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A RENEWAL THEOREM IN HIGHER DIMENSIONS

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Summary

Let F be a probability distribution on d-dimensional Euclidean space R^d with mean 0 and finite $2[d/2]$-th moment. Let $U\{A\}=\sum_{n=1}^{\infty} F^{n*}\{A\}$, where F^{n*} denotes the n-fold convolution of F and A is a measurable set on R^d. The purpose of this paper is to give an asymtotic expression for $U\{A+x\}$ as $|x|\rightarrow\infty$, in case that F is nonlattice and $d\geqq 3$.

1. Introduction and the statement of the result

Let F be a probability distribution on R^d. For any measurable set A put

$$U\{A\}=\sum_{n=1}^{\infty} F^{n*}\{A\},$$

where F^{n*} denotes the n-fold convolution of F. A random walk associated with F is transient, if for any bounded set A

$$U\{A\}<\infty.$$

For transient random walk of $d\geqq 2$, it is well known

$$\lim_{|x|\rightarrow\infty} U\{A+x\}=0.$$

For lattice distributions it was shown by F. Spitzer [2] and P. Ney and F. Spitzer [1] that for aperiodic d-dimensional random walk ($d\geqq 3$) with mean 0 and finite second moments, such that for each n, $|x|^{d-2}p_n(0,x)\rightarrow 0$ when $|x|\rightarrow\infty$, the Green function has the asymtotic behavior

$$G(0,x)\sim c_d|Q|^{-1/2}(x, Q^{-1}x)^{1-d/2}, \text{ when } |x|\rightarrow\infty.$$

Here $G(0,x)=\sum_{n=1}^{\infty} p_n(0,x)$, $p_n(0,x)$ denotes the probability that a particle starting at the origin will be at the point x at time n, Q is the covariance matrix of $p(0,x)$, Q^{-1} is its inverse, and $|Q|$ is the determinant of Q, and the constants c_d are positive and depend on the dimension.

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Our aim is to obtain the asymtotic expression of the Green function in case of a nonlattice distribution with mean 0 and finite $2[d/2]$-th moment.

Let ϕ denote the characteristic function of F. We say that F is nonlattice if

$$(1.1)\qquad |\phi(y)|<1, \qquad y\in R^d-\{0\}.$$

In our case the quadratic form is given by

$$(1.2)\qquad Q(y)=\int_{R^d}(x, y)^2 F\{dx\}.$$

For $x=(x_1, \cdots, x_d)\in R^d$ and $h>0$, let $P_n(x, h)$ be the measures assigned by F^{n*} to the set

$$\{y=(y_1, \cdots, y_d)|x_k\leqq y_k\leqq x_k+h \text{ for } 1\leqq k\leqq d\}.$$

For a fixed $\nu>0$, we take a bounded set A as

$$A=\{y=(y_1, \cdots, y_d)|0\leqq y_k<\nu \text{ for } 1\leqq k\leqq d\}.$$

Noting that

$$U\{A+x\}=\sum_{n=1}^{\infty} P_n(x, \nu),$$

we get the following

THEOREM. If F satisfies the conditions below;

(1.3) $d\geqq 3$,

(1.4) F is nonlattice,

$$(1.5)\qquad \int x\, F\{dx\}=0,$$

$$(1.6)\qquad \int |x|^{2[d/2]}F\{dx\}<\infty,$$

then

$$(1.7)\qquad U\{A+x\}\sim\frac{\nu^d\Gamma(d/2)}{(d-2)\pi^{d/2}|Q|^{1/2}}(x, Q^{-1}x)^{1-d/2}, \text{ as } |x|\to\infty.$$

Here Q is the covariance matrix of F, Q^{-1} is its inverse, and $|Q|$ is the determinant of Q.

2. Preliminaries

Before the proof we prepare two lemmas.

LEMMA 1. (C. Stone [4]) If F is a nonlattice distribution with mean 0 and second moment, then for each $\nu>0$

(2.1) $$\lim_{n\to\infty}[(2n\pi)^{d/2}P_n(x,\nu)-\nu^d|Q|^{-1/2}e^{-1/2n\,(x,Q^{-1}x)}]=0,$$

uniformly for all $x\in R^d$.

