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COINCIDENCE OF THE RIGHT JACOBSON RADICAL AND THE LEFT JACOBSON RADICAL IN A GAMMA RING

By

Shoji KYUNO

1. Introduction.

The notion of a Γ-ring was first introduced by Nobusawa [1]. The class of Γ-rings contains not only all rings but also all Hestenes ternary rings. Recently, Coppage and Luh [2] introduced the notions of the right Jacobson radical and other radicals and obtained some basic radical properties and their inclusion relations. The left Jacobson radical can be defined similarly and it is naturally asked if the right Jacobson radical coincides with the left one. In [2], they say, "It is unlikely that the left Jacobson radical is equal to the right one", but they show that if a Γ-ring satisfies the descending chain conditions on both left ideals and right ones then the right Jacobson radical and the left one coincide.

The aim of this note is to prove that the right Jacobson radical and the left one coincide without assuming any condition on a Γ-ring.

2. Preliminaries.

Let M and Γ be additive abelian groups. If for all $a, b, c \in M$ and $\gamma, \delta \in \Gamma$ the following conditions are satisfied,

(1) $a\gamma b \in M$,

(2) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\gamma+\delta)b = a\gamma b + a\delta b$, $a\gamma(b+c) = a\gamma b + a\gamma c$,

(3) $(a\gamma b)\delta c = a\gamma(b\delta c)$,

then M is called a Γ-ring. If A and B are subsets of a Γ-ring M and $\Theta \subseteq \Gamma$, we denote $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i\gamma_i b_i$ where $a_i \in A$, $b_i \in B$, and $\gamma_i \in \Theta$. For singleton subsets we abbreviate this notation, for example, $\{a\}\Theta B = a\Theta B$.

A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right and a left ideal, then we say that I is an ideal, or a two-sided ideal of M.

In the following we give the definition of the right operator ring R.

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Let M be a Γ-ring and F be the free abelian group generated by $\Gamma\times M$. Then

$$A=\{\sum_i n_i(\gamma_i, x_i)\in F \mid a\in M \Rightarrow \sum_i n_i a\gamma_i x_i=0\}$$

is a subgroup of F. Let $R=F/A$, the factor group, and denote the coset $(\gamma, x)+A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x]+[\alpha, y]=[\alpha, x+y]$ and $[\alpha, x]+[\beta, x]=[\alpha+\beta, x]$ for all $\alpha, \beta\in\Gamma$ and $x, y\in M$. We define a multiplication in R by

$$\sum_i[\alpha_i, x_i]\sum_j[\beta_j, y_j]=\sum_{i,j}[\alpha_i, x_i\beta_j y_j].$$

Then R forms a ring. If we define a composition on $M\times R$ into M by

$$a\sum_i[\alpha_i, x_i]=\sum_i a\alpha_i x_i \quad \text{for} \quad a\in M,\ \sum_i[\alpha_i, x_i]\in R,$$

then M is a right R-module, and we call R the right operator ring of a Γ-ring M.

For subsets $N\subseteq M$, $\Phi\subseteq\Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in R, where $\gamma_i\in\Phi$, $x_i\in N$. Thus, in particular, $R=[\Gamma, M]$. For $P\subseteq R$ we define $P^*=\{a\in M \mid [\Gamma, a]=[\Gamma, \{a\}]\subseteq P\}$. It follows that if P is a right (left) ideal of R, then P^* is a right (left) ideal of M. Also for any collection $\mathcal{C}$ of sets in R, $\cap_{P\in\mathcal{C}}P^*=(\cap_{P\in\mathcal{C}}P)^*$.

Similarly we can define the left operator ring L of M. For $N\subseteq M$, $\Phi\subseteq\Gamma$, we denote by $[N, \Phi]$ the set of all finite sums $\sum_i[x_i, \alpha_i]$ in L with $x_i\in N$ and $\alpha_i\in\Phi$. In particular, $L=[M, \Gamma]$. For $Q\subseteq L$ we define $Q^+=\{a\in M \mid [a, \Gamma]=[\{a\}, \Gamma]\subseteq Q\}$. It follows that if Q is a right (left) ideal of L, then Q^+ is a right (left) ideal of M. Also, for any collection $\mathcal{D}$ of sets in L,

$$\cap_{Q\in\mathcal{D}}Q^+=(\cap_{Q\in\mathcal{D}}Q)^+.$$

For all notions relevant to ring theory we refer to [5] and for all other notions to the Γ-ring we refer to [2] and [3].

