TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 339–343

A CHARACTERIZATION OF PARACOMPACTNESS OF LOCALLY LINCDLOF SPACES

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Abstract. A space X is said to have property $\mathscr{B}$ if every infinite open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that every point $x\in X$ has a neighborhood W with $|\{V\in\mathcal{V}: W\cap V\neq\emptyset\}|<|\mathcal{U}|$. It is proved that a locally Lindelöf space is paracompact iff it has property $\mathscr{B}$.

All spaces are assumed to be regular T_1.

A well-known problem posed by Arhangel'skii and Tall is: Is every locally compact normal metacompact space paracompact? The problem is affirmative if we assume $V=L$ [10] or if the space is perfectly normal [1] or boundedly metacompact [5] or locally connected [6].

In connection with this problem, in this paper we give a characterization of paracompactness for locally Lindelöf spaces by using property $\mathscr{B}$, and provide another partial answer to the problem.

Property $\mathscr{B}$ was introduced originally by Zenor [12] as a generalization of parpcompactness: a space X is said to have property $\mathscr{B}$, if for every monotone increasing open cover $\mathcal{U}=\{U_\alpha: \alpha\in\kappa\}$ (that is, $U_\alpha\subset U_\beta$ if $\alpha<\beta$) of X, there exists a monotone increasing open cover $\mathcal{V}=\{V_\alpha: \alpha\in\kappa\}$ which is a shrinking of $\mathcal{U}$, i.e., $\bar{V}_\alpha\subset U_\alpha$ for $\alpha\in\kappa$.

It is proved in [11] that a space X has property $\mathscr{B}$ iff every open cover of X of infinite cardinality κ has an open refinement $\mathcal{V}$ such that every point $x\in X$ has a neighborhood W with $|\{V\in\mathcal{V}: V\cap W\neq\emptyset\}|<\kappa$; we say such a refinement $\mathcal{V}$ is locally κ. It is known from Rudin [9] that normal spaces with property $\mathscr{B}$ are not necessarily paracompact. However, Balogh and Rudin [3] recently proved that a monotonically normal space is paracompact iff it has property $\mathscr{B}$. Using the idea in Balogh [2] we now prove the following theorem.

THEOREM 1. *A locally Lindelöf space is paracompact iff it has property* $\mathscr{B}$.

PROOF. Let X be a locally Lindelöf space with property $\mathscr{B}$. Suppose X is not paracompact. Then there exists a minimal cardinal κ such that we have

Received July 20, 1992, Revised November 5, 1992.

some open cover $\mathcal{U}$ of X of cardinality κ which has no locally finite open refinement. We will show $\mathcal{U}$ has, however, a locally finite open refinement. Let $\mathcal{U}=\{U_\alpha: \alpha\in\kappa\}$. Since X is countably paracompact and locally Lindelöf we can assume that $\kappa>\omega$ and each $\bar{U}_\alpha$ is Lindelöf. There are two cases to consider.

Case 1. κ is singular. Then $\mathrm{cf}(\kappa)=\tau<\kappa$. Let $\{\kappa_\mu: \mu\in\tau\}$ be an increasing cofinal subset of κ so that $\{\cup\mathcal{U}_{\kappa_\mu}: \mu\in\tau\}$ is a monotone increasing open cover of X, where $\mathcal{U}_\alpha=\{U_\beta: \beta\in\alpha\}$ for every $\alpha\in\kappa$. Since X has property $\mathcal{B}$, there is a monotone increasing open cover $\{V_\mu: \mu\in\tau\}$ of X such that $\bar{V}_\mu\subset\cup\mathcal{U}_{\kappa_\mu}$ for every $\mu\in\tau$. By the definition of κ, there exists a locally finite open collection $\mathcal{G}_\mu$ such that $\mathcal{G}_\mu$ refines $\mathcal{U}_{\kappa_\mu}$ and $\bar{V}_\mu\subset\cup\mathcal{G}_\mu$. Let us consider the open caver $\mathcal{G}=\cup\{\mathcal{G}_\mu: \mu\in\tau\}$ of X. Note that each member of $\mathcal{G}$ has Lindelöf closure, it is easy to check that each member of $\mathcal{G}$ meets at most τ many other members of $\mathcal{G}$. Using usual chaining argument, we may find some partition $\{\mathcal{A}_\alpha: \alpha\in A\}$ of $\mathcal{G}$ such that $(\cup\mathcal{A}_\alpha)\cap(\cup\mathcal{A}_{\alpha'})=\emptyset$ if $\alpha, \alpha'\in A$ with $\alpha\neq\alpha'$, and $|\mathcal{A}_\alpha|\leqq\tau$ for every $\alpha\in A$. By the definition of κ, $\mathcal{A}_\alpha$ has, since $\cup\mathcal{A}_\alpha$ is clopen, a locally finite open refinement $\mathcal{H}_\alpha$, so that $\cup\{\mathcal{H}_\alpha: \alpha\in A\}$ is the desired refinement of $\mathcal{U}$.

