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# MORITA EQUIVALENCE FOR RINGS WITHOUT IDENTITY

By

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In the paper [1] Abrams made a first step in extending the theory of Morita equivalence to rings without identity. He considered rings in which a set of commuting idempotents is given such that every element of the ring admits one of these idempotents as a two-sided unit, and the categories of all left modules over these rings which are unitary in a natural sense. He proved that two such module categories over the rings $R$ and $S$, say, are equivalent if and only if there exists a unitary left $R$-module $P$ which is a generator, the direct limit of a given kind of system of finitely generated projective modules, and such that $S$ is isomorphic to the ring of certain endomorphisms of $P$.

The aim of the present paper is to extend this theory in two ways: to cover a wider range of rings, and to transfer more of the classical Morita theory. Firstly, one can weaken the condition of commutativity of the idempotents in question: it suffices to require that any two of them have a common upper bound under the natural partial order (i.e., any two elements of the ring admit a common two-sided identity), a condition which is fulfilled by all regular rings (regular in the sense of Neumann). Whenever one has such a system of idempotents, then any larger system, in particular, the set of all idempotents, is also such, which is not the case for the systems of Abrams. Secondly, by a suitable modification of some homological lemmas we obtain also the two-sided characterizations of Morita equivalence, arriving thus at a complete analogy to the classical case of rings with identity. Our presentation is a combination of those in Anderson-Fuller [2], §§ 21-22, and Bass [5] (see also [6], Chapter II). This machinery allows us to avoid the elaborate construction of Abrams. As examples we describe, among others, those rings with local units which are Morita equivalent to division rings and primary rings, respectively. The Rees matrix rings studied in [4] turn out to have a natural place in this theory.

The theory we present here is a counterpart of the theory of Morita duality developed by Yamagata [10]. On the one hand, we shall use the same modified Hom-functors but for projective and not injective modules, and on the

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other hand, it turns out that every Morita equivalence class of rings with local units contains rings with enough idempotents, i.e., rings considered by Yamagata.

Notice also that the module categories we consider are full subcategories of the categories of modules over unital overrings of the respective rings with local units. Nevertheless, Sato's [8] theory of equivalence does not apply because he considers the usual Hom-functors, which does not work in our case.

## 1. Preparations.

DEFINITION 1. $R$ is a *ring with local units* if every finite subset of $R$ is contained in a subring of the form $eRe$ where $e=e^2\in R$.

We call a left module $M$ over $R$ *unitary* if $RM=M$, i.e., for each $m\in M$ there are $r_1, \cdots, r_n\in R$ and $m_1, \cdots, m_n\in M$ such that $r_1m_1+\cdots+r_nm_n=m$. If $R$ is a ring with local units then this implies that for every finite subset $M'\subset M$ there is an idempotent $e\in R$ such that $em=m$ for all $m\in M'$ By $R$ Mod we denote the category of unitary left $R$-modules together with the usual $R$-homomorphisms. Dually, Mod $R$ denotes the category of unitary right $R$-modules. Similarly to the case considered in Abrams [1], $R$ Mod (or Mod $R$) is a complete and cocomplete additive category. We call a bimodule unitary if it is unitary on both sides.

In what follows, $R$ denotes a ring with local units. The most important thing for us is to find those modules in $R$ Mod which play the role of the progenerators in the case of rings with identity. Of course, projective generators make sense in $R$ Mod for this a categorical notion; however, ${}_RR$ is neither finitely generated nor projective if $R$ has no identity, and the notion we need ought to include ${}_RR$, too. Therefore we define:

DEFINITION 2. $P\in R$ Mod is a *locally projective module* if there is a direct system $(P_i)_{i\in I}$ of finitely generated projective direct summands of $P$ together with projections $\phi_i: P\to P_i$ such that $\phi_i$ factors through $\phi_j$ whenever $i\leqq j$, and such that $\varinjlim P_i=P$. Notice that ${}_RR$ is locally projective if $R$ has local units, as $Re$ is a projective direct summand of $R$ for every idempotent $e\in R$, and the multiplication maps $\phi_e: R\to Re$ satisfy the condition on $\phi_i$ if we define $e\leqq f\Leftrightarrow ef=fe=e$.

The role of progenerators will be played by the locally projective generators in $R$ Mod. But before turning to them, we shall establish homological properties of locally projective modules. In doing so, we shall need a more restrictive notion instead of the Hom-sets. For the sake of convenience, homomorphisms of modules will be written opposite the scalars.

Notice that the usual definition of tensor product makes no use of the identity in the ring, hence it makes sense in our case, too.

The following Propositions 1.1-1.5 and 1.7 can be proved along the same lines as in the case of rings with identity (see e.g. Anderson-Fuller [2], § 20), therefore we present them without proof. Before stating Proposition 1.1, observe the following. If $R$ and $S$ are rings with local units and ${}_SN$ and ${}_SU_R$ are unitary then $\mathrm{Hom}_S(U, N)$ is a left $R$-module by putting, for $\phi\in\mathrm{Hom}_S(U, N)$ and $r\in R$, $r\phi: u\in U\mapsto(ur)\phi\in N$. The submodule $R\,\mathrm{Hom}_S(U, N)$ is the largest unitary $R$-submodule of $\mathrm{Hom}_S(U, N)$. By $R\,\mathrm{Hom}_S(U, -)$ we denote the functor induced by the mapping $N\mapsto R\,\mathrm{Hom}_S(U, N)$.

PROPOSITION 1.1. *For all* $M, M'\in R\,\mathrm{Mod}$, $m\in M$, $\phi\in\mathrm{Hom}_R(M, M')$, *put*

$$m\rho_M: r\mapsto mr \quad (r\in R)\ (\textit{thus}\ \rho: M\to R\,\mathrm{Hom}_R(R, M))$$

*and*

$$\rho_\phi:\gamma\mapsto\gamma\circ\phi \quad (\gamma\in R\,\mathrm{Hom}_R(R, M)).$$

*Then* $\rho: 1_{R\,\mathrm{Mod}}\to R\,\mathrm{Hom}_R(R, -)$ *is a natural isomorphism.*

PROPOSITION 1.2. *For all* $M, M'\in R\,\mathrm{Mod}$ *and* $\phi\in\mathrm{Hom}_R(M, M')$, *put*

$$(r\otimes m)\mu_M=rm \quad (r\in R,\ m\in M)\ (\textit{thus}\ \mu_M: R\otimes M\to M)$$

*and*

$$\mu_M: R\otimes\phi\mapsto\phi.$$

*Then* $\mu: R\underset{R}{\otimes}-\to 1_{R\,\mathrm{Mod}}$ *is a natural isomorphism.*

COROLLARY 1.3. *For all* $e^2=e\in R$ *and* $M\in R\,\mathrm{Mod}$, $eR\otimes M\cong eM$.

