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KOSTANT'S WEIGHTING FACTOR IN MACDONALD'S IDENTITIES

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I. Introduction

Macdonald's identities can be interpreted in terms of the fundamental solution, $H(x, t)$, of the heat equation on a compact Lie group G. In the notation of [2] this is

$$H(a, t)=e^{-i\pi kt/12}\eta(t)^{k}. \tag{1.1}$$

Equation (1.1) can be obtained in two ways. One due to Kostant [3] and the other due to Van Asch [5]. The purpose of this paper is to point out that a key step in each of these derivations is in fact the same. This is done in the proof of Theorem 1.1.

THEOREM 1.1. *Let P be the lattice of weights and P^* its dual. If ρ is half the sum of the positive roots, λ a dominant weight such that $\lambda=s\rho-\rho+\mu$ for $s\in W$, the Weyl group, $\sigma: \mathfrak{t}\to\mathfrak{t}^*$ the isomorphism induced by the Killing form, and $\mu\in(1/2)\sigma P^*$, then $\chi_\lambda(a)=\det s$. For all other λ, $\chi_\lambda(a)=0$, where a is an element "principal of type ρ".*

The derivation of Kostant involves rewriting Macdonald's original identities in terms of the highest weights of representations. In doing so the term $\chi_\lambda(a)$ was introduced. Here $\chi_\lambda(a)$ is the value of the character with highest weight λ on a special point a called "principal of type ρ". It is clear from Kostant's work that $\chi_\lambda(a)$ is either $+1$, -1, or 0.

Meanwhile, Van Asch [5] gave a direct proof of Macdonald's identities using the Poisson summation formula. Fegan, in [2], related this to the heat equation, a step involving writing a sum over a full lattice as a sum over the highest weights of representation. In both cases there is the need to reduce the sum over a lattice P to a sum over a sublattice. The point of this paper is to show that the changes of Kostant and Van Asch are essentially the same.

While the formula of Theorem 1.1 is essentially contained in [3] the proof

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follows the lines of the reduction of the summation in [5]. We prove the theorem in the next section. Finally we calculate $\chi_\lambda(a)$ for all the rank two two groups. In the rank one case where $\lambda\in(1/2)Z$ and $\lambda\geq 0$ the result

$$\chi_\pi(a)=\begin{cases}(-1)^\lambda & \lambda\in Z\\ 0 & \lambda\notin Z\end{cases}$$

is well known and easy to prove.

II. Proof of the theorem.

We start by reviewing the notation and terminology. Let G be a compact, semi-simple and simply connected Lie group which is simple modulo its center. Pick a maximal torus T in G, and let $\mathfrak{t}$ be the Lie algebra of T. The negative of the Killing form gives an inner product $\langle\,,\rangle$ on $\mathfrak{t}$ and hence a isomorphism $\sigma:\mathfrak{t}\to\mathfrak{t}^*$. The roots of G are elements $\alpha_i\in\mathfrak{t}^*$. Half of the roots are positive and half are negative. Let $\rho=(1/2)\sum_{\alpha>0}\alpha$ summed over the positive roots and the element a, "principal of type ρ" is given by $a=\exp(2\pi\sigma^{-1}(2\rho))$. The Weyl group is denoted by W and $P\subset\mathfrak{t}^*$ is the lattice of weights. Its dual is $P^*=\{x\in\mathfrak{t}: y(x)\in Z \text{ for all } y\in P\}$.

To study characters we introduce the formal character

$$f_x(\lambda)=\frac{\sum_{s\in W}(-1)^s\exp(2\pi i\langle s\lambda, x\rangle)}{\sum_{s\in W}(-1)^s\exp(2\pi i\langle s\rho, x\rangle)} \tag{2.1}$$

for $x\in\mathfrak{t}^*$ and λ a weight. Then the Weyl character formula is

$$\chi_\lambda(\exp(2\pi\sigma^{-1}(x)))=f_x(\lambda+\rho) \tag{2.2}$$

when λ is a highest weight. This is compatible with the notation of [2].