Case 2. κ is regular. Using property $\mathcal{B}$ find an open refinemnet $\mathcal{G}$ of $\mathcal{U}$ such that every point in X has a neighborhood V with

$$|\{G: G\in\mathcal{G}, G\cap V\neq\emptyset\}|<\kappa.$$

Clearly we may assume $\mathcal{G}=\{G_\alpha: \alpha\in\kappa\}$ with $G_\alpha\subset U_\alpha$ for every $\alpha\in\kappa$. Let us first show that

$$S=\{\alpha\in\kappa: \overline{G_\alpha^*}\backslash G_\alpha^*\neq\emptyset\}$$

is a non-stationary subset in κ, where $G_\alpha^*=\cup\{(G_\beta: \beta\in\alpha\}$ for $\alpha\in\kappa$.

Suppose the contrary that S is stationary. Then for every $\alpha\in S$, pick a point $x_\alpha\in\overline{G_\alpha^*}\backslash G_\alpha^*$ and let $s(\alpha)=\sup\{\mu\in\kappa: x_\alpha\in G_\mu\}$ which belongs to κ, since κ is regular. Define a subset C of κ by

$$C=\{\alpha\in\kappa: \beta\in S\cap\alpha \text{ implies } s(\beta)<\alpha\}.$$

Let us check that C is a c.u.b. set in κ. Indeed, if $\alpha\bar{\in}C$, then there is a $\beta\in S\cap\alpha$ with $s(\beta)\geqq\alpha$, so that $(\beta, \alpha]$ is a neighborhood of α which misses C. To see C is unbounded, let $\alpha\in\kappa$ be given, since S is stationary, we may find an $\alpha_1\in S$ such that $\alpha<\alpha_1$. Proceeding by induction, find an $\alpha_{n+1}\in S$ so that

$$\alpha_{n+1}>\sup\{s(\mu): \mu\in S, \mu\leqq\alpha_n\}.$$

Then we obtain an increasing sequence $\{\alpha_n: n\in N\}$ such that $\alpha<\sup\{\alpha_n: n\in N\}\in C$. This concludes that C is a c.u.b. set in κ. Let $S_1=S\cap C$ and for every $\alpha\in S_1$ define $m(\alpha)=\min\{\mu\in\kappa: x_\alpha\in G_\mu\}$ so that $\alpha\leqq m(\alpha)\leqq s(\alpha)$. It follows that

$x_\alpha \notin G_{m(\beta)}$ and $x_\beta \notin G_{m(\alpha)}$ whenever $\alpha, \beta \in S_1$ with $\alpha \neq \beta$. This implies that the set $P = \{x_\alpha : \alpha \in S_1\}$ consists of distinct points of X, and $\{G_{m(\alpha)} : \alpha \in S_1\}$ is an open expansion of P, i.e., $G_{m(\alpha)} \cap P = \{x_\alpha\}$ for every $\alpha \in S_1$. Now for every $\alpha \in S_1$, since $x_\alpha \in \overline{\cup \{G_\beta : \beta \in \alpha\}}$, there is a $\beta(\alpha) \in \alpha$ such that $G_{\beta(\alpha)} \cap G_{m(\alpha)} \neq \emptyset$. By Pressing Down Lemma, there are a $\beta \in \kappa$ and a stationary set $S_2 \subset S_1$ such that $\beta(\alpha) = \beta$ for all $\alpha \in S_2$, consequently $G_\beta \cap G_{m(\alpha)} \neq \emptyset$ for all $\alpha \in S_2$. This contradicts our assumption that $\overline{G}_\beta$ is Lindelöf.

Now take a c.u.b. set C_1 in κ such that $C_1 \cap S = \emptyset$ and thus G_α^* is clopen for every $\alpha \in C_1$. Define H_α for $\alpha \in C_1$ by

$$H_\alpha = G_\alpha^* \setminus \cup \{G_\mu^* : \mu \in C_1 \cap \alpha\}$$

so that $X = \cup \{H_\alpha : \alpha \in C_1\}$. Furthermore for every $\alpha \in C_1$, we have

