrt{-k}t/2) \quad (i\neq1,1^*). \end{cases}$$

In the above, a_i and b_i are arbitrary real numbers and i moves the range $\{1,\cdots,$ $m, 1^*,\cdots,m^*\}$.

§3. 2-transnormal hypersurfaces in $\overline{M}$ of positive constant holomorphic sectional curvature

First we investigate the case where the ambient space $\overline{M}$ has positive constant holomorphic sectional curvature k.

Let M be a connected complete 2-transnormal hypersurface in $\overline{M}$. Suppose that the diameter d of M as a subset of $\overline{M}$ satisfies $d<\pi/\sqrt{k}$. This implies that the cut locus $C(p)$ of $p\in M$ in $\overline{M}$ does not intersect M; namely $C(p)\cap M=\emptyset$. Throughout the rest of this paper, by principal curvatures of M we mean principal curvatures of M with respect to the inward unit normal vector.

LEMMA 3.1. *Every principal curvature of M is greater than $\sqrt{k}\cot(d\sqrt{k})$ at each point of M.*

PROOF. Fix a point $x\in M$ arbitrarily, and let $\tilde{x}$ be the antipodal point of x. Let γ be the minimizing normal geodesic from $\tilde{x}$ to x. Then the Hessian H of the function $\Lambda_{\tilde{x}}$ at x is given by

(3.1) $$H(X, Y)=2d<\{(\sqrt{k}/2)\cot(d\sqrt{k}/2)\cdot I-S_{N(x)}\}X, Y>$$
$$-d\sqrt{k}\tan(d\sqrt{k}/2)<E(x), X><E(x), Y>$$
$$\text{for } X, Y\in M_x,$$

where I denotes the identity transformation. This formula can be obtained from

(1.1) and the result of § 2. In fact, from the result of § 2, the Jacobi field $\psi(t)$ along γ satisfying the conditions $\psi(0)=0$ and $\psi(d)\in M_x$ can be expressed as

$$\psi(t)=a\cdot\sin(\sqrt{k}t)J\gamma'(t)+\sin(\sqrt{k}t/2)A(t),$$

where a is an arbitrary real number and $A(t)$ is a parallel vector field along γ satisfying the conditions $A(d)\in M_x$ and $<A(t), J\gamma'(t)>=0$. Substituting this formula into (1.1), we get the formula (3.1).

Since M is compact and 2-transnormal, $\Lambda_{\tilde{x}}$ takes its maximum at x, which is a non-degenerate critical point of $\Lambda_{\tilde{x}}$. Hence H is negative definite at x, i.e. $H(X, X)<0$ for all $X\in M_x\backslash\{0\}$. Let λ be an eigenvalue of $S_{N(x)}$ and X be a unit eigenvector corresponding to the eigenvalue λ. Then

$$H(X, X)=2d\{(\sqrt{k}/2)\cot(d\sqrt{k}/2)-\lambda\}$$
$$-d\sqrt{k}\tan(d\sqrt{k}/2)<E(x), X>^2<0,$$

from which we obtain $\lambda>\sqrt{k}\cot(d\sqrt{k})$. q.e.d.

Now, in the sequel we assume that the almost contact structure vector $E(x)$ is a principal vector having the principal curvature $\lambda(x)$ at each point x of M. Furthermore we denote by $\nu(x, X)$ the principal curvature of M at x associated with the principal vector X orthogonal to E. Then we can prove the following.

LEMMA 3.2. *At the antipodal point $\tilde{x}$ of x,*

(1)
$$\lambda(\tilde{x})=\frac{\sqrt{k}\sin(d\sqrt{k})+\lambda(x)\cos(d\sqrt{k})}{(\lambda(x)/\sqrt{k})\sin(d\sqrt{k})-\cos(d\sqrt{k})}$$

(2)
$$\nu(\tilde{x}, \tilde{X})=\frac{(\sqrt{k}/2)\sin(d\sqrt{k}/2+\nu(x, X)\cos(d\sqrt{k}/2)}{(2\nu(x, X)/\sqrt{k})\sin(d\sqrt{k}/2)-\cos(d\sqrt{k}/2)}$$

where $\tilde{X}$ is the tangent vector of M at $\tilde{x}$ given by parallel translation of X along $\gamma=\gamma(x, \tilde{x})$.