LEMMA 2. If F is nonlattice distribution with mean 0 and $2k$-th ($k\geqq 1$, integer) moment, then

(2.2) $$\lim_{n\to\infty}\left(\frac{|x|}{\sqrt{n}}\right)^{2k}[(2n\pi)^{d/2}P_n(x,\nu)-\nu^d|Q|^{-1/2}e^{-1/2n\,(x,Q^{-1}x)}]=0,$$

uniformly for all $x\in R^d$.

Following C. Stone [4], we define $g(x)$ and $\gamma(x)$, $x\in R^d$ by

$$g(x)=\left(\frac{1}{2\pi A_{2m}}\right)^d\prod_{j=1}^d\left(\frac{\sin x_j}{x_j}\right)^{2m}\quad(m\geqq k+1,\text{ integer fixed}),$$

$$\gamma(y)=\int e^{i(y,x)}g(x)dx$$

$$=\left(\frac{1}{\pi A_{2m}}\right)^d\prod_{j=1}^d\int_0^\infty\cos(y_jx_j)\left(\frac{\sin x_j}{x_j}\right)^{2m}dx_j,$$

where $A_{2m}=\dfrac{1}{\pi}\displaystyle\int_0^\infty\left(\frac{\sin x}{x}\right)^{2m}dx.$

Set $|y|=(y_1^2+\cdots+y^2)^{1/2}$ and $||y||=\max_{1\leq j\leq d}|y_j|$. $\gamma(y)$ is a function of class C^{2k} on R^d and $\gamma(y)\equiv 0$ on $||y||\geqq 2m$. For $a>0$ set $g_a(x)=a^{-d}g(a^{-1}x)$ and $\gamma_a(y)=\gamma_a(ay)$. Then

$$\int g_a(x)dx=1,$$

$$\int e^{i(x,y)}g_a(x)dx=\gamma_a(y).$$

Now $P_n(\cdot,h)$ is integrable and

$$\int e^{i(x,y)}P_n(\sqrt{n}\,x,\sqrt{n}\,h)dx=h^d\prod_{j=1}^d\frac{1-e^{-ihy_j}}{ihy_j}\phi^n\left(\frac{y}{\sqrt{n}}\right).$$

To complete the proof of Lemma 2 we need the following two propositions.

PROPOSITION 1.

(2.3) $$\lim_{n\to\infty}\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]=\Delta^k[e^{-1/2\,Q(y)}],\ y\in R^d;$$

(2.4) for an arbitrary fixed $B>0$

$$\left|\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]\right|\leqq\text{constant},\ |y|\leqq B;$$

$$\left|\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]\right| \leqq \text{constant } |y|^{2k}\left|\phi^{n-2k}\left(\frac{y}{\sqrt{n}}\right)\right|, \quad |y|\geqq 1; \tag{2.5}$$

$$\left|\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]\right| \leqq \text{constant } e^{-1/4\, Q(y)}, \ |y|\leqq\varepsilon\sqrt{n}; \tag{2.6}$$

where Δ^k is the k-th ($k=0, 1, 2, \cdots$) iteration of Laplace operator Δ, the constant in (2.4) may depend on B but is independent of n and y, the constants in (2.5) and (2.6) are both independent of n and y, and as to $\varepsilon>0$ in (2.6) see Appendix.

We can show Proposition 1 in the same way of P. Ney and F. Spitzer [1] and [2].

Using Proposition 1, we next prove the following proposition.

