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ON THE INTERVALS BETEEN CONSECUTIVE NUMBERS THAT ARE SUMS OF TWO PRIMES

By

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1. Introduction.

It is the well known conjecture of H. Cramér that

$$p_{n+1}-p_{n}\ll(\log p_{n})^{2}$$

where p_{n} is the n-th prime. In 1940 P. Erdös proposed the problem to estimate the sum

$$\sum_{p_{n}\leq x}(p_{n+1}-p_{n})^{2},$$

and A. Selberg showed that it is

$$\ll x(\log x)^{3}$$

under the Riemann hypothesis. This problem has been stimulating the several authors, vide [2, 3, 10, 11, 13].

Let (g_{n}) denote in ascending order even integers that are representable as the sum of two primes. The Goldbach conjecture is then interpreted as that

$$g_{n+1}-g_{n}=2$$

for all n. In 1952 Ju. V. Linnik [7] proved, on assuming the Rieman hypothesis, that

$$g_{n+1}-g_{n}\ll(\log g_{n})^{3+\varepsilon}$$

for any $\varepsilon>0$ and all n. Also see [1]. In this paper we shall estimate the third moment of it.

THEOREM.

$$\sum_{g_{n}\leq x}(g_{n+1}-g_{n})^{3}\ll x(\log x)^{300}.$$

COROLLARY. *For* $0\leqq\gamma<3$, *we have*

$$\sum_{g_{n}\leq x}(g_{n+1}-g_{n})^{\gamma}=(2^{\gamma-1}+o(1))x.$$

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Our assertion should be compared with the known results in Goldbach's problem. Let $E(x)$ be the number of even integers not exceeding x that may not be expressed as a sum of two primes, and $D(x)$ be the maximum of $(g_{n+1}-g_n)$ for $g_n \leqq x$. It was proved by H. L. Montgomery and R. C. Vaughan [9] that

$$E(x) \ll x^{1-\delta}$$

with some $\delta>0$. As for $D(x)$, the argument in [9] runs as follows. Suppose that one knows the equi-distribution of primes in intervals $[x, x+x^{\theta}]$ for almost all x, and in $[x, x+x^{\Theta}]$ for all x. Then,

$$D(x) \ll x^{\theta\Theta}. \tag{1.1}$$

By an elementary consideration, see section 3, we find

$$\sum_{g_n \leqq x} (g_{n+1}-g_n)^2 = 2x + O(D(x)E(x)).$$

It seems that no unconditional result leads $\theta\Theta \leqq \delta$.

Our argument is based upon Linnik's method [6, 7] and D. Wolke's trick [13]. The limitation of our estimate comes from A. E. Ingham's bound [4] for zeros of the Riemann zeta-function.

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2. Notation and Lemmas.

We use the standard notation in number theory. ρ stands for the non-trivial zeros of the Riemann zeta-function. For $1/2 \leqq \sigma \leqq 1$ and $T>0$, $N(\sigma, T)$ denotes the number of ρ such that $\sigma \leqq \mathrm{Re}(\rho)$ and $|\mathrm{Im}(\rho)| \leqq T$.

LEMMA 1. *Uniformly for $x, T \geqq 3$, we have*

$$\sum_{|\mathrm{Im}(\rho)| \leqq T} x^{\rho} = -\frac{T}{\pi} \Lambda'(x) + O(x(\log xT)^2)$$

where $\Lambda'(x)$ is equal to the von Mangoldt function if x is an integer, and $\Lambda'(x)=0$ otherwise.

This is a formula of E. Landau [5]. Though his estimate is not uniform for x, it is easy to alter the proof of [5] to be suitable for our aim. The following Lemma 2 is due to Ingham [4] and Montgomery [8, Theorem 1]. Lemma 3 follows from [8, Theorem 2].

LEMMA 2. *For $T>2$, we have*

$$N(\sigma, T)\ll T^{\lambda(\sigma)(1-\sigma)}(\log T)^{13}$$

where

$$\lambda(\sigma)=\begin{cases} \dfrac{3}{2-\sigma} & \text{if } 1/2\leqq\sigma\leqq 4/5 \\ \dfrac{2}{\sigma} & \text{if } 4/5\leqq\sigma\leqq 1 . \end{cases}$$

LEMMA 3. *If $9/10\leqq\sigma\leqq 1$, then*

$$N(\sigma, T)\ll T^{(2-c)(1-\sigma)}(\log T)^{13}$$

where c is a positive absolute constant.