(*) either $H_\alpha = \emptyset$ or $H_\alpha = G_\alpha^* \setminus G_{\mu(\alpha)}^*$ for some $\mu(\alpha) \in C_1 \cap \alpha$. In fact, if $H_\alpha \neq \emptyset$ then there is an $x \in H_\alpha$, and thus there is $\gamma \in \alpha$ such that $x \in G_\gamma$ and $x \notin G_\mu^*$ for any $\mu \in C_1 \cap \alpha$. This shows $(\gamma, \alpha) \cap C_1 = \emptyset$, because if there is some $\mu \in (\gamma, \alpha) \cap C_1$, then $x \in G_\gamma \subset G_\mu^*$ which is impossible. Define $\mu(\alpha) = \sup \{\mu \leqq \gamma : \mu \in C_1\}$ which belongs to C_1. Then for every $\mu \in C_1 \cap \alpha$, since $(\gamma, \alpha) \cap C_1 = \emptyset$, we must have $\mu \leqq \gamma$. This implies $\mu \leqq \mu(\alpha)$ from which it follows that $H_\alpha = G_\alpha^* \setminus G_{\mu(\alpha)}^*$, i.e., (*) holds. By the definition of κ, we can find, for every $\alpha \in C_1$, a locally finite open cover of $\mathcal{H}_\alpha$ of H_α such that every member of $\mathcal{H}_\alpha$ is contained in some member of $\mathcal{U}$, so that $\cup \{\mathcal{H}_\alpha : \alpha \in C_1\}$ is, since X is now the union of the disjoint clopen collection $\{H_\alpha : \alpha \in C_1\}$, a locally finite open refinement of $\mathcal{U}$. Thus the proof is complete.

In [9], by proving that the Navy's space has property $\mathcal{B}$, Rudin shows that normality plus property $\mathcal{B}$ does not imply paracompactness. But the Navy's space is metacompact [7], in connection with Arhangel'skii and Tall's problem, it is natural to ask if the Navy's space is locally compact. But our Theorem 1 even shows that

COROLLARY 1. *The Navy's space is not locally Ldelöf.*

Also from Theorem 1 the problem of Arhangel'skii and Tall can be stated as follows:

PROBLEM 1. *Does every locally compact normal metacompact space have property* $\mathcal{B}$?

However note that normal metacompact spaces do not necessarily have property $\mathcal{B}$, see Example 4.9 (ii) in [4] or [8] for such a counterexample.

With a modification of proof of Theorem 1 we can prove Arhangel'skii's result mentioned above, even we have

THEOREM 2. *Locally Lindelöf perfectly normal metacompact spaces are paracompact.*

PROOF. Since normal metacompact spaces are shrinking (thus countably paracompact), κ and a point-finite open cover $\mathcal{G}=\{G_\alpha: \alpha\in\kappa\}$ can be defined in the same way as Theorem 1. Clearly we need only consider the case of κ being regular, and it suffices to prove that

$$S=\{\alpha\in\kappa: \overline{\bigcup_{\beta<\alpha} G_\beta}\setminus\bigcup_{\beta<\alpha} G_\beta\neq\emptyset\}$$

is non-stationary.

Suppose indirectly that S is stationary. As in the proof of Theorem 1, define $m(\alpha)\in\kappa$ for every $\alpha\in S$. Without loss of generality, we may assume that there is a $\beta\in\kappa$ such that

$$G_{m(\alpha)}\cap\overline{G}_\beta\neq\emptyset$$

for all $\alpha\in S$.

For every $n\in\omega$ let

$$X_n=\{x\in X: \mathrm{ord}(x, \mathcal{G})\leqq n\}.$$

Then X_n is closed in X. Let

$$S_n=\{\alpha\in S: G_{m(\alpha)}\cap\overline{G}_\beta\cap X_n\neq\emptyset\}$$

so that $S=\bigcup_{n\in\omega}S_n$ and thus there is a minimal $n\in\omega$ with $|S_n|=\kappa$.

Since

$$\overline{G}_\beta\cap X_n=\overline{G}_\beta\cap X_n\cap(X\setminus(\overline{G}_\beta\cap X_{n-1}))\cup(\overline{G}_\beta\cap X_{n-1}),$$

we can assume that

$$G_{m(\alpha)}\cap\overline{G}_\beta\cap X_n\cap(X\setminus(\overline{G}_\beta\cap X_{n-1}))\neq\emptyset$$

for all $\alpha\in S_n$.

Now every point in $\overline{G}_\beta\cap X_n\cap(X\setminus(\overline{G}_\beta\cap X_{n-1}))$ has a neighborhood which meets $\overline{G}_{m(\alpha)}\cap\overline{G}_\beta\cap X_n$ for at most finitely may $\alpha\in S_n$. Since X is perfrect, the set $\overline{G}_\beta\cap X_n\cap(X\setminus(\overline{G}_\beta\cap X_{n-1}))$ is Lindelöf, and hence

$$G_{m(\alpha)}\cap\overline{G}_\beta\cap X_n\cap(X\setminus(\overline{G}_\beta\cap X_{n-1}))\neq\emptyset$$

for at most countably many $\alpha\in S_n$, a contradiction proving S is non-stationary. Thus the proof is complete.

Note that normal submetacompact spaces are shrinking [11], but we do not know whether in Theorem 2 metacompactness can be replaced by submetacompactness, that is

PLOBLEM 2. *Are locally Lindelöf perfectly normal and submetacompact spaces paracompact?*

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