PROPOSITION 2. Let

$$V_n(x, h, a)=|x|^{2k}\int g_a(x-y)P_n(\sqrt{n}\, y, \sqrt{n}\, h)dy. \tag{2.7}$$

For arbitrary fixed positive numbers ν and λ, set $h=\nu/\sqrt{n}$ and $a=\lambda/\sqrt{n}$. Then

$$\begin{aligned} V_n(x, h, a)&=(-1)^k\left(\frac{\nu}{2\pi\sqrt{n}}\right)^d\int e^{-i(x,y)}\Delta^k[e^{-1/2\, Q(y)}]dy+o(n^{-d/2}) \\ &=|x|^{2k}\frac{\nu^d}{(2\pi n)^{d/2}|Q|^{1/2}}\, e^{-1/2\,(x, Q^{-1}x)}+o(n^{-d/2}). \end{aligned} \tag{2.8}$$

PROOF. Put $\prod_{j=1}^{d}(1-e^{-ihy_j})(ihy_j)^{-1}=f_h(y)$, then we have by Fubini's theorem and Green's theorem

$$\begin{aligned} V_n(x, h, a)&=\left(\frac{h}{2\pi}\right)^d|x|^{2k}\int_{||y||\leqq 2ma^{-1}} e^{-i(x,y)}\gamma_a(y)f_h(y)\phi\left(\frac{y}{\sqrt{n}}\right)dy \\ &=\left(\frac{h}{2\pi}\right)^d|x|^{2k}\int_{|y|\leqq 4\sqrt{d}\, ma^{-1}} e^{-i(x,y)}\gamma_a(y)f_h(y)\phi^n\left(\frac{y}{\sqrt{n}}\right)dy \\ &=(-1)^k\left(\frac{h}{2\pi}\right)^d\int_{|y|\leqq 4\sqrt{d}\, ma^{-1}} \Delta^k[e^{-i(x,y)}]\gamma_a(y)f_h(y)\phi^n\left(\frac{y}{\sqrt{n}}\right)dy \\ &=(-1)^k\left(\frac{h}{2\pi}\right)^d\int_{|y|\leqq 4\sqrt{d}\, ma^{-1}} e^{-i(x,y)}\Delta^k\left[\gamma_a(y)f_h(y)\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]dy. \end{aligned}$$

Let

$$I_1=\int_{|y|\leqq B} e^{-i(x,y)}\left(\gamma_a(y)f_h(y)\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]-\Delta^k[e^{-1/2\, Q(y)}]\right)dy;$$

$$I_2=\int_{B<|y|\leq\varepsilon\sqrt{n}} e^{-i(x,y)}\gamma_a(y)f_h(y)\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]dy\,;$$

$$I_3=\int_{\varepsilon\sqrt{n}<|y|\leq 4\sqrt{d}\,ma^{-1}} e^{-i(x,y)}\gamma_a(y)f_h(y)\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]dy\,;$$

$$I_4=-\int_{|y|>B} e^{-i(x,y)}\Delta^k[e^{-1/2\,Q(y)}]dy\,;$$

$$I_5=\left(\int_{|y|\leq B}+\int_{B<|y|\leq\varepsilon\sqrt{n}}+\int_{\varepsilon\sqrt{n}<|y|\leq 4\sqrt{d}\,ma^{-1}}\right)e^{-i(x,y)}H(y)dy\,;$$

where $H(y)=\Delta^k\left[\gamma_a(y)f_h(y)\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]-\gamma_a(y)f_h(y)\Delta^k\left[\phi^n\left(\frac{y}{\sqrt{n}}\right)\right]$. $H(y)$ is a polynomial of partial derivatives of $\gamma_a(y)f_h(y)\phi^n\left(\frac{y}{\sqrt{n}}\right)$ of the first $2k$ order, each term involving at least the first partial derivatives of $\gamma_a(y)f_h(y)$. Note that

$$(I_1+I_2+I_3+I_4+I_5)\times(-1)^k\left(\frac{h}{2\pi}\right)^d$$
$$=V_n(x,h,a)-(-1)^k\left(\frac{h}{2\pi}\right)^d\int e^{-i(x,y)}\Delta^k[e^{-1/2\,Q(y)}]dy.$$

In order to prove (2.8) it is sufficient to show that

$$\lim_{n\to\infty} I_m=0\ (1\leqq m\leqq 5)\ \text{uniformly for all}\ x\in R^d.$$

Since $|\gamma_a(y)|\leqq 1$, $|f_h(y)|\leqq 1$, and $\lim_{n\to\infty}\gamma_a(y)=\lim_{n\to\infty}f_h(y)=1$, it follows from (2.3) and (2.4) that

