TSUKUBA J. MATH.
Vol. 4 (1980). 99—105

SOME METRICS ON A $(4r+3)$-SPHERE AND SPECTRA

By

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Let (S^{m}, g) be a unit sphere in a Euclidean $(m+1)$-space. In a paper [4] the author gave an orthogonal decomposition of the eigenspace V_{k} corresponding to the k-th eigenvalue of the Laplacian acting on functions on (S^{2n+1}, g). This decomposition is related to the Hopf fibration: $S^{2n+1}\rightarrow CP^{n}$, where CP^{n} denotes the complex projective space, and we can define a 1-parameter family of Riemannian metrics $g(t)$ such that $g(0)=g$ and the spectrum of each $(S^{2n+1}, g(t))$ is calculatable by this decomposition. In §2 we give a brief review on $(S^{2n+1}, g(t))$.

The analogous decomposition of V_{k} is possible for the Hopf fibration: $S^{4r+3}\rightarrow QP^{r}$, where QP^{r} denotes the quaternion projective space. The decomposition is given by Proposition 3.1. We define a 1-parameter family of Riemannian metrics $g(t)$ on S^{4r+3} such that $g(0)=g$ and the volume element with respect to $g(t)$ is unchanged when t varies. Then the first eigenvalue ${}^{(t)}\lambda_{1}$ of the Laplacian ${}^{(t)}\Delta$ on $(S^{4r+3}, g(t))$ is given by Proposition 3.2, and we see that ${}^{(t)}\lambda_{1}\rightarrow 0$ as $t\rightarrow 0$ and ${}^{(t)}\lambda_{1}\rightarrow\infty$ as $t\rightarrow\infty$. The multiplicity of ${}^{(t)}\lambda_{1}$ is given by Proposition 4.1.

Results of [2]$\sim$[5] show that the generalization of Hersch type theorem on S^{2} to S^{m} $(m\geqq 3)$ or to some homogeneous spaces is impossible. These metrics in [2]$\sim$[5] are related to 1-dimensional distributions on manifolds. Metrics on S^{4r+3} in this paper are related to 3-dimensional distributions. And they give new examples of compact Riemannian manifolds whose spectra are calculatable.

§1. Preliminaries.

Let ξ_{1}, ξ_{2} and ξ_{3} be Killing vector fields on a Riemannian manifold (M, g) of dimension m such that

(1.1) $\qquad \xi_{1}, \xi_{2}$ and ξ_{3} are orthonormal,

(1.2) $\qquad [\xi_{1}, \xi_{2}]=2\xi_{3}, \quad [\xi_{2}, \xi_{3}]=2\xi_{1}, \quad [\xi_{3}, \xi_{1}]=2\xi_{2}.$

By η_{1}, η_{2} and η_{3} we denote the 1-forms dual to ξ_{1}, ξ_{2} and ξ_{3} with respect to g. Then we obtain

Received December 6, 1979

(1.3) $$L_{\xi_1}\eta_1=0\,,\quad L_{\xi_1}\eta_2=2\eta_3\,,\quad L_{\xi_1}\eta_3=-2\eta_2\,,$$

and the corresponding relations for L_{ξ_2} and L_{ξ_3}, where L_X denotes the Lie derivation by a vector field X.

We define a Riemannian metric $*g$ by

(1.4) $$*g=\alpha g+(\beta-\alpha)(\eta_1\otimes\eta_1+\eta_2\otimes\eta_2+\eta_3\otimes\eta_3)$$

for some positive numbers α and β. By ∇ and $*\nabla$ we denote the Riemannian connections by g and $*g$, respectively.

By $\{\exp tX\}$ we denote the local 1-parameter group of local transformations generated by a vector field X. Since each ξ_i is a unit Killing vector field, $\{(\exp t\xi_i)p\}$ is a geodesic in (M, g) for each point p of M. By (1.3) and (1.4) ξ_i is also a Killing vector field with respect to $*g$ of constant length $\beta^{1/2}$. In particular, for each point p of M, $\{(\exp t\xi_i)p\}$ is also a geodesic with respect to $*g$.

