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IDEALS ON ω WHICH ARE OBTAINED FROM HAUSDORFF-GAPS

By

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Let $\mathcal{G}$ be a Hausdorff gap in ${}^{\omega}\omega$. Hart and Mill [2] defined the ideal $I_{\mathcal{G}}$ which is the family of all subsets of ω whose restriction of $\mathcal{G}$ is filled. In this paper, we shall show two results (Theorems 1, 6) about these ideals.

Our notions and terminology follow the usual use in set theory. Let X be a subset of ω and f, g functions from X to ω. g dominates f (denoted by $f \prec g$), if $\{n \in X;\ g(n) \leqq f(n)\}$ is finite. Let κ and λ be infinite cardinals. A pair of sequence $\langle\langle f_{\alpha} | \alpha<\kappa\rangle | \langle g_{\beta} | \beta<\lambda\rangle\rangle$ is called a (κ, λ)-gap, if the following (1), (2) are satisfied.

(1) $f_{\alpha}, g_{\beta} : \omega \to \omega$, for any $\alpha<\kappa$, $\beta<\lambda$.

(2) $f_{\alpha} \prec f_{\gamma} \prec g_{\delta} \prec g_{\beta}$, for any $\alpha<\gamma<\kappa$, $\beta<\delta<\lambda$.

A (κ, λ)-gap $\langle\langle f_{\alpha} | \alpha<\kappa\rangle | \langle g_{\beta} | \beta<\lambda\rangle\rangle$ is unfilled, if there does not exist a function $h : \omega \to \omega$ such that, for all $\alpha<\kappa$, $\beta<\lambda$, $f_{\alpha} \prec h \prec g_{\beta}$. We call an unfilled (ω_1, ω_1)-gap a Hausdorff gap (H-gap). The following fact is well-known.

FACT. For any regular cardinals κ and λ with $(\kappa, \lambda) \neq (\omega_1, \omega_1)$, there exists a generic extension W such that W preserves all cardinals and, in W, there are no unfilled (κ, λ)-gap.

In contrast to this fact, the following theorem holds about H-gaps.

THEOREM (Hausdorff [1, Theorem 4.3]). *There is an H-gap.*

Let $\mathcal{G} = \langle\langle f_{\alpha} | \alpha<\omega_1\rangle | \langle g_{\alpha} | \alpha<\omega_1\rangle\rangle$ be a (ω_1, ω_1)-gap. Following [2], we define the ideal $I_{\mathcal{G}}$ by

$$I_{\mathcal{G}} = \{x \subset \omega;\ \exists h : x \to \omega \forall \alpha<\omega_1 (f_{\alpha} \upharpoonright x \prec h \prec g_{\alpha} \upharpoonright x)\}.$$

It is easy to see that

$$\omega \in I_{\mathcal{G}} \text{ if and only if } \mathcal{G} \text{ is filled,}$$

$$\mathrm{Fin} = \{x \subset \omega;\ x \text{ is finite}\} \subset I_{\mathcal{G}}.$$

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In this paper, we shall show two result about these ideals $I_{\mathcal{G}}$.

THEOREM 1. *Assume the Continume Hypothesis (CH). For any ideal I with* $\mathrm{Fin}\subset I$, *there exists an (ω_1, ω_1)-gap $\mathcal{G}$ such that $I=I_{\mathcal{G}}$.*

We need the several lemmas and corollaries to show Theorem 1. Let $\Gamma=\{h;\ \exists x\subset\omega\,(h:\ x\to\omega)\}$. For any $f,\ g\in\Gamma$, $f\ll g$ means that, for any $k<\omega$, $\{n\in \mathrm{dom}(f)\cap\mathrm{dom}(g);\ g(n)<f(n)+k\}$ is finite. For any $X,\ Y\subset\Gamma$, $X\ll Y$ means that, for all $f\in X$ and $g\in Y$, $f\ll g$.

LEMMA 2. *Let X, Y be countable subsets of ${}^{\omega}\omega$, $X\neq\emptyset$, and $X\ll Y$. Then there exists an $h:\ \omega\to\omega$ such that $X\ll\{h\}\ll Y$.*

PROOF. The case of $Y=\emptyset$ is clear. So, we may asumme that $Y\neq\emptyset$. Take an enumeration $\langle f_j|j<\omega\rangle$ of X, and an enumeration $\langle g_j|j<\omega\rangle$ of Y. For any $k<\omega$, since $X\ll Y$, it holds that

$$\lim_{n\to\omega}(\min\{g_i(n);\ i\leqq k\}-\max\{f_j(n);\ j\leqq k\})=\omega.$$