In sections 3 and 4 we use the convention $L=\log X$. For a real x, write $e(x)=e^{2\pi ix}$. $*$ and $\hat{}$ mean that $f*g(x)=\int_{-\infty}^{+\infty}f(x-y)g(y)dy$ and $\hat{f}(x)=\int_{-\infty}^{+\infty}f(y)\cdot e(-xy)dy$, respectively. The implied constants in O and $\ll$ are absolute, except for the proof of Corollary.

3. Reduction of the problem

In this section we first reduce the proof of Theorem to that of Lemma 4 below. Lemma 4 will be verified in section 5. Next we derive Corollary from Theorem. Put $d_n=g_{n+1}-g_n$, for simplicity.

PROOF OF THEOREM. It is sufficient to prove

$$F(x)=\sum_{x'<g_n\leqq x} d_n^3\ll x(\log x)^{300}$$

for all large x and $x'=(5/7)x$. We have

$$F(x)\ll \sum_{\substack{x'<g_n\leqq x \\ d_n\leqq(\log x)^{150}}} d_n^3+(\log x)\sup_{\delta>(\log x)^{150}}\sum_{\substack{x'<g_n\leqq x \\ \delta<d_n\leqq 2\delta}} d_n^3 .$$

Because of (1.1), $\delta\leqq D(x)<x^{1/6}$. Put

$$\Gamma(x, \delta)=\{g_n : x'<g_n\leqq x,\ \delta<d_n\leqq 2\delta,\ g_{n+1}\leqq x\} .$$

Then,

$$\text{(3.1)}\qquad F(x)\ll(\log x)^{300}\sum_{g_n\leqq x} d_n+(\log x)\sup_{(\log x)^{150}<\delta<x^{1/6}}\Big(\sum_{g_n\in\Gamma(x,\delta)} d_n^3+\delta^2 D(x)\Big)$$

$$\ll x(\log x)^{300}+(\log x)\sup_{(\log x)^{150}<\delta<x^{1/6}}\delta^2\sum_{g_n\in\Gamma(x,\delta)} d_n .$$

Here we state our main lemma.

LEMMA 4. *Let X be a large parameter,*

$$(5/2)X\leqq x\leqq(7/2)X \quad and \quad (1/2)(\log X)^{150}<\Delta<X^{1/6}.$$

There exists a function $R(x, \Delta)$ such that

$$\int_{(5/2)X}^{(7/2)X}|R(x, \Delta)|^2dx\ll X^3(\log X)^{299} \tag{3.2}$$

and

$$\sum_{\substack{X<m,\, n\leq 2X \\ x-\Delta<m+n\leq x}}\sum \Lambda(m)\Lambda(n)=\Delta(X-|x-3X|)+O(\Delta X(\log X)^{-4})+R(x, \Delta), \tag{3.3}$$

uniformly for X, x and Δ.

Now, if $t\in[(g_n+g_{n+1})/2, g_{n+1})$ for $g_n\in\Gamma(x, \delta)$ then

$$t-\frac{\delta}{2}>\frac{g_n+g_{n+1}}{2}-\frac{d_n}{2}=g_n.$$

Namely the interval $(t-\delta/2, t]$ contains no sum of two primes. By (3.3) in Lemma 4 with $(7/2)X=x$ and $2\Delta=\delta$, we therefore have

$$R(t, \delta/2)=-\frac{\delta}{2}\left(\frac{2}{7}x-\left|t-\frac{6}{7}x\right|\right)+O(\delta x(\log x)^{-4})$$

for all $t\in[(g_n+g_{n+1})/2, g_{n+1})$ with $g_n\in\Gamma(x, \delta)$. Since these intervals are mutually disjoint, we have

$$\sum_{g_n\in\Gamma(x,\delta)}\left(g_{n+1}-\frac{g_n+g_{n+1}}{2}\right)(\delta x)^2\ll\sum_{g_n\in\Gamma(x,\delta)}\int_{(g_n+g_{n+1})/2}^{g_{n+1}}|R(t, \delta/2)|^2dt$$

$$\leqq\int_{x'}^{x}|R(t, \delta/2)|^2dt.$$

Hence (3.2) in Lemma 4 yields that

$$\delta^2\sum_{g_n\in\Gamma(x,\delta)}d_n\ll x(\log x)^{299}$$

uniformly for δ, $(\log x)^{150}<\delta<x^{1/6}$. Combining this with (3.1) we obtain

$$F(x)\ll x(\log x)^{300},$$

as required.