LEMMA 1.1. *Let $\{x(t)\}$ be a geodesic with arclength parameter t in (M, g). If $\{x(t)\}$ is orthogonal to each ξ_i at some point $x(t_0)$, then $\{x(t)\}$ is also a geodesic with respect to $*g$.*

PROOF. Since each ξ_i is a Killing vector field, the geodesic $\{x(t)\}$ is orthogonal to ξ_i at $x(t_0)$ if and only if it is orthogonal to ξ_i at each point $x(t)$. Let e be a unit vector field defined on an open neighborhood U of a piece l of $\{x(t)\}$, such that e is orthogonal to each ξ_i on U and satisfies $e=dx(t)/dt$ on l. Then $\nabla_e e=0$ holds on l. By an identity defining $*\nabla$ we get

$$2*g(*\nabla_e e, Z)=2e\cdot *g(e, Z)-2*g(e, [e, Z])$$

on l, where Z denotes a vector field on U. Since $*g(e, Z)=\alpha g(e, Z)$, we get $*g(*\nabla_e e, Z)=\alpha g(\nabla_e e, Z)=0$ on l. This shows that l in U and hence $\{x(t)\}$ in (M, g) is a geodesic with respect to $*g$. Q. E. D.

Let Δ and $*\Delta$ be the Laplacians with respect to g and $*g$, respectively. Then we get

LEMMA 1.2. *For a function f on M,*

$$*\Delta f=\alpha^{-1}\Delta f+(\beta^{-1}-\alpha^{-1})(L_{\xi_1}L_{\xi_1}+L_{\xi_2}L_{\xi_2}+L_{\xi_3}L_{\xi_3})f\,.$$

PROOF. Let p be a point of M and let $\{\xi_1, \xi_2, \xi_3, e_4, \cdots, e_m\}$ be a field of orthonormal frames with respect to g defined on an open neighborhood of p such that $\{(\exp te_j)p,\ |t|<\varepsilon\}$ is a geodesic for each $j=4, \cdots, m$. Since $(\Delta f)(p)$ is given by the sum of the second derivatives of f at p with respect to the arclength parameter t along mutually orthogonal m geodesics passing through p, we get

$$(\Delta f)(p)=\sum_i (L_{\xi_i}L_{\xi_i}f)(p)+\sum_j (L_{e_j}L_{e_j}f)(p).$$

$\{(\exp t\beta^{-1/2}\xi_i)p\}$ and $\{(\exp t\alpha^{-1/2}e_j)p\}$ are geodesics with arclength parameter t with respect to $*g$. Therefore,

$$(^*\Delta f)(p)=\beta^{-1}\sum_i (L_{\xi_i}L_{\xi_i}f)(p)+\alpha^{-1}\sum_j (L_{e_j}L_{e_j}f)(p),$$

and we get the identity.

§ 2. A review on $(S^{2n+1}, g(t))$.

Let (S^{2n+1}, g) be a unit sphere of dimension $m=2n+1$ in a Euclidean space E^{m+1}. E^{m+1} is considered as a complex Euclidean space CE^{n+1} and so let $(x^\alpha, y^\alpha; \alpha=1, \cdots, n+1)$ be a natural coordinate system in E^{m+1}. Then we have an almost complex structure J such that $J(x^\alpha, y^\alpha)=(y^\alpha, -x^\alpha)$. If one considers a point $x=(x^\alpha, y^\alpha)$ in S^m as a unit vector in CE^{n+1}, and Jx as a tangent vector at x to S^m, we get a vector field ξ on S^m. The 1-form η dual to ξ with respect to g on S^m is a contact structure on S^m. ξ is a Killing vector field and called sometimes a Sasakian structure on (S^m, g).