3. Jacobson radicals.

Let M be a Γ-ring and R be its right operator ring and L be its left operator ring.

The right Jacobson radical of R, written as $J_r(R)$, is defined as the set of all elements of R which annihilate all the irreducible right R-modules. If G is a right R-module, $Ann_R(G)$ is defined as the set $\{r\in R \mid Gr=0\}$. Thus, we have $J_r(R)=\cap Ann_R(G)$, where this intersection runs over all irreducible right R-modules G.

Similarly, for the left operator ring L we have $J_r(L)=\cap Ann_L(S)$, where this intersection runs over all irreducible right L-modules S and $Ann_L(S)=\{l\in L \mid Sl=0\}$.

Ordinary ring theory shows

THEOREM 1. $J_l(L)=J_r(L)$, where $J_l(L)$ denotes the left Jacobson radical of L and $J_r(L)$ denotes the right Jacobson radical of L.

In [3], the author introduced the notion of right Γ-ring M-module (merely, we refer to them as right M-modules), and gave the definition of the right Jacobson radical of a Γ-ring M in the following:

DEFINITION. The right Jacobson radical of a Γ-ring M, written as $J_r(M)$, is the set of all elements of M which annihilate all the irreducible right M-modules. If M has no irreducible right M-modules, we put $J_r(M)=M$.

It was shown in [3] that $J_r(M)$ coincided with the right Jacobson radical defined by using quasi-regularity in [2]. In [3], we proved the following:

THEOREM 2. ([3] Theorem 3.1, also [2] Theorem 8.2)
If M is a Γ-ring and R is the right operator ring of M, then

$$J_r(M)=J_r(R)^*.$$

We can define the left Jacobson radical $J_l(M)$ of a Γ-ring M and by the similar fashions as in the right Jacobson radical we have

THEOREM 3. If M is a Γ-ring and L is the left operator ring of M, then $J_l(M)=J_l(L)^+$.

From Theorem 1 and Theorem 3, we have

THEOREM 4. $J_l(M)=J_l(L)^+=J_r(L)^+$.

4. Irreducible right R-modules and irreducible right L-modules.

We show the following theorem. The major part of its proof, i.e., the existence of $[G,\Gamma]$ owes to Luh ([4] Theorem 1).

THEOREM 5. There exists an injection φ from the set $\mathcal{A}$ of all irreducible right R-modules to the set $\mathcal{B}$ of all irreducible right L-modules.

PROOF. Let G be an arbitrary irreducible right R-module. Let A be the free abelian additive group generated by the set of ordered pairs (g,γ), where $g\in G$, $\gamma\in\Gamma$, and let B the subgroup of elements $\sum_i m_i(g_i,\gamma_i)\in A$, where m_i are integers such that $\sum_i m_i g_i[\gamma_i,x]=0$ for all $x\in M$. Denote by $[G,\Gamma]$ the factor group A/B and, without causing any ambiguity, by $[g,\gamma]$ the coset $(g,\gamma)+B$. Every element in $[G,\Gamma]$ therefore can be expressed as a finite sum $\sum_i[g_i,\gamma_i]$. $[G,\Gamma]$ forms a right L-module with the definition

$$\sum_i[g_i,\gamma_i]\sum_j[x_j,\beta_j]=\sum_{i,j}[g_i[\gamma_i,x_j],\beta_j]$$

for $\sum_i[g_i, \gamma_i]\in[G, \Gamma]$ and $\sum_j[x_j, \beta_j]\in L$.

To see that $[G, \Gamma]$ is irreducible, let $\sum_i[g_i, \gamma_i]$ be an arbitrary non-zero element in $[G, \Gamma]$. Then the set $G'=\{\sum_i g_i[\gamma_i, x] \mid x\in M\}$ is a non-zero R-submodule of G. Since G is irreducible, $G'=G$. For any $\sum_j[g_j', \gamma_j']\in[G, \Gamma]$, we may write $g_j'=\sum_i g_i[\gamma_i, x_j]$ where $x_j\in M$. Thus, $\sum_j[g_j', \gamma_j']=\sum_j[\sum_i g_i[\gamma_i, x_j], \gamma_j']=\sum_i[g_i, \gamma_i]\sum_j[x_j, \gamma_j']$ $\in\sum_i[g_i, \gamma_i]L$. Hence, $[G, \Gamma]$ is irreducible.