PROPOSITION 1.4. *Let* $\theta: {}_RU_S\to{}_RV_S$ *be a bimodule homomorphism between unitary bimodules* ${}_RU_S$ *and* ${}_RV_S$ *where* $R$ *and* $S$ *are rings with local units. Put, for all* $M, M'\in R\,\mathrm{Mod}$ *and* $\phi\in\mathrm{Hom}_R(M, M')$,

$$\eta_M:\gamma\mapsto\theta\circ\gamma \quad (\gamma\in S\,\mathrm{Hom}_R(V, M)),$$

$$\eta_\phi:\delta\mapsto\theta\circ\delta \quad (\delta\in S\,\mathrm{Hom}_R(V, \phi)).$$

*Then* $\eta: S\,\mathrm{Hom}_R(V, -)\to S\,\mathrm{Hom}_R(U, -)$ *is a natural transformation between two functors from* $R\,\mathrm{Mod}$ *to* $S\,\mathrm{Mod}$. *Moreover, if* $\theta$ *is an isomorphism then* $\eta$ *is a natural isomorphism.*

Before stating the next proposition, observe the following. If $N_S$ and ${}_RU_S$ are unitary, then $\mathrm{Hom}_S(N, U)$ is a left $R$-module if we put, for all $\phi\in\mathrm{Hom}_S(N, U)$ and $r\in R$, $r\phi: n\in N\mapsto r(\phi n)$. Then $R\,\mathrm{Hom}_S(N, U)$ is a unitary left $R$-module. Similarly, if ${}_RM$ is any unitary left $R$-module then $\mathrm{Hom}_R(M, U)$ is naturally a right $S$-module and hence $\mathrm{Hom}_R(M, U)S$ is a unitary right $S$-module. Further, notice that ${}_S(K\underset{R}{\otimes}M)\in S\,\mathrm{Mod}$ whenever ${}_SK_R$ and ${}_RM$ are unitary modules.

PROPOSITION 1.5. *For every triple* $({}_RP, {}_RU_S, {}_SM)$ *such that* ${}_RP$ *is a finitely generated projective module, there is an isomorphism of abelian groups*

$$\eta : \mathrm{Hom}_R(P, U)\underset{S}{\otimes}M \to \mathrm{Hom}_R(P, U\underset{S}{\otimes}M)$$

*defined via*

$$\eta(\gamma\otimes m) : p\mapsto p\gamma\otimes m$$

*that is natural in each of the three variables P, U, M.*

COROLLARY 1.6. *For every triple of unitary modules* $({}_RP_S, {}_RU_S, {}_SM)$ *such that* ${}_RP$ *is locally projective and* $Pf$ *is a finitely generated left R-module for all* $f^2=f\in S$, *there is an isomorphism of left S-modules*

$$\eta : S\,\mathrm{Hom}_R(P, U)S\underset{S}{\otimes}M \to S\,\mathrm{Hom}_R(P, U\underset{S}{\otimes}M)$$

*defined via*

$$(\gamma\otimes m)\eta : p\mapsto p\gamma\otimes m$$

*that is natural in each of the three variables P, U, M.*

PROOF. It is routine to verify that $\eta$ is a homomorphism which is natural in each variable. Next we show that $\eta$ is injective. In fact, assume $(\Sigma\gamma_i\otimes m_i)\eta=0$. Since $\gamma_i\in S\,\mathrm{Hom}_R(P, U)S$, there is an idempotent $f^2=f\in S$ with $f\gamma_i f=\gamma_i$ for all $i$. By assumption the left $R$-submodule $Pf$ is finitely generated, hence it is contained in a finitely generated projective direct summand $P'$ of $P$. By $P'=P'f\oplus P'(1-f)=Pf\oplus P'(1-f)$, where $P'(1-f)=\{p\in P' \mid P'f=0\}$, we obtain that $Pf$ is also projective. By $(\Sigma\gamma_i\otimes m_i)\eta=0$, the homomorphism $\phi' : Pf\to U\underset{S}{\otimes}M : pf\mapsto\Sigma p\gamma_i\otimes m_i$ is trivial. Therefore by Proposition 1.5 we have $\Sigma\gamma_i\otimes m_i=0$ in $\mathrm{Hom}_R(Pf, U)\underset{S}{\otimes}M$, but $\mathrm{Hom}_R(Pf, U)=f\,\mathrm{Hom}_R(P, U)S$, hence $\Sigma\gamma_i\otimes m_i$ must be zero in $S\,\mathrm{Hom}(P, U)S\underset{S}{\otimes}M$. For proving the surjectivity of $\eta$, if $\phi$ is any element in $S\,\mathrm{Hom}_R(P, U\underset{S}{\otimes}M)$, then there is an idempotent $f^2=f\in S$ with $f\phi=\phi$. Consider the restriction $\phi'$ of $\phi$ to $Pf$ which is a finitely generated projective direct summand of ${}_RP$. By Proposition 1.5, there is an element $\Sigma\gamma_i'\otimes m_i$ of $\mathrm{Hom}_R(Pf, U)\underset{S}{\otimes}M$ which corresponds to $\phi'$. Extend $\gamma_i'$ to a $\gamma_i$ defined on $P$ by putting $(P(1-f))\gamma_i=0$. Now it is clear that $(\Sigma\gamma_i\otimes m_i)\eta=f\phi=\phi$, and we are done.