PROOF. Let $Y(t)$ be the vector field along γ given by

$$Y(t)=\{\cos(\sqrt{k}t)-(\lambda(x)/\sqrt{k})\sin(\sqrt{k}t)\}J\gamma'(t).$$

Then $Y(t)$ is an (M, x)-Jacobi field along γ, i.e. $Y(t)$ is the Jacobi field satisfying the boundary conditions;

$$Y(0)\in M_x \quad \text{and} \quad S_{\gamma'(0)}Y(0)+Y'(0)\in M_x^{\perp},$$

where $M_x^{\perp}$ denotes the orthogonal complement to M_x in $\overline{M}_x$. Since M is transnormal, every (M, x)-Jacobi field along γ is also an $(M, \tilde{x})$-Jacobi field along γ.

Thus the above $Y(t)$ must satisfy the following boundary condition;

$$S_{\gamma'(d)}Y(d)+Y'(d)\in M_x^\perp .$$

From this it follows that

$$S_{-\gamma'(d)}\{(\lambda(x)/\sqrt{k})\sin(d\sqrt{k})-\cos(d\sqrt{k})\}J\gamma'(d)$$

$$-\{\sqrt{k}\sin(d\sqrt{k})+\lambda(x)\cos(d\sqrt{k})\}J\gamma'(d)$$

$$\in M_{\hat{x}}\cap M_{\frac{\perp}{x}}=\{0\},$$

which together with Lemma 3.1 implies the first assertion.

Next we take the (M, x)-Jacobi field $Z(t)$ along γ such that

$$Z(t)=\{\cos(\sqrt{k}t/2)-(2\nu(x, X)/\sqrt{k})\sin(\sqrt{k}t/2)\}X(t),$$

where $X(t)$ is a parallel vector field along γ with $X(0)=X$. Then we can obtain the result by the same method as that of the proof of (1). Note here, however, that $\nu>(\sqrt{k}/2)\cot(d\sqrt{k}/2)$, i.e. $(2\nu/\sqrt{k})\sin(d\sqrt{k}/2)-\cos(d\sqrt{k}/2)>0$, which is obtained by the proof of Lemma 3.1 and the fact that $<E(x), X>=0$.   q.e.d.

With these lemmas prepared, we shall prove Theorem 3.3.

THEOREM 3.3. *Let M be a connected complete 2-transnormal hypersurface in a simply connected complete Kaehler manifold $\overline{M}$ of positive constant holomorphic sectional curvature k, and $d=d(M)$ be the diameter of M as a subset of $\overline{M}$. For each $x\in M$ the almost contact structure vector $E(x)$ is assumed to be a principal vector with principal curvature $\lambda(x)$. If $d<\pi/\sqrt{k}$ and if $\lambda(x)$ and the other principal curvatures $\nu(x)$ satisfy the following conditions at each $x\in M$;*

$$\lambda(x)\geqq\sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k})$$

$$(\text{or }\lambda(x)\leqq\sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k}))$$

and

$$\nu(x)\geqq\sqrt{k}(1+\cos(d\sqrt{k}/2))/2\sin(d\sqrt{k}/2)$$

$$(\text{or }\nu(x)\leqq\sqrt{k}(1+\cos(d\sqrt{k}/2))/2\sin(d\sqrt{k}/2)),$$

then M is C-umbilic and furthermore M is a geodesic hypershere with radius $d/2$.

PROOF. We consider only the following case;

$$\lambda(x)\geqq\sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k})$$

and

$$\nu(x)\geqq\sqrt{k}(1+\cos(d\sqrt{k}/2))/2\sin(d\sqrt{k}/2) .$$

The proofs for three other cases are accomplished in a similar way.