(2.9) $$\lim_{n\to\infty} I_1=0\quad\text{uniformly for all}\ x\in R^d.$$

Using (2.6) for each given $\varepsilon_1>0$, we can choose $B>0$ independent of x such that

(2.10) $$|I_2|<\varepsilon_2.$$

Nextly we prove that there exists a positive constant δ independent of n (but may depend on ε, m, and λ) such that

(2.11) $$|I_3|\leqq\text{constant}\ n^{d/2+k}(1-\delta)^{n-2k}.$$

Indeed since we can choose $\delta>0$ such that

$$\left|\phi\left(\frac{y}{\sqrt{n}}\right)\right|<1-\delta\ \text{for}\ \varepsilon\sqrt{n}<|y|\leqq 4m\sqrt{d}\,\lambda^{-1}\sqrt{n}$$

Using (2.5) we have

$$|I_3| \leqq \text{constant } (1-\delta)^{n-2k} \int_{|y| \leqq 4m\sqrt{d}\lambda^{-1}\sqrt{n}} |y|^{2k} dy.$$

Now choosing B sufficiently large we get

(2.12) $$|I_4| < \varepsilon_2$$

for a given fixed $\varepsilon_2 > 0$. Finally we prove

(2.13) $$\lim_{n\to\infty} I_5 = 0 \quad \text{uniformly for all } x \in R^d.$$

Note that

$$\lim_{n\to\infty} \frac{\partial^{|r|}}{\partial y_1^{r_1} \cdots \partial y_d^{r_d}} (\gamma_a(y) f_h(y)) = 0 \quad \text{if } |r| \neq 0.$$

Then it follows that

$$\lim_{n\to\infty} \int_{|y| \leqq B} e^{-i(x,y)} H(y) dy = 0 \quad \text{uniformly for all } x \in R^d,$$

because the derivatives of $\phi^n\left(\frac{y}{\sqrt{n}}\right)$ and $\gamma_a(y) f_h(y)$ are uniformly bounded on every compact set. Furthermore using the estimations similar to (2.5) and (2.6) for the derivative of $\phi^n\left(\frac{y}{\sqrt{n}}\right)$, we have

$$\lim_{n\to\infty} \int_{B<|y| \leqq \varepsilon\sqrt{n}} e^{-i(x,y)} H(y) dy = 0, \quad \lim_{n\to\infty} \int_{\varepsilon\sqrt{n}<|y| \leqq 4ma^{-1}\sqrt{d}} e^{-i(x,y)} H(y) dy = 0$$

uniformly for all $x \in R^d$. That completes the proof.

PROOF OF LEMMA 2. The proof of Lemma 2 is as same as C. Stone's [4]. But for completeness we repeat it here.

Put $p_k(x) = |x|^{2k} (2\pi)^{-d/2} |Q|^{-1/2} e^{-1/2\,(x, Q^{-1}x)}$ and $p = \max_{x \in R^d} p_k(x)$. Since $p_k(x)$ is uniformly continuous, there is an $h_1 > 0$ such that $|p_k(x) - p_k(y)| \leqq 1/4\,\varepsilon$ if $||x-y|| \leqq h_1$. We choose a $\delta > 0$ such that $(1+2\delta)^d \leqq 4/3$, $(1+2\delta)^d - 1 \leqq \varepsilon_1$, $(1-2\delta)^d - 1 \geqq -\varepsilon_1$, and

$$\int_{||x||>1/\delta} g(x) dx \leqq \varepsilon_2,$$

where ε_1 and ε_2 are positive numbers satisfying

$$\left(p + \varepsilon_1 p + \frac{1}{2}\varepsilon\right)(1-\varepsilon_2)^{-1} - p \leqq \varepsilon$$

and

$$\varepsilon_1 p + \varepsilon_2(p + \varepsilon) \leqq \frac{1}{2}\varepsilon.$$