PROPOSITION 1.7. *Let $P_S$, $_RU_S$ and $_RM$ be unitary modules such that $P_S$ is finitely generated and projective. Then there is an isomorphism of abelian groups*

$$\eta : P\underset{S}{\otimes}S\ \mathrm{Hom}_R(U, M) \to \mathrm{Hom}_R(\mathrm{Hom}_S(P, U), M)$$

*defined via*

$$\eta(p\otimes\gamma) : \delta\in \mathrm{Hom}_S(P, U)\mapsto(\delta p)\gamma$$

*that is natural in each of the three variables $P, U, M$.*

COROLLARY 1.8. *Let $_RP_S$, $_RU_S$ and $_RM$ be unitary modules such that $P_S$ is locally projective and $eP$ is a finitely generated right $S$-module for all idempotents $e^2=e\in R$. Then there is an isomorphism of left $R$-modules*

$$\eta : P\underset{S}{\otimes}S\ \mathrm{Hom}_R(U, M) \to R\ \mathrm{Hom}_R(R\ \mathrm{Hom}_S(P, U)R, M)$$

*defined via*

$$(p\otimes\gamma)\eta : \delta\in R\ \mathrm{Hom}_S(P, U)R\mapsto(\delta p)\gamma$$

*that is natural in each of the three variables $P, U, M$.*

PROOF. It is routine to verify that $\eta$ is a homomorphism which is natural in $P$, $U$ and $M$. Assume now $(\Sigma p_i\otimes\gamma_i)\eta=0$. Since $_RP$ is unitary, there is an idempotent $e\in R$ with $ep_i=p_i$ for all $i$. From our assumption it follows that $eP$ is a finitely generated projective direct summand of $P_S$. Since $(\Sigma p_i\otimes\gamma_i)\eta=0$, the element $\phi$ of $\mathrm{Hom}_R(\mathrm{Hom}_S(eP, U), M)$ defined by $\delta\phi=\Sigma(\delta p_i)\gamma_i$, $\delta\in\mathrm{Hom}_S(eP, U)$, is zero. Hence we can apply Proposition 1.7 and obtain that the element $\Sigma p_i\otimes\gamma_i$ is zero in $eP\underset{S}{\otimes}S\ \mathrm{Hom}_R(U, M)$ and therefore it must be zero in $P\underset{S}{\otimes}S\ \mathrm{Hom}_R(U, M)$, too. The surjectivity of $\eta$ is seen as in the proof of Corollary 1.6.

COROLLARY 1.9. *Let $_RP_S$ and $_SN$ be unitary modules such that $P_S$ is locally projective and $eP$ is a finitely generated right $S$-module for every idempotent $e\in R$. Then there is an isomorphism of left $R$-modules*

$$\eta : P\underset{S}{\otimes}N\to R\ \mathrm{Hom}_S(\mathrm{Hom}_S(P, S)R, N)$$

*defined via*

$$(p\otimes n)\eta : \delta\in\mathrm{Hom}_S(P, S)R\mapsto(\delta p)n$$

*that is natural in $P$ and $N$.*

PROOF. Putting $R=S$, $U=S$ and $N=M$ in Proposition 1.7, we obtain that $\eta : P\underset{S}{\otimes}N\to\mathrm{Hom}_S(\mathrm{Hom}_S(P, S), N)$ is an isomorphism of abelian groups which is natural in $P$ and $N$, provided that $P_S$ is finitely generated and projective. In the general case, it is straightforward to check that $\eta$ is a homomorphism of left $R$-modules which is natural in $P$ and $N$. Next we show that $\eta$ is injective. In fact, assume $(\Sigma p_i \otimes n_i)\eta=0$. Since ${}_RP$ is unitary, there is an idempotent $e \in R$ such that $ep_i=p_i$ for all $i$, and $eP$ is a projective right $S$-module for $P_S$ is locally projective. Now the element $\phi$ of $\mathrm{Hom}_S(\mathrm{Hom}_S(eP, S), N)$ defined by $\delta\phi=\Sigma(\delta p_i)n_i$ is zero since $(\Sigma p_i \otimes n_i)\eta=0$, and by the remark made at the beginning of the proof we have that $\Sigma p_i \otimes n_i$ is the zero element in $eP \otimes_S N$, hence it is the zero in $P \otimes_S N$, too. The surjectivity of $\eta$ is seen as in the proof of Corollary 1.6.

LEMMA 1.10. *Let ${}_RP$ be a locally projective generator, and let $S$ be a subring of* $\mathrm{End}_RP$ *having local units such that* $P \in \mathrm{Mod}\, S$, $S\, \mathrm{End}_RP=S$, *and $Pf$ is a finitely generated submodule of ${}_RP$ for every idempotent $f \in S$. Then*

1) *$P_S$ is a locally projective generator,*

2) *the mapping $\lambda: R \to \mathrm{End}_SP: r \mapsto \lambda_r$, where $\lambda_r: p \mapsto rp$, is an embedding of $R$ into* $\mathrm{End}_SP$ *such that* $(\mathrm{End}_SP)(\lambda(R))=\lambda(R)$.

PROOF. By the assumption, $P$ can be considered as a unitary $R$-$S$-bimodule. Since ${}_RP$ is a generator, it generates $Re$ for any idempotent $e \in R$, i.e., there are a natural number $n$ and a unitary left $R$-module $P'$ such that $P^n \cong Re \oplus P'$. Then it follows

$$(\mathrm{End}_RP)^n \cong \mathrm{Hom}_R(P^n, P) \cong \mathrm{Hom}_R(Re, P) \oplus \mathrm{Hom}_R(P', P) \cong eP \oplus \mathrm{Hom}_R(P', P).$$

This fact implies, since $\mathrm{End}_RP$ is a ring with identity, that $eP$ is a finitely generated right $\mathrm{End}_RP$-module, i.e., there are finitely many elements $p_1, \cdots, p_k \in eP$ such that every element $p \in eP$ can be expressed as $p=p_1\phi_1+\cdots+p_k\phi_k$ where $\phi_1, \cdots, \phi_k \in \mathrm{End}_RP$. On the other hand, we know that $P \in \mathrm{Mod}\, S$, hence there is an idempotent $f \in S$ with $p_i f=p_i$ for all $i$ and then $p=p_1(f\phi_1)+\cdots+p_k(f\phi_k)$. Since $f\phi_1, \cdots, f\phi_k$ are contained in $S$ by the assumption, we see that $eP$ is a finitely generated right $S$-module. Since $eP$ is a projective right $\mathrm{End}_RP$-module and every right $S$-module can be considered as a right $\mathrm{End}_RP$-module, we deduce immediately that $eP$ is a projective right $S$-module, too. Furthermore, if $e_1, e_2 \in R$ are idempotents such that $e_1 \leqq e_2$, then the map $\phi_{e_1}: P \to e_1P: p \mapsto e_1p$ factors through the corresponding map $\phi_{e_2}$. All this shows that $P_S$ is locally projective.

For any idempotent $f \in S$, $Pf$ is a finitely generated projective left $R$-module by the assumptions, hence there is an idempotent $e \in R$ with $Pf \oplus T \cong (Re)^n$ for a natural number $n$ and a unitary left $R$-module $T$. Therefore

$$(eP)^n \cong [\mathrm{Hom}_R(Re, p)]^n \cong \mathrm{Hom}_R((Re)^n, P) \cong \mathrm{Hom}_R(Pf, P) \oplus \mathrm{Hom}_R(T, P)$$
$$\cong fS \oplus \mathrm{Hom}_R(T, P)$$

which implies, since $S$ has local units, that $P_S$ is a generator.