The spectrum of the Laplacian Δ acting on functions on (S^m, g) is given by

$$\text{(2.1)} \qquad \operatorname{Spec}(S^m, g)=\{\lambda_k=k(m+k-1);\ k=0, 1, \cdots\},$$

where the multiplicity $\mu(k)$ of λ_k is given by

$$\text{(2.2)} \qquad \mu(k)={}_{m+k}C_k-{}_{m+k-2}C_{k-2}, \quad k\geqq 2,$$

and $\mu(0)=1$, $\mu(1)=m+1$.

We define a 1-parameter family of Riemannian metrics $g(t)$ by

$$\text{(2.3)} \qquad g(t)=t^{-1}g+(t^{m-1}-t^{-1})\eta\otimes\eta.$$

Then the volume element with respect to $g(t)$ is unchanged when t varies, and the Laplacian ${}^{(t)}\Delta$ is given by

$$\text{(2.4)} \qquad {}^{(t)}\Delta f=t\Delta f+(t^{1-m}-t)L_\xi L_\xi f.$$

Let V_k denote the eigenspace corresponding to the k-th eigenvalue λ_k of Δ. $L_\xi L_\xi$ induces a symmetric linear transformation of V_k with respect to the usual inner product and we see that $L_\xi L_\xi$ has non-positive eigenvalues $-\theta^2$, where $\theta=k, k-2, \cdots, k-2[k/2]$ (where $[k/2]$ is the integral part of $k/2$). V_k has the following orthogonal decomposition

$$\text{(2.5)} \qquad V_k=V_{k,k}+V_{k,k-2}+\cdots+V_{k;k-2[k/2]}$$

such that $f \in V_{k,\theta}$ satisfies $L_\xi L_\xi f + \theta^2 f = 0$. Thus,

$$ {}^{(t)}\Delta f + [tk(m+k-1) + (t^{1-m} - t)\theta^2] f = 0 \tag{2.6} $$

for $f \in V_{k,\theta}$. The first eigenvalue ${}^{(t)}\lambda_1$ of ${}^{(t)}\Delta$ is $(2n+t^{-m})t$ for $t^{-m} \leqq m+3$ and $4(n+1)t$ for $t^{-m} \geqq m+3$. Consequently, ${}^{(t)}\lambda_1 \to 0$ as $t \to 0$ and ${}^{(t)}\lambda_1 \to \infty$ as $t \to \infty$.

Since $\dim V_{2,0} = n(n+2)$ and $\dim V_{1,1} = m+1$, the multiplicity of ${}^{(t)}\lambda_1$ is equal to $(m^2+6m+1)/4$ for $t^{-m} = m+3$. This is bigger than the multiplicity $m+1$ of the first eigenvalue ${}^{(0)}\lambda_1 = m$ with respect to the standard metric.

§ 3. $(S^{4r+3}, g(t))$.

Let (S^{4r+3}, g) be a unit sphere in $E^{4(r+1)}$. $E^{4(r+1)}$ is considered as a product space $Q \times \cdots \times Q$ of $r+1$ copies of the space of quaternions. Let $(x^\alpha, y^\alpha, z^\alpha, w^\alpha;\ \alpha = 1, \cdots, r+1)$ be a natural coordinate system in $E^{4(r+1)}$. Let $\{I, J, K\}$ be the quaternion structure of $E^{4(r+1)}$. If one considers a point $x = (x^\alpha, y^\alpha, z^\alpha, w^\alpha)$ of S^{4r+3} as a unit vector in $E^{4(r+1)}$ and

$$ Ix = (y^\alpha, -x^\alpha, w^\alpha, -z^\alpha), $$
$$ Jx = (z^\alpha, -w^\alpha, -x^\alpha, y^\alpha), $$
$$ Kx = (w^\alpha, z^\alpha, -y^\alpha, -x^\alpha) $$

as tangent vectors at x to S^{4r+3}, we get a field of orthonormal vectors ξ_1, ξ_2 and ξ_3 on S^{4r+3}. We put