Set $i=(1,\cdots,1)\in R^d$. By Proposition 2 we can find $N>0$ such that for $n\geqq N$ and $x\in R^d$

$$V_n\left(x-\frac{\delta\nu}{\sqrt{n}}i, \frac{(1+2\delta)\nu}{\sqrt{n}}, \frac{\delta^2\nu}{\sqrt{n}}\right)\leqq\left(\frac{(1+2\delta)\nu}{\sqrt{n}}\right)^d p\left(x-\frac{\delta\nu}{\sqrt{n}}i\right)+\varepsilon\left(\frac{\nu}{\sqrt{n}}\right)^d \tag{2.14}$$

$$\leqq\left(\frac{(1+2\delta)\nu}{\sqrt{n}}\right)^d\left(p_k(x)+\frac{1}{4}\varepsilon\right)+\varepsilon\left(\frac{\nu}{\sqrt{n}}\right)^d$$

$$\leqq\left(\frac{\nu}{\sqrt{n}}\right)^d\left(p_k(x)+\varepsilon_1 p+\frac{1}{2}\varepsilon\right),$$

and

$$V_n\left(x+\frac{\delta\nu}{\sqrt{n}}i, \frac{(1-2\delta)\nu}{\sqrt{n}}, \frac{\delta^2\nu}{\sqrt{n}}\right)\geqq\left(\frac{(1-2\delta)\nu}{\sqrt{n}}\right)^d p_k\left(x+\frac{\delta\nu}{\sqrt{n}}i\right)-\frac{1}{4}\varepsilon\left(\frac{\nu}{\sqrt{n}}\right)^d \tag{2.15}$$

$$\geqq\left(\frac{(1-2\delta)\nu}{\sqrt{n}}\right)^d\left(p_k(x)-\frac{1}{4}\varepsilon\right)-\frac{1}{4}\varepsilon\left(\frac{\nu}{\sqrt{n}}\right)^d$$

$$\geqq\left(\frac{\nu}{\sqrt{n}}\right)^d\left(p_k(x)-\varepsilon_1 p-\frac{1}{2}\varepsilon\right).$$

Now

$$P_n\left(\sqrt{n}\left(x-\frac{\delta\nu}{\sqrt{n}}i-y\right), (1+2\delta)\nu\right)\geqq P_n(\sqrt{n}\,x, \nu), \quad ||y||\leqq\frac{\delta\nu}{\sqrt{n}} \tag{2.16}$$

and

$$P_n\left(\sqrt{n}\left(x+\frac{\delta\nu}{\sqrt{n}}i-y\right), (1-2\delta)\nu\right)\leqq P_n(\sqrt{n}\,x, \nu), \quad ||y||\leqq\frac{\delta\nu}{\sqrt{n}}. \tag{2.17}$$

By (2.16) we get

$$V_n\left(x-\frac{\delta\nu}{\sqrt{n}}i, \frac{(1+2\delta)\nu}{\sqrt{n}}, \frac{\delta^2\nu}{\sqrt{n}}\right) \tag{2.18}$$

$$\geqq|x|^{2k}\int_{||y||\leqq\delta\nu/\sqrt{n}} g_{\frac{\delta^2\nu}{\sqrt{n}}}(y)P_n\left(\sqrt{n}\left(x-\frac{\delta\nu}{\sqrt{n}}i-y\right), (1+2\delta)\nu\right)dy$$

$$\geqq|x|^{2k}P_n(\sqrt{n}\,x, \nu)\int_{y\leqq\delta\nu/\sqrt{n}} g_{\frac{\delta^2\nu}{\sqrt{n}}}(y)dy$$

$$\geqq(1-\varepsilon_2)|x|^{2k}P_n(\sqrt{n}\,x, \nu).$$

Therefore by (2.14) and (2.18) we get

$$|x|^{2k}P_n(\sqrt{n}\,x, \nu)\leqq\left(\frac{\nu}{\sqrt{n}}\right)^d(p_k(x)+\varepsilon_1 p+\varepsilon)(1-\varepsilon_2)^{-1} \tag{2.19}$$

$$\leqq\left(\frac{\nu}{\sqrt{n}}\right)^d(p_k(x)+\varepsilon).$$

On the other hand

$$V_n\left(x+\frac{\delta\nu}{\sqrt{n}}i, \frac{(1-2\delta)\nu}{\sqrt{n}}, \frac{\delta^2\nu}{\sqrt{n}}\right)$$