PROOF OF COROLLARY. With the notation in section 1, we easily see that

$$\sum_{g_n\leq x}d_n=x+O(D(x)),$$

and

$$\sum_{g_n\leq x}1=\frac{1}{2}x+O(1)-E(x).$$

By subtraction, we have

(3.4) $$\sum_{\substack{g_n\leqq x\\ d_n>2}} d_n \ll D(x)+E(x) \ll E(x)\,,$$

or

(3.5) $$\sum_{\substack{g_n\leqq x\\ d_n>2}} 1=\frac{1}{2}x+O(E(x))\,.$$

Now, if $0\leqq\gamma\leqq 1$ then

$$\sum_{g_n\leqq x} d_n{}^{\gamma}=2^{\gamma}\sum_{\substack{g_n\leqq x\\ d_n=2}} 1+O\Big(\sum_{\substack{g_n\leqq x\\ d_n>2}} d_n\Big)=2^{\gamma-1}x+O(E(x))$$

by (3.4) and (3.5). It is known [12; Kap. VI. Satz 7.1] that

(3.6) $$E(x)\ll x(\log x)^{-A}$$

for any $A>0$. Hence we get Corollary in case $0\leqq\gamma\leqq 1$.

Suppose $1<\gamma<3$. Let D be a positive constant, which will be specified later. Then,

$$\sum_{g_n\leqq x} d_n{}^{\gamma}=\sum_{d_n=2}+\sum_{2<d_n\leqq(\log x)D}+\sum_{d_n>(\log x)D}$$

$$=2^{\gamma-1}x+O(E(x))+O\Big((\log x)^{(\gamma-1)D}\sum_{\substack{g_n\leqq x\\ d_n>2}} d_n\Big)+O\Big((\log x)^{-(3-\gamma)D}\sum_{g_n\leqq x} d_n{}^3\Big)$$

$$=2^{\gamma-1}x+O(E(x)(\log x)^{(\gamma-1)D}+x(\log x)^{300-(3-\gamma)D})$$

because of (3.4), (3.5) and Theorem. On taking $D=301/(3-\gamma)$ we get, by (3.6), that

$$\sum_{g_n\leqq x} d_n{}^{\gamma}=2^{\gamma-1}x+O(x(\log x)^{-1})\,,$$

as required.

4. Proof of Lemma 4, preliminaries.

We begin with modifying the explicit formula:

(4.1) $$\sum_{n\leqq x}\Lambda(n)=x-\sum_{|\mathrm{Im}(\rho)|\leqq T}\frac{x^{\rho}}{\rho}+O\Big(\Big(1+\frac{x}{T}\Big)(\log xT)^2\Big)$$

uniformly for x, $T\geqq 3$. For $T\geqq 3$, define

(4.2) $$q_n=q_n(T)=\int_{n-1/2}^{n+1/2}\sum_{|\mathrm{Im}(\rho)|\leqq T} y^{\rho-1}dy$$

if $n\leqq 5$, and $q_n=0$ otherwise. Moreover we determine $r_n=r_n(T)$ by the relation

(4.3) $$\Lambda(n)=1-q_n-r_n\,.$$

Lemma 1 then gives

$$q_n,\ r_n \ll (\log nT)^2. \tag{4.4}$$

For large x, it follows from the prime number theorem, (4.1) and (4.2) that

$$\begin{aligned}\sum_{n\leqq x} q_n &= \sum_{|\mathrm{Im}(\rho)|\leqq T}\left(\frac{([x]+1/2)^{\rho}}{\rho}+O\left(\frac{1}{|\rho|}\right)\right)\\ &\ll x\exp(-(\log x)^{1/2})+\left(1+\frac{x}{T}\right)(\log xT)^2.\end{aligned} \tag{4.5}$$

Similarly,

$$\sum_{n\leqq x} r_n \ll \left(1+\frac{x}{T}\right)(\log xT)^2 \tag{4.6}$$

by (4.1), (4.2) and (4.3).