So, we can take a sequence of natural numbers n_k (for $k<\omega$) such that

$$n_k<n_{k+1}$$

and

$$\forall n\in[n_k,\ n_{k+1})(\min\{g_i(n);\ i\leqq k\}-\max\{f_j(n);\ j\leqq k\}\geqq 2k).$$

Define $h:\ \omega\to\omega$ by

$$h(n)=\max\{f_j(n);\ j\leqq k\}+k,\ \text{if } n\in[n_k,\ n_{k+1}).$$

It is easy to see that $X\ll\{h\}\ll Y$. □

COROLLARY 3. *Let $X, Y\subset\Gamma$. Suppose that $|X|\leqq\omega$, $|Y|\leqq\omega$, $X\ll Y$, and $\exists f\in X(f:\ \omega\to\omega)$. Then, there exists an $h:\ \omega\to\omega$ such that $X\ll\{h\}\ll Y$.*

PROOF. For each $f\in X$, define $f_*:\ \omega\to\omega$ by

$$f_*(n)=\begin{cases} f(n), & \text{if } n\in\mathrm{dom}(f),\\ 0, & \text{otherwise}.\end{cases}$$

By Lemma 2, there exists $g:\ \omega\to\omega$ such that $\{f_*;\ f\in X\}\ll\{g\}$. For each $f\in Y$, define $f^*:\ \omega\to\omega$ by

$$f^*(n)=\begin{cases} f(n), & \text{if } n\in\mathrm{dom}(f),\\ g(n), & \text{otherwise}.\end{cases}$$

Then, since $\{f_*;\ f\in X\}\ll\{f^*;\ f\in Y\}$, there exists $h:\ \omega\to\omega$ such that $\{f_*;\ f\in$

$X\}\ll\{h\}\ll\{f^*;\ f\in Y\}$, by Lemma 2. This h is as required. □

COROLLARY 4. *Let X, Y, Z be countable subsets of Γ such that $X\ll Z$, $Z\ll Y$, $X\ll Y$, and $\exists f\in X(f:\omega\to\omega)$. Then, there exist g, $h:\omega\to\omega$ such that $X\ll\{h\}\ll Z$ and $Z\ll\{g\}\ll Y$ and $h\ll g$.*

PROOF. Since $X\ll Z\cup Y$, by Corollary 3, we can take $h:\omega\to\omega$ such that $X\ll\{h\}\ll Z\cup Y$. Then $Z\cup\{h\}\ll Y$ and we can take $g:\omega\to\omega$ such that $Z\cup\{h\}\ll\{g\}\ll Y$. □

LEMMA 5. *Let b be an infinite subset of ω and $s: b\to\omega$. Suppose that X, $Y\subset{}^{\omega}\omega$ and $Z\subset\Gamma$ satisfy that*

(2.1) $\quad X\neq\emptyset\ \&\ |X|\leqq\omega\ \&\ |Y|\leqq\omega\ \&\ |Z|\leqq\omega\ \&\ X\ll Y\ \&\ X\ll Z\ll Y,$

(2.2) $\quad \forall h\in Z(b\cap\mathrm{dom}(h)$ *is finite*$)$.

Then, there are f, $g:\omega\to\omega$ such that

(2.3) $$X\ll\{f\}\ll Z\ll\{g\}\ll Y \quad \textit{and} \quad f\ll g,$$

(2.4) $$f\restriction b\not< s \quad \textit{or} \quad s\not< g\restriction b.$$

PROOF. Set $a=\omega\setminus b$. By using Corollary 4, take f_1, $g_1: a\to\omega$ such that

$$X\restriction a\ll\{f_1\}\ll Z\ll\{g_1\}\ll Y\restriction a \quad \text{and} \quad f_1\ll g_1.$$

Take f_2, g_2: $b\to\omega$ such that

$$X\restriction b\ll\{f_2\}\ll\{g_2\}\ll Y\restriction b \quad \& \quad f_2\not< s \quad \text{or} \quad s\not< g_2$$

and set

$$f=f_1\cup f_2, \qquad g=g_1\cup g_2.$$

Then, f and g are as required. □

PROOF OF THEOREM 1. Let I be an ideal on ω such that $\mathrm{Fin}\subset I$.

The case of that $\omega\in I$ has no problem. So, we may assume that $\omega\notin I$. Set $\mathcal{X}=\{s;\ \exists x\subset\omega(x\notin I\ \&\ s: x\to\omega)\}$. By CH, take an enumeration $\langle s_\alpha|\alpha<\omega_1\rangle$ of $\mathcal{X}$ and an enumeration $\langle a_\alpha|\alpha<\omega_1\rangle$ of I. For each $\alpha<\omega_1$, let $b_\alpha=\mathrm{dom}(s_\alpha)$. By induction on $\alpha<\omega_1$, we shall take f_α, $g_\alpha:\omega\to\omega$ and $h_\alpha: a_\alpha\to\omega$ which satisfy the following (1)~(4).

