

A NOTE ON THE HEREDITARY PROPERTIES IN THE PRODUCT SPACE

By

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In this note we shall investigate some hereditary properties of a subspace of a product space.

Let X_α be a topological space for each $\alpha \in I$ and A be a subset of I . p_A is the projection: $\prod_{\alpha \in I} X_\alpha \rightarrow \prod_{\alpha \in A} X_\alpha$, i. e. $p_A(x)$ is the restricted function of x whose domain is A . A is co-countable if $I - A$ is countable.

The family of sets is linked if each pair of its members has non-empty intersection. The space has (K) -property (precaliber \aleph_1) (caliber \aleph_1), if any uncountable family of non-empty open subsets of X includes an uncountable subfamily which is linked (has the finite intersection property) (has non-empty intersection). [2]

THEOREM. *Let X_α be second-countable for $\alpha \in I$ and X be a subspace of $\prod_{\alpha \in I} X_\alpha$ and ϕ be one of the properties:*

1) the countable chain condition, 2) (K) -property, 3) precaliber \aleph_1 , 4) caliber \aleph_1 , 5) the separability, 6) the Lindelöf property.

Then, X satisfies the hereditarily ϕ if and only if for any subspace Y of X , there exists co-countable subset A of I such that $p_A''Y$ satisfies ϕ .

LEMMA 1. (N. A. Sanin) [1] *Let Γ be an uncountable set of finite sets, then Γ includes an uncountable subfamily Δ which is quasi-disjoint i. e. $x \cap y \subseteq \bigcap \Delta$ for each different x and y of Δ .*

See [1] for the proof.

LEMMA 2. *Let f be a continuous function whose domain is X and X satisfies ϕ in the theorem. Then, the range of f also satisfies ϕ .*

Proof. Easy to check.

Let $\{V_n^\alpha; n < \omega\}$ be a base of X_α . Then, $\{p_A^{-1}V_{n_1}^{\alpha_1} \times \cdots \times V_{n_m}^{\alpha_m}; A = \{\alpha_1, \dots, \alpha_m\}, m < \omega\} = \mathcal{C}\mathcal{V}$ is a base of $\prod_{\alpha \in I} X_\alpha$. The domain of the basic open set $V (= p_A^{-1}V_{n_1}^{\alpha_1} \times$

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$\dots \times V_{\frac{\alpha}{m}}^{\alpha}$ is A , which is finite, and is denoted by $\text{dom } V$.

LEMMA 3. Let θ be an uncountable subfamily of \mathcal{CV} . Then, θ includes an uncountable subfamily Φ which has the following properties:

- a) $\{\text{dom } V; V \in \Phi\}$ is quasi-disjoint,
- b) $p'_A V = p'_A W$ for each $V, W \in \Phi$, where $A = \bigcap \{\text{dom } V; V \in \Phi\}$.

PROOF. By Lemma 1, θ includes an uncountable subfamily θ' such that $\{\text{dom } V; V \in \theta'\}$ is quasi-disjoint.

Let $A = \bigcap \{\text{dom } V; V \in \theta'\}$. Then, $\{p'_A V; V \in \theta'\}$ is countable. Hence, some uncountable subfamily Φ of θ' has the properties in the lemma.

PROOF OF THEOREM. The necessity is clear and so we shall prove the sufficiency. Suppose that X does not satisfy the hereditarily ψ . Then, there exists a subset $\{x_\alpha; \alpha < \omega_1\}$ and a family $\{O_\alpha; \alpha < \omega_1\}$ of open subsets of X such that $x_\alpha \in O_\alpha$ for each $\alpha < \omega_1$ and

- i) $x_\alpha \notin O_\beta$ for any $\beta \neq \alpha$,
- ii) for any uncountable subset S of ω_1 , there exists a pair α, β of S ;
 $O_\alpha \cap O_\beta \cap \{x_\alpha; \alpha < \omega_1\} = \emptyset$,
- iii) for any uncountable subset S of ω_1 , there exists a finite subset F of S ;
 $\bigcap_{\alpha \in F} O_\alpha \cap \{x_\alpha; \alpha < \omega_1\} = \emptyset$,
- iv) for any uncountable subset S of ω_1 , $\bigcap_{\alpha \in S} O_\alpha \cap \{x_\alpha; \alpha < \omega_1\} = \emptyset$,
- v) $x_\alpha \notin O_\beta$ for any $\beta > \alpha$, or
- vi) $x_\alpha \notin O_\beta$ for any $\beta < \alpha$, according to ψ is 1), 2), 3), 4), 5) or 6), respectively.

We may take the above $O_\alpha (\alpha < \omega_1)$ from \mathcal{CV} . By Lemma 3, without a loss of generality we can assume that $\{O_\alpha; \alpha < \omega_1\}$ satisfies the conditions a) and b) of Lemma 3.

Now, we apply the assumption and Lemma 2 to $\{x_\alpha; \alpha < \omega_1\}$. Then, there exists a co-countable subset A of I such that $p'_A \{x_\alpha; \alpha < \omega_1\}$ satisfies ψ and $\bigcap \{\text{dom } O_\alpha; \alpha < \omega_1\} \cap A$ is empty. Since $\{\text{dom } O_\alpha; \alpha < \omega_1\}$ is quasi-disjoint, we may assume $\text{dom } O_\alpha - \bigcap \{\text{dom } O_\alpha; \alpha < \omega_1\} \subseteq A$ for $\alpha < \omega_1$. There exists(s)

- i)' α such that $p_A(x_\alpha) \in p'_A O_\beta \cap p'_A O_\gamma$ for some $\beta \neq \gamma$,
- ii)' an uncountable subset S of ω_1 such that
 $p'_A O_\alpha \cap p'_A O_\beta \cap p'_A \{x_\alpha; \alpha < \omega_1\} \neq \emptyset$ for each distinct $\alpha, \beta \in S$,
- iii)' an uncountable subset S of ω_1 such that $\bigcap_{\alpha \in F} p'_A O_\alpha \cap p'_A \{x_\alpha; \alpha < \omega_1\} \neq \emptyset$

for any finite $F \subseteq S$,

iv)' α and an uncountable subset S of ω_1 such that $p_A(x_\alpha) \in \bigcap_{\alpha \in S} p_A'' O_\alpha$,

v)' α such that $p_A(x_\alpha) \in p_A'' O_\beta$ for some $\beta > \alpha$, or

vi)' α such that $p_A(x_\alpha) \in p_A'' O_\beta$ for some $\beta < \alpha$,

according that ψ is 1), 2), 3), 4), 5) or 6), respectively.

By the assumption of A and the fact $x_\alpha \in O_\alpha, p_A(x_\alpha) \in p_A'' O_\beta$ holds if and only if $x_\alpha \in O_\beta$ holds, for each α, β . So, i)', \dots , or vi)' contradicts to i), \dots , or vi) respectively.

Now, the proof is complete.

Since the hereditary separability is equivalent to the hereditary caliber- \aleph_1 -property, it is a little interesting to compare the two cases 4) and 5) in the theorem.

References

- [1] Juhász, I., Cardinal Functions in Topology, Amsterdam (1971).
- [2] Kunen, K. and Tall, F.D., Between Martin's Axiom and Souslin's Hypothesis, Fundamenta Mathematicae CII. 3 (1979).

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