Since ${}_RP$ is a generator, ${}_RR$ is a sum of homomorphic images of ${}_RP$, but $R$ as a ring has local units, and so $\mathrm{ann}_R(P)=0$ must hold. This implies that the mapping $\lambda$, which is clearly a homomorphism, is an embedding. In what follows we shall identify $R$ with the subring $\lambda(R)$ of $\mathrm{End}_S P$.

In order to see $(\mathrm{End}_S P)R=R$, take any $\rho \in (\mathrm{End}_S P)R$. Since $R$ has local units, there is an idempotent $e \in R$ such that $\rho e=\rho$. As $eP$ is a finitely generated right $S$-module, we have $eP=p_1S+\cdots+p_nS$. Let $K$ denote the submodule of ${}_RP^n$ generated by $(p_1, \cdots, p_n)$. Since ${}_RP$ is a generator, $K$ is a sum of homomorphic images of ${}_RP^n$, i.e., $(p_1, \cdots, p_n)=x_1\phi_1+\cdots+x_k\phi_k$ where $x_1, \cdots, x_k \in P^n$ and $\phi_1, \cdots, \phi_k : P^n \to K$. As $P \in \mathrm{Mod}\, S$ and $S$ has local units, each $x_i$ is contained in a $(Pf)^n$, $f^2=f \in S$, so we can replace each $\phi_i$ by $f\phi_i$. Now $f\phi_i : P^n \to K$ can be considered as an $n \times n$ matrix with entries from $f\,\mathrm{End}_R P$, by one of our assumptions we have $S\,\mathrm{End}_R P=S$, hence each $f\phi_i$ can be considered as an element of $S_n$, the ring of $n \times n$ matrices over $S$. All this shows that $\rho=\rho e$ can be considered as an endomorphism of $(P^n)_{S_n}$ and therefore we have

$$\rho(p_1, \cdots, p_n)=\rho(x_1(f\phi_1)+\cdots+x_k(f\phi_k))=(\rho x_1)(f\phi_1)+\cdots+(\rho x_k)(f\phi_k),$$

and here $(\rho x_i)(f\phi_i) \in P^n f\phi_i \subseteq K$ for $i=1, \cdots, k$. Hence $\rho(p_1, \cdots, p_n) \in K= R(p_1, \cdots, p_n)$, thus we have that $\rho=\rho e=re$ for some $r \in R$.

## 2. The Morita equivalence.

THEOREM 2.1. *Let $R$ and $S$ be equivalent rings with local units via inverse equivalences $G : R\,\mathrm{Mod} \to S\,\mathrm{Mod}$ and $H : S\,\mathrm{Mod} \to R\,\mathrm{Mod}$. Set*

$$P=H({}_SS) \quad \text{and} \quad Q=G({}_RR).$$

*Then $P$ and $Q$ are naturally unitary bimodules ${}_RP_S$ and ${}_SQ_R$ such that*

1) *${}_RP$, $P_S$, ${}_SQ$, $Q_R$ are locally projective generators and* $S\,\mathrm{End}_R P=S=(\mathrm{End}_R Q)S$, $(\mathrm{End}_S P)R=R=R\,\mathrm{End}_S Q$;
2) ${}_RP_S \cong \mathrm{Hom}_R(Q, R)S \cong R\,\mathrm{Hom}_S(Q, S)$, ${}_SQ_R \cong \mathrm{Hom}_S(P, S)R \cong S\,\mathrm{Hom}_R(P, R)$;
3) $G \cong S\,\mathrm{Hom}_R(P, -)$, $H \cong R\,\mathrm{Hom}_S(Q, -)$;
4) $G \cong Q \otimes_R -$, $H \cong P \otimes_S -$;
5) *identifying ${}_SQ_R$ with $S\,\mathrm{Hom}_R(P, R)$ and $S$ with $S\,\mathrm{Hom}_R(P, P)$ (see 2 and 1 above), consider the bilinear products*

$$(-, -): P\times Q\to R: (p, q)=pq\in R,$$

$$\langle -, -\rangle: Q\times P\to S: \langle q, p\rangle=(-, q)p\in S;$$

*then* $P\otimes_S Q$ *and* $Q\otimes_R P$ *become rings if we put* $(p_1\otimes q_1)(p_2\otimes q_2)=p_1\otimes\langle q_1, p_2\rangle q_2$ *and* $(q_1\otimes p_1)(q_2\otimes p_2)=q_1\otimes(p_1, q_2)p_2$, *and we have* $R\cong P\otimes_S Q$ *and* $S\cong Q\otimes_R P$.

PROOF. By Proposition 1.1, $R\cong R\,\mathrm{Hom}_R(R, R)$, moreover, this is also an isomorphism of rings; furthermore, $G$ yields the isomorphism of rings $\mathrm{End}_R({}_RR)\cong\mathrm{End}_S({}_SQ)$, and therefore $Q$ can be considered as a right $R$-module. In order to show that $Q_R$ is unitary, take an arbitrary element $q\in Q$. Since $Q=\cup\{G(Re): e^2=e\in R\}$, there is an idempotent $e\in R$ such that $q\in G(Re)$. Now the right translation $\rho_e\in\mathrm{End}_SQ$ induced by $e$ acts as an identity on $G(Re)$, hence $qe=q$. Similarly, $P$ is a unitary right $S$-module. It is clear that $Qe=G(Re)$ is a finitely generated left $S$-module.

Since ${}_RR$ is a locally projective generator, the same holds for ${}_SQ$, too. In the same way, ${}_RP$ is a locally projective generator. Now we can apply Lemma 1.10 and obtain that $P_S$ and $Q_R$ are also locally projective generators and it holds $R\,\mathrm{End}_SQ=R=(\mathrm{End}_SP)R$ and $S=S\,\mathrm{End}_RP=(\mathrm{End}_RQ)S$, and our first claim is proven.

Next we turn to the proof claim 3). Since $G$ and $H$ are equivalences, for every $M\in R\,\mathrm{Mod}$ we have the left $S$-isomorphism

$$\mathrm{Hom}_S(S, G(M))\cong\mathrm{Hom}_R(H(S), M)=\mathrm{Hom}_R(P, M).$$

Furthermore, by Proposition 1.1, $G(M)\cong S\,\mathrm{Hom}_S(S, G(M))$ is a natural isomorphism in $M$. All this shows that $G\cong S\,\mathrm{Hom}_R(P, -)$ and similarly $H\cong R\,\mathrm{Hom}_S(Q, -)$, and claim 3) is proven.