(1) $$f_\xi< f_\alpha\ll g_\alpha< g_\xi, \qquad \text{for any} \quad \xi<\alpha.$$

(2) $$f_\alpha\restriction a_\xi\ll h_\xi\ll g_\alpha\restriction a_\xi, \qquad \text{for any} \quad \xi<\alpha.$$

(3) $$f_\alpha\restriction b_\alpha\not< s_\alpha \quad \text{or} \quad s_\alpha\not< g_\alpha\restriction b_\alpha.$$

(4) $$f_\alpha\restriction a_\alpha \ll h_\alpha \ll g_\alpha\restriction a_\alpha .$$

Assume that we could take such f_α, g_α, h_α (for $\alpha<\omega_1$). By (1),

$$\mathcal{G}=\langle\langle f_\alpha \mid \alpha<\omega_1\rangle \mid \langle g_\alpha \mid \alpha<\omega_1\rangle\rangle$$

is a gap. By (2), it holds that

$$f_\alpha\restriction a_\beta \prec h_\beta \prec g_\alpha\restriction a_\beta, \qquad \text{for any} \quad \alpha, \beta<\omega_1 .$$

So, it holds that, for all $\beta<\omega_1$, $a_\beta\in I_{\mathcal{G}}$ (i.e., $I\subset I_{\mathcal{G}}$). And by (3), we have that $I_{\mathcal{G}}\subset I$.

It remains to show that we can take such f_α, g_α, h_α (for $\alpha<\omega_1$).

Suppose that $\alpha<\omega_1$ and defined f_ξ, g_ξ, h_ξ (for $\xi<\alpha$) satisfying (1)~(4). Since it holds that

$$b_\alpha\notin I \ \& \ \{a_\xi ; \xi<\alpha\}\subset I \ \& \ \mathrm{Fin}\subset I,$$

we can take $b\subset b_\alpha$ such that

$$b \text{ is infinite and } b\cap a_\xi \text{ is finite for each } \xi<\alpha .$$

By Lemma 5, take $f_\alpha, g_\alpha : \omega\to\omega$ such that

$$f_\xi \prec f_\alpha \ll g_\alpha \prec g_\xi \qquad \text{for all} \quad \xi<\alpha,$$

$$f_\alpha\restriction a_\xi \gg h_\xi \ll g_\alpha\restriction a_\xi \qquad \text{for all} \quad \xi<\alpha,$$

$$f_\alpha\restriction b \not\prec s_\alpha\restriction b \quad \text{or} \quad s_\alpha\restriction b \not\prec g_\alpha\restriction b,$$

and take $h_\alpha : a_\alpha\to\omega$ such that

$$f_\alpha\restriction a_\alpha \ll h_\alpha \ll g_\alpha\restriction a_\alpha .$$

These f_α, g_α, h_α satisfy (1)~(4). ■

Here, we remark that the assumption of CH in Theorm 1 is necessary. To see this, let V be a ground model which satisfies that $2^\omega=2^{\omega_1}$. Then, in V, there exists an ideal which is not obtained from any (ω_1, ω_1)-gaps, since the cardinality of the family of ideals on ω is greater than the cardinality of the family of (ω_1, ω_1)-gaps. Which ideals are obtained from (ω_1, ω_1)-gaps, under the assumption of $\neg$CH? The following theorem deals a case whose model is obtained by a simple generic extension.

THEOREM 6. *Assume CH. Let κ be a cardinal such that $\kappa^\omega=\kappa$ and P be the partial ordering $\{p ; \exists x\subset\kappa(|x|<\omega \ \& \ p : x\to 2)\}$ which adjoins κ-many Cohen reals. Then, in V^P, it holds that the family $\{I_{\mathcal{G}} ; \mathcal{G}$ is an H-gap$\}$ consists of all ideals I such that $\omega\notin I$ and $\mathrm{Fin}\subset I$ and I are $\leqq\omega_1$-generated.*

We need the following lemma and corollary to show Theorem 6. Let Q be the partial ordering $\{q; \exists x\subset\omega(|x|<\omega \,\&\, q: x\to 2)\}$ which adjoins a Cohen real.