Now, on choosing

$$T=\frac{X}{\Delta}L^8,$$

we consider the sum in question:

$$G=\sum\sum_{\substack{X<m,\, n\leqq 2X\\ x-\Delta<m+n\leqq x}} \Lambda(m)\Lambda(n).$$

By (4.3),

$$\Lambda(m)\Lambda(n)=1+q_m q_n-(q_m+q_n)-r_m\Lambda(n)-\Lambda(m)r_n-r_m r_n.$$

Accordingly,

$$G=G_1+G_2-2G_3-2G_4-G_5, \qquad \text{say.} \tag{4.7}$$

$$\begin{aligned}G_1&=\sum\sum_{\substack{X<m,\, n\leqq 2X\\ x-\Delta<m+n\leqq x}} 1\\ &=\sum_{\substack{X<m\leqq 2X\\ x-2X<m<x-\Delta-X}} \#\{n:\ x-m-\Delta<n\leqq x-m\}+O(\Delta^2)\\ &=\Delta\sum_{\substack{X<m\leqq 2X\\ x-2X<m\leqq x-X}} 1+O(X)\\ &=\Delta(X-|x-3X|)+O(X).\end{aligned} \tag{4.8}$$

On writing

$$Z(y)=Z(y,\ T)=\sum_{|\mathrm{Im}(\rho)|\leqq T} y^{\rho-1}, \tag{4.9}$$

we have

$$G_2 = \sum\sum_{\substack{X<m,\, n\leq 2X \\ x-\Delta<m+n\leq x}} \int_{m-1/2}^{m+1/2}\int_{n-1/2}^{n+1/2} Z(u)Z(v)du\,dv$$
$$= \iint_D Z(u)Z(v)du\,dv\,, \qquad \text{say.}$$

We replace the domain D by

(4.10) $$\boldsymbol{D}=\boldsymbol{D}(X,\ x,\ \Delta)=\{(u,\ v)\in[X,\ 2X]^2:\ x-\Delta\leqq u+v\leqq x\}.$$

The resulting error is

(4.11) $$\ll \iint_{(D\cup\boldsymbol{D})\setminus(D\cap\boldsymbol{D})} |Z(u)Z(v)|du\,dv \ll XL^4\,,$$

because of Lemma 1.

(4.12) $$G_3 = \sum\sum_{\substack{X<m,\, n\leq 2X \\ x-\Delta<m+n\leq x}} q_m$$
$$= \sum_{\substack{X<m\leq 2X \\ x-2X<m\leq x-\Delta-X}} q_m(\Delta+O(1))+O(\Delta^2L^2)$$
$$\ll \Delta \sup_{M\leq 2X}\Big|\sum_{m\leq M} q_m\Big| + XL^2$$
$$\ll \Delta XL^{-4}\,,$$

by (4.4) and (4.5). Also, (4.4) and (4.6) give that

(4.13) $$G_4 = \sum\sum_{\substack{X<m,\, n\leq 2X \\ x-\Delta<m+n\leq x}} \Lambda(m)r_n$$
$$\ll \sum_{m\leq 2X}\Lambda(m) \sup_{N\leq 2X}\Big|\sum_{n\leq N} r_n\Big|$$
$$\ll XL^2\Big(1+\frac{X}{T}\Big)L^2$$
$$\ll \Delta XL^{-4}\,.$$

Similarly,

(4.14) $$G_5 \ll \Delta XL^{-4}\,.$$

On summing up the above estimates (4.7)-(4.14) we obtain

(4.15) $$G=\Delta(X-|x-3X|)+O(\Delta XL^{-4})+\iint_{\boldsymbol{D}} Z(u)Z(v)du\,dv$$

where Z and $\boldsymbol{D}$ are defined by (4.9) and (4.10), respectively.

5. Proof of Lemma 4.

Put

$$N^{+}(\sigma, T)=T^{\lambda(\sigma)(1-\sigma)}L^{13}$$

where λ is defined in Lemma 2. Since

$$T^{\lambda(1/2)}=T^{\lambda(1)}=\left(\frac{X}{\Delta}L^{8}\right)^{2}<X^{2}L^{-280}<\left(\frac{X}{\Delta}L^{8}\right)^{12/5}=T^{\lambda(3/4)}<T^{\lambda(4/5)},$$

there exist r and t such that $1/2<r<3/4$, $4/5<t<1$ and

$$T^{\lambda(r)}=T^{\lambda(t)}=X^{2}L^{-280}.$$

Define $s=\min(t, 9/10)$, and $\boldsymbol{I}=[r, s)$. We then see

(5.1) $$T^{\lambda(\sigma)}\leqq X^{2}L^{-280} \quad \text{for all } \sigma\in[1/2, 9/10)\setminus\boldsymbol{I},$$

and

(5.2) $$T^{\lambda(\sigma)}\geqq X^{2}L^{-280} \quad \text{for all } \sigma\in\boldsymbol{I}.$$

Now, we divide the sum $Z(y)$, which is defined by (4.9).