Now we have

$${}_SQ_R={}_SG(R)_R\cong S\,\mathrm{Hom}_R(P, R)\cong S\,\mathrm{Hom}_R(P, \mathrm{Hom}_S(P, P)R),$$

and there is also an $S$-$R$-bimodule isomorphism $\eta$ between $S\,\mathrm{Hom}_R(P, \mathrm{Hom}_S(P, P)R)$ and $\mathrm{Hom}_S(P, S\,\mathrm{Hom}_R(P, P))R\cong\mathrm{Hom}_S(P, S)R$ defined by

$$\eta(\gamma)\in\mathrm{Hom}_S(P, S\,\mathrm{Hom}_R(P, P))R: a\mapsto\eta(\gamma)a\in S\,\mathrm{Hom}_R(P, P): b\in P\mapsto(a\gamma)b$$

for every element $\gamma\in S\,\mathrm{Hom}_R(P, \mathrm{Hom}_S(P, P)R)$. (For proving that $\eta$ is an isomorphism, notice that its inverse is $b\eta^{-1}(\alpha): a\in P\mapsto a(\alpha b)$.) Hence we get ${}_SQ_R\cong\mathrm{Hom}_S(P, S)R$. Similarly we have ${}_RP_S\cong\mathrm{Hom}_R(Q, R)S\cong R\,\mathrm{Hom}_S(Q, S)$.

Now Proposition 1.4 and Corollary 1.9 together with claims 2) and 3) proven just above yield

$$H \cong R\,\mathrm{Hom}_S(Q, -) \cong R\,\mathrm{Hom}_S(\mathrm{Hom}_S(P, S)R, -) \cong P \underset{S}{\otimes} -$$

and similarly $G \cong Q \underset{R}{\otimes} -$.

To prove 5), consider the mapping

$$\lambda : P \underset{S}{\otimes} Q \to R : \sum p_i \otimes q_i \mapsto \sum (p_i, q_i).$$

It is clear that $\lambda$ is a homomorphism of abelian groups. Next,

$$\begin{aligned}\lambda[(p_1\otimes q_1)(p_2\otimes q_2)] &= \lambda[p_1 \otimes \langle q_1, p_2\rangle q_2] = (p_1, \langle q_1, p_2\rangle q_2) = (p_1\langle q_1, p_2\rangle, q_2)\\ &= ((p_1, q_1)p_2, q_2) = (p_1, q_1)(p_2, q_2) = \lambda(p_1\otimes q_1)\lambda(p_2\otimes q_2),\end{aligned}$$

hence $\lambda$ is a ring homomorphism. Since ${}_RP$ is a generator, ${}_RR$ is a sum of homomorphic images of $P$, so every $r \in R$ can be written as a finite sum $r = \sum p_i\phi_i$, $p_i \in P$, $\phi_i \in \mathrm{Hom}_R(P, R)$. Now $P_S$ is unitary and $S$ has local units, hence there is an idempotent $f \in S$ such that $p_i = p_i f$ for all $i$. Therefore we can replace $\phi_i$ by $f\phi_i = q_i \in Q$, and then $r = \sum p_i q_i = \lambda(\sum p_i \otimes q_i)$. Thus the mapping $\lambda$ is surjective. Finally, suppose that $\sum (p_i, q_i) = 0$. Since $Q_R$ is unitary, there is an $e \in R$ such that $q_i e = q_i$ for all $i$, and by the surjectivity of $\lambda$, $e$ can be written as $\sum (p'_j, q'_j)$. Now we have $\sum p_i \otimes q_i = \sum p_i \otimes q_i (\sum p'_j, q'_j) = \sum_{i,j} p_i \otimes q_i (p'_j, q'_j)$. At this point, notice that for any $p' \in P$ and $q, q' \in Q$, $q(p', q') \in Q \subseteq \mathrm{Hom}(P, R)$ and $\langle q, p'\rangle q' \in Q \subseteq \mathrm{Hom}(P, R)$, and for all $p \in P$ it holds $p(q(p', q')) = (p, q)(p', q') = ((p, q)p', q') = (p\langle q, p'\rangle, q') = (p, \langle q, p'\rangle q')$, hence $q(p', q') = \langle q, p'\rangle q'$. Therefore we can continue:

$$\begin{aligned}\sum p_i \otimes q_i &= \sum_{i,j} p_i \otimes \langle q_i, p'_j\rangle q'_j = \sum_{i,j} p_i\langle q_i, p'_j\rangle \otimes q'_j = \sum_{i,j} (p_i, q_i)p'_j \otimes q'_j\\ &= \sum_j \Big(\sum_i p_i, q_i\Big) p'_j \otimes q'_j = 0,\end{aligned}$$

which proves that $\lambda$ is injective.

Finally we consider $\rho : Q \underset{R}{\otimes} P \to S : \sum q_i \otimes p_i \mapsto \sum \langle q_i, p_i\rangle$. It is clear that $\rho$ is a homomorphism of abelian groups. The fact that $\rho$ is a ring homomorphism and the injectivity of $\rho$ are proven in the same way as was done for $\lambda$ above. To prove the surjectivity, consider first an idempotent $f \in S$. Since $Pf$ is a finitely generated projective left $R$-module, there are a $P' \in R\,\mathrm{Mod}$, an idempotent $e \in R$, and a natural number $n$ such that $Pf \oplus P' \cong (Re)^n$. Denote by $p_1, \cdots, p_n$ the canonical image of the basis $(e)_1, \cdots, (e)_n$ of $(Re)^n$. Then every element $p \in Pf$ admits a unique decomposition $p = (r_1 e)p_1 + \cdots + (r_n e)p_n$. Denote by $q_i$ ($i = 1, \cdots, n$) the mapping which assigns to each $p \in Pf$ the corresponding element $r_i e$. Clearly, this $q_i$ is a homomorphism from $Pf$ to $Re \subseteq R$. We extend $q_i$ to the whole of $P$ by putting $(P(1-f))q_i = 0$ (here $1-f$ makes sense for $f$ is an endomorphism of ${}_RP$) and denote this extended mapping also by $q_i$.