$$ \xi_1^\# = \sum_\alpha (y^\alpha \partial/\partial x^\alpha - x^\alpha \partial/\partial y^\alpha + w^\alpha \partial/\partial z^\alpha - z^\alpha \partial/\partial w^\alpha), $$
$$ \xi_2^\# = \sum_\alpha (z^\alpha \partial/\partial x^\alpha - w^\alpha \partial/\partial y^\alpha - x^\alpha \partial/\partial z^\alpha + y^\alpha \partial/\partial w^\alpha), $$
$$ \xi_3^\# = \sum_\alpha (w^\alpha \partial/\partial x^\alpha + z^\alpha \partial/\partial y^\alpha - y^\alpha \partial/\partial z^\alpha - x^\alpha \partial/\partial w^\alpha). $$

Then each ξ_i is the restriction of $\xi_i^\#$ on $E^{4(r+1)}$ to S^{4r+3}. ξ_1, ξ_2 and ξ_3 are Killing vector fields and satisfy (1.1) and (1.2). The 3-dimensional distribution defined by $\{\xi_1, \xi_2, \xi_3\}$ is integrable and each integral submanifold is isometric to a unit 3-sphere. This gives the Hopf fibration: $S^{4r+3} \to QP^r$.

We define a 1-parameter family of Riemannian metrics $g(t)$ by

$$ g(t) = t^{-1} g + (t^{4r/3} - t^{-1})(\eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3) \tag{3.1} $$

where η_1, η_2 and η_3 are defined similarly as in § 1. The volume element with respect to $g(t)$ is unchanged when t varies. By Lemma 1.2 the Laplacians ${}^{(t)}\Delta$ and Δ with respect $g(t)$ and g are related by

$$ {}^{(t)}\Delta f = t\Delta f + (t^{-4r/3} - t) L f, \tag{3.2} $$

where we have put $L=\sum_i L_{\xi_i}L_{\xi_i}$.

Let V_k denote the eigenspace corresponding to the k-th eigenvalue of the Laplacain Δ on (S^{4r+3}, g). Since each $L_{\xi_i}L_{\xi_i}$ induces a symmetric transformation of V_k, L is also a symmetric transformation of V_k. Every eigenvalue of L is real and non-positive.

PROPOSITION 3.1 *For a non-negative integer k, V_k has the orthogonal decomposition;*

$$V_k=W_{k,k}+W_{k,k-2}+\cdots+W_{k,k-2[k/2]}$$

such that $f\in W_{k,\theta}$ satisfies

$${}^{(t)}\Delta f+[tk(4r+k+2)+(t^{-4r/3}-t)\theta(\theta+2)]f=0\,.$$

PROOF. V_k is identified with the space of harmonic homogeneous polynomials of degree k in $E^{4(r+1)}$. Let F be an element of V_k. We put $L^{\#}=\sum_i L_{\xi_i^{\#}}L_{\xi_i^{\#}}$. Then $L^{\#}F|S^{4r+3}=Lf$ holds, where $|$ denotes the restriction and $f=F|S^{4r+3}$. Let

$$V_k=V_{k;1}+V_{k;2}+\cdots+V_{k;\nu}$$

be the orthogonal decomposition of V_k into eigenspace with respect to $L^{\#}$ or L such that $Lf+\omega_h f=0$ for $f\in V_{k;h}$, where $1\leqq h\leqq \nu$. We take a point x of S^{4r+3}. Then the integral submanifold W of the distribution $\{\xi_1, \xi_2, \xi_3\}$ passing through x is isometric to a unit 3-sphere (S^3, g). The restriction of L to W is the usual Laplacian Δ on (S^3, g). So, the eigenvalue ω_h of L must be an eigenvalue of Δ on (S^3, g) and hence it is of the form $\theta(\theta+2)$. Since F is of degree k, the degree of its restriction to W is one of $k, k-2, \cdots, 0$ (for k=even) or 1 (for k=odd). Thus,

$$\theta(\theta+2)=k(k+2),\quad (k-2)k,\ \cdots,\ (k-2[k/2])(k+2-2[k/2])\,.$$

Then Proposition 3.1 follows from (2.1) and (3.2).