LEMMA 7. *Let* $\mathcal{G}=\langle\langle f_\alpha|\alpha<\omega_1\rangle|\langle g_\alpha|\alpha<\omega_1\rangle\rangle$ *be an H-gap. Then, it holds that*

$$V^Q\models \text{“}I_{\mathcal{G}} \text{ is the ideal generated by } (I_{\mathcal{G}})^V\text{”}.$$

PROOF. Set $l=(I_{\mathcal{G}})^V$. Since $V^Q\models$ “$l\subset I_{\mathcal{G}}$”, it suffices to show that

$$\Vdash_Q \forall x\in I_{\mathcal{G}}\exists y\in l(x\subset y).$$

To show this, let

$$q\in Q \;\&\; x: Q\text{-name} \;\&\; q\Vdash x\in I_{\mathcal{G}}.$$

Take a Q-name h such that

$$q\Vdash h: x\to\omega \;\&\; \forall\alpha<\omega_1(f_\alpha\restriction x\prec h\prec g_\alpha\restriction x).$$

For each $\alpha<\omega_1$, take $q_\alpha\leqq q$ and $n_\alpha<\omega$ such that

$$q_\alpha\Vdash\forall k\in x\setminus n_\alpha(f_\alpha(k)<h(k)<g_\alpha(k)).$$

Since $|Q\times\omega|=\omega$, there exist $r\in Q$ and $m<\omega$ such that

$$A=\{\alpha<\omega_1;\; q_\alpha=r \;\&\; n_\alpha=m\} \text{ is cofinal in } \omega_1.$$

Set $y=\{k<\omega;\; m\leqq k \;\&\; \exists r'\leqq r(r'\Vdash k\in x)\}$. It holds that $r\Vdash x\subset y\cup m$.

CLAIM 1. For any α, $\beta\in A$ and any $k\in y$, $f_\alpha(k)+1<g_\beta(k)$.

PROOF OF CLAIM 1. Let α, $\beta\in A$ and $k\in y$. Take $r'\leqq r$ such that

$$r'\Vdash k\in x.$$

Since $k\geqq m$, we have that $r'\Vdash f_\alpha(k)<h(k)<g_\beta(k)$ which implies $f_\alpha(k)+1<g_\beta(k)$

QED OF CLAIM 1.

By using Claim 1, define $h': y\to\omega$ by

$$h'(k)=\max\{f_\alpha(k);\; \alpha\in A\}+1.$$

Then, it holds that $\forall\alpha<\omega_1(f_\alpha\restriction y\prec h'\prec g_\alpha\restriction y)$ and we get $y\in l$. □

COROLLARY 8. *Let* $\mathcal{G}=\langle\langle f_\alpha|\alpha<\omega_1\rangle|\langle g_\alpha|\alpha<\omega_1\rangle\rangle$ *be an H-gap. Then it holds*

$$V^P\models \text{“}I_{\mathcal{G}} \text{ is the ideal generated by } (I_{\mathcal{G}})^V\text{”}.$$

PROOF. This follows from Lemma 7 and the fact that

$$V^P \cap \mathcal{P}(\omega) \subset \bigcup \{V^{P\restriction a};\ a \in V\ \&\ a \subset \kappa\ \&\ |a| \leqq \omega\}. \quad \square$$

PROOF OF THEOREM 6. First we shall show that, in V^P,

$$\forall \mathcal{G}: H\text{-gap } (I_{\mathcal{G}} \text{ is } \leqq \omega_1\text{-generated}).$$

So, let $\mathcal{G}$ be a P-name such that, $V^P \models \mathcal{G}$ is an H-gap. Take an $A \in V$ such that

$$A \subset \kappa \quad \& \quad |A| \leqq \omega_1 \quad \& \quad \mathcal{G} \in V^{P\restriction A}.$$

Since $V^{P\restriction A} \models \mathrm{CH}$, we have

$$V^{P\restriction A} \models I_{\mathcal{G}} \text{ is } \leqq \omega_1\text{-generated}.$$

Since $P \cong (P\restriction A) \times (P\restriction(\kappa \setminus A))$ and $P \cong P\restriction(\kappa \setminus A)$, by Corollary 8,

$$V^P \models I_{\mathcal{G}} \text{ is } \leqq \omega_1\text{-generated}.$$

To show the reverse implication, let l be a P-name such that

$$V^P \models \omega \notin l \text{ and } l \text{ is } \leqq \omega_1\text{-generated and } \mathrm{Fin} \subset l.$$

Take an $S \in V^P$ such that

$$V^P \models |S| \leqq \omega_1 \text{ and } l \text{ is generated by } S.$$

Then, there exists an $A \in V$ such that

$$A \subset \kappa, \quad |A| \leqq \omega_1 \text{ and } S \in V^{P\restriction A}.$$

Since $V^{P\restriction A} \models \mathrm{CH}$, there is a $\mathcal{G} \in V^{P\restriction A}$ such that

$$V^{P\restriction A} \models \mathcal{G} \text{ is an } H\text{-gap and } I_{\mathcal{G}} \text{ is generated by } S.$$

By Corollary 8, $V^P \models I_{\mathcal{G}} = l$. ■

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