By the definition of $q_i$ we have, for every $p \in P$, $(p, q_i) = (pf, q_i)$ and $pf = \sum(pf, q_i)p_i$. Therefore $p\sum\langle q_i, p_i\rangle = \sum(p, q_i)p_i = pf$ for all $p \in P$, hence $f = \sum\langle q_i, p_i\rangle$. Finally, if $s \in S$ and $p \in P$ are arbitrary, then $s = fs$ for an idempotent $f = \langle q_i, p_i\rangle \in S$, and then $ps = pfs = p(\sum\langle q_i, p_i\rangle s) = \sum(p, q_i)(p_i s) = p\sum\langle q_i, p_i s\rangle$, i.e., $s = \sum\langle q_i, p_i s\rangle$, and we are done.

The usual definition of a Morita context makes no use of the identities of the rings, hence it makes sense in our case. Now we have:

THEOREM 2.2. *Let $R$, $S$, ${}_RP_S$, ${}_SQ_R$, $(\,,\,): P \times Q \to R$, $\langle\,,\,\rangle: Q \times P \to S$ be a Morita context where $R$, $S$ are rings with local units and $P$, $Q$ are unitary bimodules. Then $P \underset{S}{\otimes} - : S\,\mathrm{Mod} \to R\,\mathrm{Mod}$ and $Q \underset{R}{\otimes} - : R\,\mathrm{Mod} \to S\,\mathrm{Mod}$ are equivalences inverse to each other if and only if both $(\,,\,)$ and $\langle\,,\,\rangle$ are surjective.*

PROOF. If $P \underset{S}{\otimes} -$ and $Q \underset{R}{\otimes} -$ are inverse equivalences then the surjectivity of $(\,,\,)$ and $\langle\,,\,\rangle$ follows from 5) in Theorem 2.1. Conversely, if these mappings are surjective then they induce surjective bimodule homomorphisms from ${}_R(P \underset{S}{\otimes} Q)_R$ to $R$ and from ${}_S(Q \underset{R}{\otimes} P)_S$ to $S$. Next we see that these homomorphisms are also injective. Indeed, let $\sum(p_i, q_i) = 0$. Since $Q_R$ is unitary, there is an $e \in R$ such that $q_i e = q_i$ for all $i$, and by the surjectivity of $(\,,\,)$, $e$ can be written as $\sum(p'_j, q'_j)$. Now we have $\sum p_i \otimes q_i = \sum p_i \otimes q_i(\sum p'_j, q'_j) = \sum_{i,j} p_i \otimes q_i(p'_j, q'_j) = \sum_{i,j} p_i \otimes \langle q_i, p'_j\rangle q'_j = \sum_{i,j} p_i\langle q_i, p'_j\rangle \otimes q'_j = \sum_{i,j}(p_i, q_i)p'_j \otimes q'_j = \sum_j(\sum_i(p_i, q_i))p'_j \otimes q'_j = 0$. The injectivity of $\langle\,,\,\rangle$ is proved dually. Now we obtain, for every $M \in R\,\mathrm{Mod}$ and $N \in S\,\mathrm{Mod}$, $P \underset{S}{\otimes} (Q \underset{R}{\otimes} M) \cong (P \underset{S}{\otimes} Q) \underset{R}{\otimes} M \cong R \underset{R}{\otimes} M \cong M$ and similarly $Q \underset{R}{\otimes} (P \underset{S}{\otimes} N) \cong N$.

REMARK. In Taylor [9] Morita contexts with surjective mappings are shown to yield Morita equivalence, and vice versa, for central separable algebras over a commutative ring with identity. However, central separable algebras need not have local units and the converse implication does not hold either.

COROLLARY 2.3. *For any rings $R$, $S$ with local units, $R\,\mathrm{Mod}$ and $S\,\mathrm{Mod}$ are equivalent if and only if $\mathrm{Mod}\,R$ and $\mathrm{Mod}\,S$ are equivalent.*

Next we proceed to characterize Morita equivalence in a way similar to the case of rings with identity. Conform to that terminology, call a unitary bimodule ${}_RM_S$ *balanced* if the canonical homomorphisms $S \to \mathrm{End}_R M$ and $R \to \mathrm{End}_S M$ are injective and, identifying $R$ and $S$ with the corresponding subrings of endomorphisms of $M$, it holds $S\,\mathrm{End}_R M = S$ and $(\mathrm{End}_S M)R = R$.

THEOREM 2.4. *Let $R$, $S$ be rings with local units and $G: R\,\mathrm{Mod} \to S\,\mathrm{Mod}$, $H: S\,\mathrm{Mod} \to R\,\mathrm{Mod}$ be additive functors. Then $G$ and $H$ are equivalences inverse* *to each other if and only if there exists a unitary bimodule* ${}_RP_S$ *such that*

1) *both* ${}_RP$, $P_S$ *are locally projective generators,*
2) ${}_RP_S$ *is balanced,*
3) $G\cong S\,\mathrm{Hom}_R(P,-)$ *and* $H\cong P\underset{S}{\otimes}-$.

*Moreover, if* $P$ *satisfies these conditions then, putting* $Q=S\,\mathrm{Hom}_R(P, R)$, ${}_SQ_R$ *is a balanced bimodule, both* ${}_SQ$ *and* $Q_R$ *are locally projective generators,* $H\cong R\,\mathrm{Hom}_S(Q,-)$ *and* $G\cong Q\underset{R}{\otimes}-$.

PROOF. The necessity of the conditions as well as the final assertion follow from Theorem 2.1. To prove the sufficiency, let $M\in R\,\mathrm{Mod}$ be arbitrary. Then we have

$$HG(M)\cong P\underset{S}{\otimes}S\,\mathrm{Hom}_R(P, M)\overset{\text{Cor. 1.8}}{\cong}R\,\mathrm{Hom}_R(R\,\mathrm{Hom}_S(P, P)R, M)\overset{\text{Prop. 1.3}}{\cong}$$
$$\cong R\,\mathrm{Hom}_R(R, M)\overset{\text{Prop. 1.1}}{\cong}M.$$

On the other hand, for any $N\in S\,\mathrm{Mod}$,

$$GH(N)\cong S\,\mathrm{Hom}_R(P, P\underset{S}{\otimes}N)\overset{\text{Cor. 1.6}}{\cong}S\,\mathrm{Hom}_R(P, P)S\underset{S}{\otimes}N\cong S\underset{S}{\otimes}N\overset{\text{Prop. 1.2}}{\cong}N.$$

Following Abrams [1], now we present a concrete way to construct rings with local units Morita equivalent to a given ring $R$ of this kind.