PROPOSITION 3.2. *The first eigenvalue ${}^{(t)}\lambda_1$ of ${}^{(t)}\Delta$ is given by*

$$\begin{aligned} {}^{(t)}\lambda_1&=8(r+1)t \qquad \textit{for}\quad 4(r+2)\leqq 3t^{-(4r+3)/3} \\ &=4rt+3t^{-4r/3} \quad \textit{for}\quad 4(r+2)\geqq 3t^{-(4r+3)/3} \end{aligned}$$

and ${}^{(t)}\lambda_1\to 0$ as $t\to 0$; ${}^{(t)}\lambda_1\to\infty$ as $t\to\infty$.

PROOF. This follows from the table of possibility of eigenvalues given in Proposition 3.1 and the fact that $W_{1,1}$ and $W_{2,0}$ are non-empty (cf. (i), (iv) in §4).

§4. On $W_{k,\theta}$.

In this section we denote $x^\alpha, \cdots, w^\alpha$ by $x_\alpha, \cdots, w_\alpha$.

(i) $V_1 = W_{1,1}$ and $\dim W_{1,1} = 4(r+1)$.

(ii) $W_{k,k}$ is non-empty. In fact, let F be a harmonic homogeneous polynomial of degree k such that $F = F(x_1, y_1, z_1, w_1)$. Let E^4 be defined by $x_\alpha = y_\alpha = z_\alpha = w_\alpha = 0$; $\alpha = 2, \cdots r+1$, and put $S^3 = S^{4r+3} \cap E^4$. Since

$$L^\# F | S^3 = \Delta(F | S^3) = -k(k+2)(F | S^3)$$

and since $k(k+2)$ is the possible maximum eigenvalue of L of V_k, we get $F \in W_{k,k}$.

(iii) For $k = 2q =$ even, $W_{k,0}$ is non-empty. In fact, $\dim W_{k,0}$ is equal to the multiplicity of the q-th eigenvalue of the Laplacian on the base manifold QP^r of the Hopf fibration.

(iv) $\dim W_{2,0} = r(2r+3)$ and $W_{2,0}$ is spanned by

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 - x_\alpha^2 - y_\alpha^2 - z_\alpha^2 - w_\alpha^2; \quad 2 \leqq \alpha \leqq r+1,$$
$$x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta + w_\alpha w_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha y_\beta - y_\alpha x_\beta - z_\alpha w_\beta + w_\alpha z_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha z_\beta + y_\alpha w_\beta - z_\alpha x_\beta - w_\alpha y_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha w_\beta - y_\alpha z_\beta + z_\alpha y_\beta - w_\alpha z_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1.$$

In fact, let P_2 denote the space of homogeneous polynomials of degree 2 in $E^{4(r+1)}$. $\dim P_2 = {}_{4r+5}C_2$. $L^\#$ acts on P_2 with two eigenvalues 0 and -8. Put $P_{2,0} = \{F \in P_2;\ L^\# F = 0\}$. Then $\dim P_{2,0} = (r+1)(2r+1)$ and $P_{2,0}$ is spanned by

$$x_\alpha^2 + y_\alpha^2 + z_\alpha^2 + w_\alpha^2; \quad 1 \leqq \alpha \leqq r+1,$$
$$x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta + w_\alpha w_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha y_\beta - y_\alpha x_\beta - z_\alpha w_\beta + w_\alpha z_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha z_\beta + y_\alpha w_\beta - z_\alpha x_\beta - w_\alpha y_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha w_\beta - y_\alpha z_\beta + z_\alpha y_\beta - w_\alpha x_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1.$$