$$=|x|^{2k}\int_{y\leq\delta\nu/\sqrt{n}} g_{\frac{\delta^2\nu}{\sqrt{n}}}(y)P_n\left(\sqrt{n}\left(x+\frac{\delta\nu}{\sqrt{n}}i-y\right),(1-2\delta)\nu\right)dy$$

$$+|x|^{2k}\int_{y\geq\delta\nu/\sqrt{n}} g_{\frac{\delta^2\nu}{\sqrt{n}}}(y)P_n\left(\sqrt{n}\left(x+\frac{\delta\nu}{\sqrt{n}}i-y\right),(1-2\delta)\nu\right)dy$$

$$=J_1+J_2.$$

By (2.17) we get

$$J_1\leq|x|^{2k}P_n(\sqrt{n}\,x,\nu).$$

Noting that the equality

$$V_n\left(x+\frac{\delta\nu}{\sqrt{n}}i, \frac{(1-2\delta)\nu}{\sqrt{n}}, \frac{\delta^2\nu}{\sqrt{n}}\right)$$

$$=\left(\frac{(1-2\delta)\nu}{\sqrt{n}}\right)^d p_k\left(x+\frac{\delta\nu}{\sqrt{n}}i\right)+o(n^{-d/2})$$

$$=\left(\frac{\nu}{\sqrt{n}}\right)^d p_k(x)+o(n^{-d/2})$$

holds by (2.8), we can see that

$$J_2\leq\left(\frac{\nu}{\sqrt{n}}\right)^d(p+\varepsilon)\varepsilon_2.$$

Therefore we get

$$V_n\left(x+\frac{\delta\nu}{\sqrt{n}}i, \frac{(1-2\delta)\nu}{\sqrt{n}}, \frac{\delta^2\nu}{\sqrt{n}}\right) \tag{2.20}$$

$$\leq|x|^{2k}P_n(\sqrt{n}\,x,\nu)+\left(\frac{\nu}{\sqrt{n}}\right)^d(p+\varepsilon)\varepsilon_2.$$

Thus by (2.15) and (2.20) we obtain

$$|x|^{2k}P_n(\sqrt{n}\,x,\nu)\geq\left(\frac{\nu}{\sqrt{n}}\right)^d(p_k(x)-\varepsilon). \tag{2.21}$$

Since ε is independent of x we may replace x by $x/\sqrt{n}$ in the inequalities (2.19) and (2.21). Thus the proof of Lemma 2 is complete.

PROOF OF THE THEOREM. By Lemma 1 and Lemma 2 we have for $x\neq0$

(2.22) $$|x|^{d-2}P_n(x, \nu)=|x|^{d-2}(2n\pi)^{-d/2}\nu^d|Q|^{-1/2}e^{-1/2n(x, Qx^{-1}x)} + |x|^{d-2}n^{-d/2}E_1(n, x),$$

and

(2.23) $$|x|^{d-2}P_n(x, \nu)=|x|^{d-2}(2n\pi)^{-d/2}\nu^d|Q|^{-1/2}e^{-1/2n(x, Q^{-1}x)} + |x|^{d-2-2k}n^{-d/2+k}E_2(n, x),$$

respectively. Here $k=[d/2]$. Both of the error terms $E_1(n, x)$ and $E_2(n, x)$ have the property of tending to zero as $n\rightarrow\infty$, uniformly in x.

Let us investigate the asymptotic behavior of

$$|x|^{d-2}U\{A+x\}=\sum_{n=1}^{\infty}|x|^{d-2}P_n(x, \nu)$$

as $|x|\rightarrow\infty$. Set

$$S(x)=(2\pi)^{-d/2}\nu^d|Q|^{-1/2}|x|^{d-2}\sum_{n=1}^{\infty}n^{-d/2}e^{-1/2n(x, {}^{-1}x)}.$$