For a locally projective module $P$, the endomorphisms of each $P_i$ extend to endomorphisms of $P$ when composed by $\phi_i$, and in this way the endomorphism rings of the components $P_i$ form a direct system of subrings of $\mathrm{End}_RP$. Their limit $S=\varinjlim\mathrm{End}_RP_i$ consists exactly of those endomorphisms of $P$ which factor through one of the projections $\phi_i$. The ring $S$ has local units because if the endomorphism $s\in S$ factors through $\phi_i$ then, choosing a $P_j$ which contains $P_i$ and the image of $s$ (notice that the latter is finitely generated hence such a $P_j$ exists), the projection $\phi_j$ is a unit to $s$. Now it is clear that $P\in\mathrm{Mod}\,S$ and $S\,\mathrm{End}_RP=S$. If, in addition, ${}_RP$ is a generator then by Lemma 1.10 we obtain that $P_S$ is also a locally projective generator and ${}_RP_S$ is balanced. Then Theorem 2.4 says that the functors $S\,\mathrm{Hom}_R(P,-)$ and $P\underset{S}{\otimes}-$ are inverse equivalences between $R\,\mathrm{Mod}$ and $S\,\mathrm{Mod}$. Furthermore, it is also clear from the above that, for any $M\in R\,\mathrm{Mod}$, $S\,\mathrm{Hom}_R(P, M)$ consists exactly of those $R$-homomorphisms from $P$ to $M$ which factor through one of the $\phi_i$. Therefore $S\,\mathrm{Hom}_R(P, M)$ is, as an abelian group, just the direct limit of the $\mathrm{Hom}_R(P_i, M)$. Thus we have:

THEOREM 2.5 (cf. Abrams [1]). *Two rings $R$, $S$ with local units are Morita equivalent if and only if there exists a locally projective generator $_RP$ such that, using the notation above, $S\cong\varinjlim \mathrm{End}_R P_i$.*

REMARK. Let $R$ be an arbitrary ring with local units, and consider the module $_RP=\bigoplus_{e^2=e\in R} Re$. (Notice that if $e, f$ are idempotents with $Re=Rf$ then this left ideal appears (at least) twice in the decomposition of $P$.) Clearly, $_RP$ is a locally projective module. By Theorem 2.5, the ring $S=\varinjlim \mathrm{End}_R Re$ is Morita equivalent to $R$. To every idempotent $e\in R$ we can assign the endomorphism $\theta_e$ of $P$ defined to act identically on the direct component $Re$ and as a zero on all other components. Clearly, the $\theta_e$ are orthogonal idempotents in $S$, and by the definition of $S$ we have

$$S=\bigoplus_{e^2=e\in R}\theta_e S=\bigoplus_{e^2=e\in R} S\theta_e .$$

This shows that $S$ is a ring with enough idempotents in the sense of Fuller [7]. Thus every Morita equivalence class of rings with local units contains rings with enough idempotents (which are even more special than the rings considered in Abrams [1]). A theory of Morita duality for rings with enough idempotents is presented in Yamagata [10].

## 3. Examples and applications

Of course, all the examples given in Abrams [1] are examples for our theory, too; we are not going to list them again.

EXAMPLE 1. Every regular ring is a ring with local units (but not necessarily in the sense of Abrams [1]). Indeed, let $a_1, \cdots, a_n$ be arbitrary elements of a regular ring $R$. Then there is a $g=g^2\in R$ such that $a_i g=a_i$, $i=1, \cdots, n$, further there is an $f=f^2\in R$ such that $fa_i=a_i$, $i=1, \cdots, n$, and $fg=g$. Putting $e=:g+f-gf$, it is straightforward to check that $e^2=e$ and $a_i e=a_i=ea_i$, $i=1, \cdots, n$.

PROPOSITION 3.1. *If $R$ and $S$ are Morita equivalent rings with local units and $R$ is regular then $S$ is also regutar.*

PROOF. By Theorem 2.5, $S$ is a direct limit of endomorphism rings of finitely generated projective $R$-modules. Since $R$ is regular, all these rings are regular, too, and the same holds for $S$, being the union of these endomorphism rings.

EXAMPLE 2. Let $R$ be a ring with identity, $S$ be a Rees matrix ring over $R$ with canonical decomposition $S\cong Se\otimes_{eSe}eS$, $e^2=e\in S$, $eSe\cong R$ (for the definitions of the notions occurring in this example, see Anh-Márki [4]). If $S$ is finitely orthogonal with respect to $e$, then $S$ is obviously a ring with local units. Now $Se$ is a finitely generated projective left $S$-module. For any $M\in S\,\mathrm{Mod}$ and $m\in M$, consider the mapping $\rho_m: Se\to M: se\mapsto sem$. These $\rho_m$'s together define a homomorphism from $(Se)^{(M)}$ to $M$ whose image is $SeM=Se(SM)=(SeS)M=SM=M$. This proves that $Se$ is a generator for $S\,\mathrm{Mod}$. By Theorem 2.5, $S$ is then Morita equivalent to $\mathrm{End}_S(Se)\cong eSe\cong R$.

In what follows, a ring $S$ as in Example 2 will be called a *finitely orthogonal Rees matrix ring*.

Next, observe that § 21 in Anderson-Fuller [2] makes no use of the identity in the given rings, all the results (and proofs) presented there are valid for our module categories, too. Thus we have:

PROPOSITION 3.2 (cf. [2], Corollary 21.9). *Let $R$ and $S$ be equivalent rings with local units. Then $R$ is primitive or a ring with zero Jacobson radical if and only if $S$ is such.*

PROPOSITION 3.3 (cf. [2], Proposition 21.11). *Equivalent rings with local units have isomorphic lattices of ideals; in particular, one of them is simple if and only if so is the other.*

We can also prove the following.

PROPOSITION 3.4. *Let $R$ and $S$ be equivalent rings with local units. If both $R$ and $S$ are commutative then they are isomorphic.*

PROOF. Consider the unitary bimodule ${}_RP_S$ given in Theorem 2.1. Since $R$ is commutative, for any idempotent $e\in R$, $R$ is the direct sum of the rings $eR$ and $(1-e)R$, and we have $P\cong{}_ReP\oplus_R(1-e)P={}_{eR}eP\oplus_{(1-e)R}(1-e)P$. Now $S$, being a ring of certain endomorphisms of ${}_RP$, also decomposes into a direct sum $S_1\oplus S_2$, and again by Theorem 2.1, $S_1$ and $S_2$ are equivalent to $eR$ and $(1-e)R$, respectively. By the construction of $P$, $eP$ is finitely generated as an $S$-module, hence also as an $S_1$-module. Then $eR\cong\mathrm{End}\,eP_{S_1}$, and since $S_1$ is commutative, we obtain an embedding of $S_1$ into $eR$, but then $eP$ is finitely generated as an $eR$-module, hence also as an $R$-module. Herefrom we conclude that $rP$ is a finitely generated $R$-module for every $r\in R$, hence by the commutativity of $R$, $r$ can be considered as an element of $S$, and similarly, every element of $S$ can be considered as an element of $R$, whence the assertion follows.