By Lemma 3.2, we get

$$\lambda(\bar{x}) = \frac{\sqrt{k}\sin(d\sqrt{k}) + \lambda(x)\cos(d\sqrt{k})}{(\lambda(x)/\sqrt{k})\sin(d\sqrt{k}) - \cos(d\sqrt{k})}$$

$$\geq \sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k}),$$

from which we obtain

$$\lambda(x) \leq \sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k}).$$

Hence the first inequality is reduced to

$$\lambda \equiv \sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k})$$

In a similar way, the second inequality is reduced to

$$\nu \equiv \sqrt{k}(1+\cos(d\sqrt{k}/2))/2\sin(d\sqrt{k}/2).$$

Hence M is C-umbilic.

Now, (M, x)-Jacobi fields along γ, which is the minimizing normal geodesic from x to $\bar{x}$,

$$Y(t) = (\cos(\sqrt{k}t) - (\lambda/\sqrt{k})\sin(\sqrt{k}t))J\gamma'(t)$$

and

$$Z(t) = (\cos(\sqrt{k}t/2) - (2\nu/\sqrt{k})\sin(\sqrt{k}t/2))X(t)$$

in the proof of Lemma 3.2 attain zero when $t=d/2$. Hence $\gamma(d/2)$ is a focal point with multiplicity 2m-1. Then the focal locus of M consists of a single point α, since for a sufficiently small neighborhood U a focal point with a base point $p \in U$ coincides one another. Hence M consists of the point x such that $d_{\bar{M}}(x, \alpha) = d/2$. namely M is a geodesic hypersphere with radius $d/2$.   q. e. d.

REMARK. By the same method as above, we can get the following inequalities;

$$\sqrt{k}\cot(d\sqrt{k}) \leq \inf_{x \in M}\lambda(x)$$

$$\leq \sqrt{k}(1+\cos(d\sqrt{k}))/\sin(d\sqrt{k})$$

$$\leq \sup_{x \in M}\lambda(x)$$

$$(\sqrt{k}/2)\cot(d\sqrt{k}/2) \leq \inf_{x \in M, <X, E>=0}\nu(x, X)$$

$$\leq \sqrt{k}(1+\cos(d\sqrt{k}/2))/2\sin(d\sqrt{k}/2)$$

$$\leq \sup_{x \in M, <X, E>=0}\nu(x, X)$$

§4. Compact 2-transnormal hypersurfaces in $\overline{M}$ of negative constant holomorphic sectional curvature

This section is concerned with the case that the constant holomorphic sectional curvature k of $\overline{M}$ is negative. Then we can obtain the following Theorem 4.1 in the same way as Theorem 3.3.

Theorem 4.1. *Let M be a compact connected 2-transnormal hypersurface in a simply connected complete Kaehler manifold $\overline{M}$ of negative constant holomorphic sectional curvature k. Assume that for each $x \in M$ the almost contact structure vector $E(x)$ is a principal vector with principal curvature $\lambda(x)$. If the principal curvature $\lambda(x)$ and the other principal curvatures $\nu(x)$ satisfy the following conditions at each $x \in M$;*

$$\lambda(x) \geqq \sqrt{-k}(1 + \cosh(d\sqrt{-k})) / \sinh(d\sqrt{-k})$$

$$(\text{or } \lambda(x) \leqq \sqrt{-k}(1 + \cosh(d\sqrt{-k})) / \sinh(d\sqrt{-k}))$$

and

$$\nu(x) \geqq \sqrt{-k}(1 + \cosh(d\sqrt{-k})) / 2\sinh(d\sqrt{-k}/2)$$

$$(\text{or } \nu(x) \leqq \sqrt{-k}(1 + \cosh(d\sqrt{-k}/2)) / 2\sinh(d\sqrt{-k}/2)),$$

where d denotes the diameter of M as a subset of $\overline{M}$, then M is C-umbilic and furthermore M is a geodesic hypersphere with radius $d/2$.

To prove Theorem 4.1, we need the following lemmas instead of Lemma 3.1 and Lemma 3.2.