Put $(x, Q^{-1}x)^{-1}=\Delta$, then

$$S(x)=\frac{(2\pi)^{-d/2}\nu^d|Q|^{-1/2}|x|^{d-2}}{(x, Q^{-1}x)^{d/2-1}}\sum_{n=1}^{\infty}(n\Delta)^{-d/2}e^{-(2n\Delta)^{-1}}\Delta.$$

Since $\Delta\rightarrow 0$ as $|x|\rightarrow\infty$, the sum on the righthand side tends the convergent improper Riemann integral

$$\int_0^{\infty}t^{-d/2}e^{-1/2t}dt=\frac{2^d\Gamma(d/2)}{d-2}.$$

Therefore

(2.24) $$S(x)\sim\nu^d\pi^{-d/2}(d-2)^{-1}\Gamma(d/2)|Q|^{-1/2}(x, Q^{-1}x)^{1-d/2}|x|^{d-2} \text{ as } |x|\rightarrow\infty.$$

We now only have to explain why the error terms do not contribute to our result. We shall use (2.23) for the range $1\leqq n\leqq[|x|^2]$. Since the contribution of the principal terms in (2.24) is positive, we have to show that

(2.25) $$\lim_{|x|\rightarrow\infty}|x|^{d-2-2k}\sum_{n=1}^{[|x|^2]}n^{-d/2+k}|E_2(n, x)| + \lim_{|x|\rightarrow\infty}|x|^{d-2}\sum_{n=[|x|^2]+1}^{\infty}n^{-d/2}|E_1(n, x)|=0.$$

From (1.6) any finite number of terms in the first sum is zero. We choose M so large that $\sup|E_2(n, x)|<\varepsilon$ whenever $n\geqq M$. Then

$$|x|^{d-2-2k}\sum_{n=M}^{[|x|^2]}n^{-d/2+k}|E_2(n, x)|$$

$$\leqq \varepsilon |x|^{d-2-2k} \sum_{n=M}^{[|x|^2]} n^{-d/2+k}$$

$$\leqq \varepsilon |x|^{d-2-2k} \sum_{n=1}^{[|x|^2]} n^{-d/2+k} \leqq \varepsilon k_1$$

for some positive k_1 independent of ε and x. Since ε is arbitrary. the first limit in (2.25) is zero. The second limit is also zero since

$$|x|^{d-2} \sum_{n=[|x|^2]+1}^{\infty} n^{-d/2} |E_1(n, x)| \leqq |x|^{d-2} \sup_{n>[|x|^2]} |E_1(n, x)| \sum_{n=[|x|^2]+1}^{\infty} n^{-d/2}$$

$$\leqq k_2 \sup_{n>[|x|^2)} |E_1(n, x)|,$$

where k_2 is a positive constant independent of x. This completes the proof.

Appendix

(2.3) can be shown by expanding the derivative on the left and then taking limits $n \to \infty$.

Since

$$\left| \frac{\partial}{\partial y_j} \phi\left(\frac{y}{\sqrt{n}}\right) \right| = \frac{1}{\sqrt{n}} \left| \int \left(e^{i\left(x, \frac{y}{\sqrt{n}}\right)} - 1\right) x_j F\{dx\} \right|$$

$$\leqq \frac{|y|}{n} \int |x|^2 F\{dx\},$$

and for $|r| \geqq 2$

$$\left| \frac{\partial^{|r|}}{\partial y_1^{r_1} \cdots \partial y_d^{r_d}} \phi\left(\frac{y}{\sqrt{n}}\right) \right| \leqq n^{-r/2} \int |x_1^{r_1} \cdots x_d^{r_d}| F\{dx\},\ r_1 + \cdots + r_d = |r|,$$

we get (2.4) and (2.5).

Next, using the fact that

$$\frac{1-\phi(y)}{Q(y)} = \frac{1}{2} \quad \text{as } |y| \to 0 \quad \text{(see P7.7 of [2])},$$

we see that ε can be chosen sufficiently small so that

$$\left| \phi^n\left(\frac{y}{\sqrt{n}}\right) \right| \leqq e^{-1/4\, Q(y)} \qquad \text{for} \quad |y| \leqq \varepsilon \sqrt{n}.$$

Then we have (2.6) immediately.

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