Contrary to the case of unital rings, it is not true that if $R$ and $S$ are equivalent rings with local units then their centres must be isomorphic. In fact, if $R$ is any ring with local units, $N$ is a countably infinite set and $R_N^f$ denotes the ring of $N\times N$ matrices over $R$ with finitely many non-zero entries, the $R_N^f$ is a finitely orthogonal Rees matrix ring over $R$ (see [4]), hence it is Morita equivalent to $R$ by Example 2, and $R_N^f$ is centreless. (We thank Dr. G. Abrams for calling our attention to this simple example.)

Now we characterize rings which are Morita equivalent to rings of certain 'nice' kinds. The first result is essentially Corollary 4.3 in Abrams [1].

PROPOSITION 3.5. *A ring $R$ with local units is Morita equivalent to a ring with identity if and only if there exists an idempotent $e\in R$ with $R=ReR$. If this is the case then $R$ is Morita equivalent to $eRe$.*

The proof is the same as that of Corollary 4.3 in [1], therefore it is omitted here.

PROPOSITION 3.6. *A ring with local units is Morita equivalent to a division ring if and only if it is a simple ring with minimal one-sided ideals.*

PROOF. Let $R$ be a ring with local units which is Morita equivalent to a division ring $D$. By Proposition 3.5 there is an idempotent $e\in R$ such that $eRe$ and $R$ are Morita equivalent. Given any finite subset $X$ of $R$, $X\cup\{e\}$ has a local unit $f$; then $X\subseteq fRf$, $R\supseteq RfR\supseteq RefR=ReR=R$, so $R=RfR$, and by Proposition 3.5 $fRf$ is Morita equivalent to $R$, hence also to $D$. Now $fRf$, being a ring with identity Morita equivalent to the division ring $D$, must be isomorphic to a full matrix ring over $D$. Theorem 1 in Anh [3] tells us now that $R$ is a simple ring with minimal one-sided ideals.

Conversely, if $R$ is a simple ring with minimal one-sided ideals then it is regular, hence a ring with local units. On the other hand, for any primitive idempotent $e\in R$, $eRe$ is a division ring and $ReR=R$. By Proposition 3.5, $R$ is then Morita equivalent to $eRe$.

REMARK. Notice that, by a result of E. Hotzel (see Corollary 3.5 in [4]), simple rings with minimal one-sided ideals are just the finitely orthogonal Rees matrix rings over division rings.

By a *primary ring* $A$ we mean a ring with identity whose factor $A/J(A)$ by its Jacobson radical is a simple artinian ring such that idempotents can be lifted. If, moreover, $A/J(A)$ is a division ring then $A$ is said to be a *local* *ring*. *A* ring $R$ is said to be a *strongly locally matrix ring* over a (unital) ring $S$ if for every finite subset $U \subseteqq R$ there is an idempotent $e \in R$ such that $U \subseteqq eRe$ and $eRe$ is isomorphic to the matrix ring $S_n$ for some $n$.

PROPOSITION 3.7. *A ring $R$ with local units is Morita equivalent to a primary ring if and only if $R$ is isomorphic to a strongly locally matrix ring over a local ring. If, in addition, ${}_RR$ and $R_R$ are projective modules then $R$ is isomorphic to a finitely orthogonal Rees matrix ring over a local ring.*

PROOF. Let $R$ be Morita equivalent to a primary ring. By Proposition 3.5 there is an idempotent $e \in R$ such that $eRe$ and $R$ are Morita equivalent, hence $eRe$ is Morita equivalent to a primary ring $S$. Now both $eRe$ and $S$ are rings with identity, hence $eRe$ isomorphic to $fS_nf$ for an idempotent $f$ in a full matrix ring $S_n$ over $S$. Here $J(fS_nf)=fJ(S_n)f$, so $fS_nf/J(fS_nf)=fS_nf/fJ(S_n)f \cong \bar{f}(S_n/J(S_n))\bar{f}$ where $\bar{f}$ denotes the image of $f$ under the canonical homomorphism of $S_n$ corresponding to $J(S_n)$, the last ring is obviously simple and artinian, and it is also clear that the idempotents can be lifted. Therefore $eRe$ is itself a primary ring, hence there is an idempotent $g \in eRe$ such that $gRg$ is a local ring. Now we have $(eRe)g(eRe)=eRe$ and, by $R=ReR$, also $R=ReR=ReReR=ReRegeReR=ReRgReR=RgR$. Hence, by Proposition 3.5, the bimodule ${}_RRg_{gRg}$ induces Morita equivalence between $R$ and $gRg$. Furthermore, similarly to the case treated in Example 2, $Rg \in R\,\mathrm{Mod}$ is a finitely generated projective generator. Then by Lemma 1.10, $Rg_{gRg}$ is a locally projective generator. The canonical components of $Rg_{gRg}$ are free modules, being finitely generated projectives over a local ring. Therefore the endomorphism ring of each of them is a finite matrix ring over $gRg$, and the assertion follows from Theorem 2.5.— The converse is obvious by Proposition 3.5.

Suppose now that, in addition, ${}_RR$ and $R_R$ are projective. Then in the proof above, $Rg_{gRg}$ and ${}_{gRg}gR$ are also projective, hence they are free modules, for $gRg$ is a local ring. Now [4], Theorem 3.1 says that $R=RgR$ is a Rees matrix ring over $gRg$, and it is finitely orthogonal for $R$ has local units.

PROPOSITION 3.8. *A ring $S$ with local units is Morita equivalent to a two-sided perfect local ring $R$ if and only if $S$ is an orthogonal Rees matrix ring over $R$.*

PROOF. Suppose that $R$ and $S$ are Morita equivalent. By Theorem 2.1, there exist locally projective bimodules ${}_RP_S$ and ${}_SQ_R$ such that $S \cong Q \otimes_R P$. Since $R$ is perfect, ${}_RP$ and $Q_R$ are projective, and since $R$ is local, projectives are free. Now the assertion follows from [4], Theorem 3.1. The converse is obvious by Example 2.

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