LEMMA 2.1. *If* $\lambda_1-\lambda_2\in(1/2)\sigma P^*$ *then* $f_{2\rho}(\lambda_1)=f_{2\rho}(\lambda_2)$.

PROOF. By hypothesis $\lambda_1-\lambda_2=\mu$ for some $\mu\in(1/2)\sigma P^*$. Let $s\in W$, the Weyl group. Then

$$\begin{aligned}\langle s\lambda_1, \rho\rangle&=\langle s\lambda_2, 2\rho\rangle+\langle s\mu, 2\rho\rangle\\ &=\langle s\lambda_2, 2\rho\rangle \operatorname{Mod} Z\end{aligned}$$

since $s\mu\in(1/2)\sigma P^*$, for all $s\in W$ we have $\langle s\mu, 2\rho\rangle\in Z$ by definition of P^*. Remember that $\rho\in P$.

Now if a is "principal of type ρ", we have

$$\chi_\lambda(a)=f_{2\rho}(\lambda) \qquad \text{thus } \chi_{\lambda_1}(a)=\chi_{\lambda_2}(a)$$

since $a=\exp(2\pi\sigma^{-1}(2\rho))$.

Now we use the following facts which are found in [5]:

(1) There is a unique orbit in $P/(1/2)\sigma P^*$ on which W acts transitively and this orbit contains a coset with representative ρ.

(2) If $\mu\in P$ defines a coset $\bar{\mu}$ in $P/(1/2)\sigma P^*$ such that the stabilizer of $\bar{\mu}$ under W is nontrivial, then there is an $s\in W$ such that $\det s=-1$ and $s\bar{\mu}=\bar{\mu}$.

From (2) it follows that if $\lambda\in P$ such that $\bar{\lambda}$ has a nontrivial stabilizer then $f_{2\rho}(\lambda)=0$. Thus only the orbit involving ρ gives nonzero results. Hence

$$\chi_\lambda(a)=f_{2\rho}(\lambda+\rho)=\det s(f_{2\rho}(\rho))=\det s$$

for $\lambda+\rho=s\rho+\mu$, $s\in W$ and $\mu\in(1/2)\sigma P^*$.

III. Tables of results for the rank two groups.

We consider the rank two groups: A_2, B_2, G_2. For each group we have two fundamental weights σ and τ. Then $\rho=\sigma+\tau$ and we can use $a=2\rho$. In each table we let $\lambda=i\sigma+j\tau$, where $i, j=0, 1, \cdots$. Thus the entry in the ith column and the jth row, counting from the lower left hand corner is the value of $\chi_\lambda(a)$. The details are taken from [1].

(1) The group A_2. The negative of the Killing forms gives:

$$\langle\sigma, \sigma\rangle=\langle\tau, \tau\rangle=1/9. \tag{3.1}$$

and

$$\langle\sigma, \tau\rangle=1/18. \tag{3.2}$$

The reader can easily see a 3×3 block which is repeated.

0	0	0	0	0	0	0	0	0
0	−1	0	0	−1	0	0	−1	0
1	0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0	0
0	−1	0	0	−1	0	0	−1	0
1	0	0	1	0	0	1	0	0

(2) The group B_2. The negative of the Killing from gives

$$\langle\sigma, \sigma\rangle=1 \tag{3.3}$$

$$\langle\sigma, \tau\rangle=\langle\tau, \tau\rangle=\frac{1}{2} \tag{3.4}$$

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0
\end{matrix}$$

As the reader can see there is a 6×6 block which is repeated.

(3) The group G_2. Here the negative of the Killing form gives

$$\langle\sigma, \sigma\rangle=1/12 \tag{3.5}$$

$$\langle\tau, \tau\rangle=1/4 \tag{3.6}$$

$$\langle\sigma, \tau\rangle=1/8 \tag{3.7}$$

$$\begin{matrix}
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{matrix}$$

Here there is a 12×4 block which is repeated.

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