Put $P_{2,2} = \{F \in P_2; L^\# F + 8F = 0\}$. Then $\dim P_{2,2} = 3(r+1)(2r+3)$ and $P_{2,2}$ is spanned by

$$x_\alpha y_\alpha,\ x_\alpha z_\alpha,\ x_\alpha w_\alpha,\ y_\alpha z_\alpha,\ y_\alpha w_\alpha,\ z_\alpha w_\alpha; \quad 1 \leqq \alpha \leqq r+1,$$
$$x_\alpha^2 - y_\alpha^2,\ x_\alpha^2 - z_\alpha^2,\ x_\alpha^2 - w_\alpha^2; \quad 1 \leqq \alpha \leqq r+1,$$
$$x_\alpha x_\beta - y_\alpha y_\beta,\ x_\alpha x_\beta - z_\alpha z_\beta,\ x_\alpha x_\beta - w_\alpha w_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$
$$x_\alpha y_\beta + y_\alpha x_\beta,\ x_\alpha y_\beta + z_\alpha w_\beta,\ x_\alpha y_\beta - w_\alpha z_\beta; \quad 1 \leqq \alpha < \beta \leqq r+1,$$

$$x_\alpha z_\beta - y_\alpha w_\beta,\ x_\alpha z_\beta + z_\alpha x_\beta,\ x_\alpha z_\beta + w_\alpha y_\beta\ ; \qquad 1 \leqq \alpha < \beta \leqq r+1\ ,$$

$$x_\alpha w_\beta + y_\alpha z_\beta,\ x_\alpha w_\beta - z_\alpha y_\beta,\ x_\alpha w_\beta + w_\alpha x_\beta\ ; \quad 1 \leqq \alpha < \beta \leqq r+1\ .$$

Since $V_2 = P_2 - \{\sum_\alpha (x_\alpha^2 + y_\alpha^2 + z_\alpha^2 + w_\alpha^2)\}$, dim $W_{2,0}$=dim $P_{2,0} - 1 = r(2r+3)$.

PROPOSITOIN 4.1. *The multiplicity of the first eigenvalue* ${}^{(t)}\lambda_1$ *of* ${}^{(t)}\Delta$ *on* $(S^{4r+3}, g(t))$ *is*

$$\begin{aligned} &r(2r+3) && \textit{for}\quad 4(r+2) < 3t^{-(4r+3)/3}\ , \\ &2r^2+7r+4 && \textit{for}\quad 4(r+2) = 3t^{-(4r+3)/3}\ , \\ &4(r+1) && \textit{for}\quad 4(r+2) > 3t^{-(4r+3)/3}\ . \end{aligned}$$

This is verified by noticing that the multiplicity of ${}^{(t)}\lambda_1$ is one of dim $W_{2,0}$, dim $W_{2,0}$+dim $W_{1,1}$, and dim $W_{1,1}$ according to the respective case.

(v) Let P_3 denote the space of homogeneous polynomials of degree 3. Then $P_3 = P_{3,1} + P_{3,3}$, where $P_{3,1} = \{F \in P_3\ ;\ L^\# F + 3F = 0\}$ and $P_{3,3} = \{F \in P_3\ ;\ L^\# F + 15F = 0\}$. dim $W_{3,1}$=dim $P_{3,1} - 4(r+1)$, and $W_{3,3} = P_{3,3}$.

Examples of elements of $W_{3,1}$ are:

$$2x_1(x_1^2+y_1^2+z_1^2+w_1^2) - 3x_1(x_2^2+y_2^2+z_2^2+w_2^2)\ ,$$

$$x_1(x_1x_2+y_1y_2+z_1z_2+w_1w_2) - y_1(-x_1y_2-w_1z_2+z_1w_2+y_1x_2)\ .$$

Examples of elements of $W_{3,3}$ are:

$$x_1(x_1^2-y_1^2-z_1^2-w_1^2)\ ,\quad x_1y_1z_1\ ,$$

$$2x_1y_1y_2+y_1^2x_2-x_1^2x_2\ .$$

If $r=1$, dim $W_{3,1}=32$ and dim $W_{3,3}=80$.

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