Lemma 4.2. *Every principal curvature of M is greater than $(\sqrt{-k}/2)\cosh (d\sqrt{-k}/2)$ at each point of M.*

Under the assumption on $E(x)$, we have

Lemma 4.3. *At the antipodal point $\tilde{x}$ of x,*

$$(1) \qquad \lambda(\tilde{x}) = \frac{-\sqrt{-k}\sinh(d\sqrt{-k}) + \lambda(x)\cosh(d\sqrt{-k})}{(\lambda(x)/\sqrt{-k})\sinh(d\sqrt{-k}) - \cosh(d\sqrt{-k})}$$

$$(2) \qquad \nu(\tilde{x}, \tilde{X}) = \frac{(-\sqrt{-k}/2)\sinh(d\sqrt{-k}/2) + \nu(x, X)\cosh(d\sqrt{-k}/2)}{(2\nu(x, X)/\sqrt{-k})\sinh(d\sqrt{-k}/2) - \cosh(d\sqrt{-k}/2)}$$

where $\tilde{X}$ is the tangent vector of M at $\tilde{x}$ given by the parallel translation of X along $\gamma(x, \tilde{x})$.

The proofs of these lemmas are quite similar to those of lemmas in §3. So we describe here only the matters worth while mentioning.

Fix a point $x \in M$ arbitrarily and consider the function Λ_x on M. Note that the cut locus $C(x)$ is empty owing to the negativity of the holomorphic sectional curvature of $\overline{M}$. At a critical point $\tilde{x}$, the Hessian H of Λ_x is given by

$$H(X,\ Y) = 2d < \{(\sqrt{-k}/2)\coth(d\sqrt{-k}/2) \cdot I - S_{-\gamma'(d)}\}X,\ Y >$$
$$+ d\sqrt{-k}\tanh(d\sqrt{-k}/2) < J\gamma'(d),\ X > < J\gamma'(d),\ Y >$$
$$\text{for}\quad X,\ Y \in M_{\tilde{x}},$$

which can be obtained from (1.1). Then we can immediately obtain Lemma 4.2. To prove (1) (resp. (2)) of Lemma 4.3, we may use the Jacobi field

$$Y(t) = \{\cosh\sqrt{-k}t - (\lambda(x)/\sqrt{-k})\sinh\sqrt{k}t\}J\gamma'(t)$$

$$(\text{resp. } Z(t) = \{\cosh(\sqrt{-k}t/2) - (2\nu(x,\ X)/\sqrt{-k})\sinh(\sqrt{-k}t/2)\}X(t)).$$

Then Theorem 4.1 can be proved in the quite similar way as Theorem 3.3.

REMARK. By easy consideration, we can get the following inequalities;

$$\sqrt{-k}\coth(d\sqrt{-k}) \leqq \inf_{x \in M}\lambda(x)$$

$$\leqq \sqrt{-k}(1 + \cosh(d\sqrt{-k}))/\sinh(d\sqrt{-k})$$

$$\leqq \sup_{x \in M}\lambda(x)$$

$$(\sqrt{-k}/2)\coth(d\sqrt{-k}/2) \leqq \inf_{x \in M,\ <X,\ E>=0}\nu(x,\ X)$$

$$\leqq \sqrt{-k}(1 + \cosh(d\sqrt{-k}/2))/2\sinh(d\sqrt{-k}/2)$$

$$\leqq \sup_{x \in M,\ <X,\ E>=0}\nu(x,\ X)$$

References

[1] Bolton, J., Transnormal hypersurface, Proc. Camb. Phil. Soc., 74 (1973), 43–48.

[2] Nishikawa, S., Transnormal hypersurfaces —Generalized constant width for Riemannian manifolds—, Tôhoku Math. J., 25 (1973), 451–459.

[3] Nishikawa, S., Compact two-transnormal hypersurfaces in a space of constant curvature, J. Math. Soc. Japan, 26 (1974), 625–635.

[4] Robertson, S. A., Generalized constant width for manifolds, Michigan Math. J., 11 (1964), 97–105.

[5] Robertson, S. A., On transnormal manifolds, Topology, 6 (1976), 117–123.

Institute of Mathematics
University of Tsukuba
Sakuramura, Niihari-gun,
Ibaraki, 305 Japan