(5.3) $$Z(y)=\sum_{\mathrm{Re}(\rho)\notin I}+\sum_{\mathrm{Re}(\rho)\in I}=z_{1}(y)+z(y), \quad \text{say}.$$

We first consider z_1. By a familiar way,

$$J=\int_{X}^{2X}|z_{1}(y)|^{2}dy\ll L^{2}\sum_{\substack{|\mathrm{Im}(\rho)|\leq T\\ \mathrm{Re}(\rho)\notin I}}X^{2\mathrm{Re}(\rho)-1}$$
$$\ll L^{2}\sum_{\substack{|\mathrm{Im}(\rho)|\leq T\\ \mathrm{Re}(\rho)<1/2}}1+L^{3}\sup_{1/2\leq\sigma\notin I}X^{2\sigma-1}N(\sigma, T).$$

Here, because of the zero-free region [12; Kap. VIII. Satz 6.2], the above supremum may be taken over $\sigma\leqq 1-\eta(T)$ only, where $\eta(T)=(\log T)^{-4/5}$. Lemmas 2 and 3 yield that

(5.4) $$J\ll L^{3}T+L^{16}X\left\{\sup_{\substack{1/2\leq\sigma<9/10\\ \sigma\notin I}}\left(\frac{T^{\lambda(\sigma)}}{X^{2}}\right)^{1-\sigma}+\sup_{9/10\leq\sigma\leq 1-\eta(T)}\left(\frac{T^{2}}{X^{2}}\right)^{1-\sigma}T^{-c(1-\sigma)}\right\}$$
$$\ll L^{11}X\Delta^{-1}+L^{16}X\{(X^{-280})^{1/10}+T^{-c\eta(T)}\}$$
$$\ll XL^{-12},$$

by (5.1).

We turn to the double integral in (4.15). Since

$$Z(u)Z(v)=z(u)z(v)+z_{1}(u)Z(v)+Z(u)z_{1}(v)-z_{1}(u)z_{1}(v),$$

$$\iint_D (Z(u)Z(v)-z(u)z(v))\,du\,dv$$

$$\ll \iint_{\substack{X\leq u,v\leq 2X \\ x-\Delta\leq u+v\leq x}} |z_1(u)|(|Z(v)|+|z_1(v)|)\,du\,dv$$

$$\ll L^2\Delta\int_X^{2X}|z_1(y)|\,dy+\Delta\int_X^{2X}|z_1(y)|^2dy$$

$$\ll L^2\Delta(X^2L^{-12})^{1/2}+\Delta XL^{-12}$$

$$\ll \Delta XL^{-4}\,,$$

by Lemma 1 and (5.4). Combining this with (4.15) we reach (3.3);

$$G=\Delta(X-|x-3X|)+O(\Delta XL^{-4})+R(x,\Delta)$$

where

$$R(x,\Delta)=\iint_D z(u)z(v)\,du\,dv\,. \tag{5.5}$$

It remains to prove (3.2). First we define $z(y)=0$ if $y\notin[X, 2X]$. Next we split up $z(y)$. Let $z_\sigma(y)$ be the partial sum of $z(y)$ restricted by $\sigma\leqq\mathrm{Re}(\rho)<\sigma(1+1/L)$. Then,

$$z(y)=\sum_{\substack{\alpha=r(1+1/L)^n\in I \\ n\geqq 0}} z_\alpha(y)\,.$$

Furthermore let $\chi(x)$ denote the characteristic function of $[0, \Delta]$. Thus we may rewrite (5.5) as

$$R(x,\Delta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\chi(x-u-v)z(u)z(v)\,du\,dv$$

$$=\chi*z*z(x)\,.$$

Now, by Plancherel's relation, we have

$$I=\int_{(5/2)X}^{(7/2)X}|R(x,\Delta)|^2dx\leqq\int_{-\infty}^{+\infty}|\chi*z*z(x)|^2dx \tag{5.6}$$