Let φ be a mapping from $\mathcal{A}$ to $\mathcal{B}$ sending an arbitrary irreducible right R-module G to $[G, \Gamma]$. If $[G, \Gamma]=[G', \Gamma]$, then $G[\Gamma, M]=G'[\Gamma, M]$. Since G and G' are irreducible right R-modules, we get $G=G[\Gamma, M]$ and $G'=G'[\Gamma, M]$. Thus, $G=G'$, and the proof is completed.

Conversely, we have

THEOREM 6. There exists an injection ψ from the set $\mathcal{B}$ of all irreducible right L-modules to the set $\mathcal{A}$ of all irreducible right R-modules.

The proof is precisely analogous to that for Theorem 5 and so we omit it.

5. The proof of that $J_r(M)=J_l(M)$.

Let $\mathcal{A}$ be the set of all irreducible right R-modules and $\mathcal{B}$ be the set of all irreducible right L-modules. Let G be an arbitrary element of $\mathcal{A}$ and $[G, \Gamma]$ be its corresponding element in $\mathcal{B}$, which is shown in the proof of Theorem 5. Let S be an arbitrary element of $\mathcal{B}$ and $[S, M]$ be its corresponding element in $\mathcal{A}$, which is assured by Theorem 6. Then we have

$$J_r(M)=J_r(R)^*=(\cap_{G\in\mathcal{A}}Ann_R(G))^*=\cap_{G\in\mathcal{A}}Ann_R(G)^*$$

and

$$J_l(M)=J_l(L)^+=J_r(L)^+=(\cap_{S\in\mathcal{B}}Ann_L(S))^+=\cap_{S\in\mathcal{B}}Ann_L(S)^+.$$

By the definitions we get

$$\begin{aligned}Ann_R(G)^* &=\{x\in M \mid [\Gamma, x]\subseteq Ann_R(G)\}\\ &=\{x\in M \mid G[\Gamma, x]=0\}\end{aligned}$$

and

$$\begin{aligned}Ann_L([G, \Gamma])^+ &=\{x\in M \mid [x, \Gamma]\subseteq Ann_L([G, \Gamma])\}\\ &=\{x\in M \mid [G, \Gamma][x, \Gamma]=0\}\\ &=\{x\in M \mid [G[\Gamma, x], \Gamma]=0\}.\end{aligned}$$

Clearly, $[G[\Gamma, x], \Gamma]=0$ if and only if $G[\Gamma, x][\Gamma, M]=G[\Gamma, x]R=0$. Also, $G[\Gamma, x]=0$ implies $G[\Gamma, x]R=0$. Conversely, if $G[\Gamma, x]R=0$, then $G[\Gamma, x]=0$. For let $J=\{g\in G \mid gR=0\}$, then J is a R-submodule of G. Since G is irreducible, J must be

G or 0. If $J=G$, then $GR=JR=0$, a contradiction. Thus, J must be 0. Hence, if $G[\Gamma, x]R=0$, then $G[\Gamma, x]\subseteq J=0$.

Therefore, we have

$$Ann_R(G)^*=Ann_L([G, \Gamma])^+.$$

Thus, we have

$$J_r(M)=\cap_{G\in\mathcal{A}}Ann_R(G)^*=\cap_{G\in\mathcal{A}}Ann_L([G, \Gamma])^+$$
$$\supseteq \cap_{S\in\mathcal{B}}Ann_L(S)^+=J_l(M).$$

By the definitions we get

$$Ann_L(S)^+=\{x\in M\mid [x, \Gamma]\subseteq Ann_L(S)\}$$
$$=\{x\in M\mid S[x, \Gamma]=0\}$$

and

$$Ann_R([S, M])^*=\{x\in M\mid [\Gamma, x]\subseteq Ann_R([S, M])\}$$
$$=\{x\in M\mid [S, M][\Gamma, x]=0\}$$
$$=\{x\in M\mid [S[M, \Gamma], x]=0\}.$$

Clearly, $[S[M, \Gamma], x]=0$ if and only if $S[M, \Gamma][x, \Gamma]=0$, which is equivalent to $S[x, \Gamma]=0$, for $S[M, \Gamma]=S$. Therefore, we have

$$Ann_L(S)^+=Ann_R([S, M])^*.$$

Thus, we have

$$J_l(M)=J_r(L)^+=\cap_{S\in\mathcal{B}}Ann_L(S)^+=\cap_{S\in\mathcal{B}}Ann_R([S, M])^*$$
$$\supseteq \cap_{G\in\mathcal{A}}Ann_R(G)^*=J_r(M).$$

Therefore, we obtain that $J_r(M)=J_l(M)$ and the proof is completed.

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Department of Technology
Tohoku Gakuin University
Tagajo, Miyagi, Japan