$$=\int_{-\infty}^{+\infty}|\widehat{\chi*z*z}(x)|^2dx$$

$$=\int_{-\infty}^{+\infty}|\hat{\chi}(x)|^2|\hat{z}(x)|^4dx\,.$$

Here we see

$$|\hat{\chi}(x)|^2=\left(\frac{\sin\pi\Delta x}{\pi x}\right)^2,$$

and, on using Hölder's inequality,

$$|\hat{z}(x)|^4 \ll L^3 \sum_\alpha |\hat{z}_\alpha(x)|^4 .$$

Therefore (5.6) becomes

$$I \ll L^4 \Delta^2 \sup_{\sigma\in I}\Big(\sup_x |\hat{z}_\sigma(x)|^2\Big)\int_{-\infty}^{+\infty} |z_\sigma(x)|^2 dx\,, \tag{5.7}$$

by Plancherel's relation again.

We proceed to estimate the square integral of z_σ.

$$\int_{-\infty}^{+\infty} |z_\sigma(y)|^2 dy = \int_X^{2X} \Big|\sum_{\substack{|\mathrm{Im}(\rho)|\leq T\\ \sigma\leq \mathrm{Re}(\rho)<\sigma(1+1/L)}} y^{\rho-1}\Big|^2 dy \tag{5.8}$$

$$\ll L^2 X^{2\sigma-1} N(\sigma, T).$$

We turn to $\hat{z}_\sigma$. The simplest saddle point method [12; Kap IX, Lemma 4.2] leads that

$$\hat{z}_\sigma(x) = \sum_{\substack{|\mathrm{Im}(\rho)|\leq T\\ \sigma\leq \mathrm{Re}(\rho)<\sigma(1+1/L)}} \int_X^{2X} y^{\mathrm{Re}(\rho)-1} \exp(i(\mathrm{Im}(\rho)\log y - 2\pi x y))dy$$

$$\ll LX^\sigma + \sum_{\substack{3<|\mathrm{Im}(\rho)|\leq T\\ \sigma\leq \mathrm{Re}(\rho)<\sigma(1+1/L)}} X^{\mathrm{Re}(\rho)} |\mathrm{Im}(\rho)|^{-1/2}$$

$$\ll LX^\sigma \Big(1+\sup_{3\leq t\leq T} t^{-1/2} N(\sigma, t)\Big).$$

We now appeal to Lemma 2. Since $\lambda(\sigma)(1-\sigma)\geq 1/2$ if $1/2\leq\sigma\leq 4/5$ and $\leq 1/2$ if $4/5\leq\sigma\leq 1$, we have that

$$\hat{z}_\sigma(x) \ll \begin{cases} LX^\sigma T^{-1/2} N^+(\sigma, T) & \text{if } 1/2\leq\sigma\leq 4/5 \\ L^{14} X^\sigma & \text{if } 4/5\leq\sigma\leq 1\,, \end{cases} \tag{5.9}$$

uniformly for x.

In conjunction with (5.7), (5.8) and (5.9) we obtain

$$I \ll L^4\Delta^2\Big(\sup_{\substack{\sigma\in I\\ \sigma\leq 4/5}} L^4 X^{4\sigma-1} T^{-1} N^+(\sigma, T)^3 + \sup_{\substack{\sigma\in I\\ \sigma\geq 4/5}} L^{30} X^{4\sigma-1} N^+(\sigma, T)\Big).$$

Notice that

$$\lambda(\sigma)(1-\sigma) = \begin{cases} 1-\dfrac{1}{3}\lambda(\sigma)(2\sigma-1) & \text{if } 1/2\leq\sigma\leq 4/5 \\ 2-\lambda(\sigma)(2\sigma-1) & \text{if } 4/5\leq\sigma\leq 1\,. \end{cases}$$

Hence, by (5.2), we conclude

$$I \ll L^{47}\Delta^2 \sup_{\sigma\in I} X^{4\sigma-1} T^{2-\lambda(\sigma)(2\sigma-1)}$$

$$\ll L^{63} X^3 \sup_{\sigma\in I}\Big(\frac{X^2}{T^{\lambda(\sigma)}}\Big)^{2\sigma-1}$$

$$\ll X^3 L^{287}\,,$$

as required.

This completes our proof.

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