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NOTES ON M-SEMIGROUPS

By

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Introduction.

Let S be a torsion-free cancellative commutative (additive) semigroup $\supsetneqq\{0\}$. Let G be the quotient group of S. We assume $G\neq S$. For each subset A of G, we set $A^{-1}=\{x\in G: x+A\subset S\}$ and $(A^{-1})^{-1}=A^{v}$. If $A^{v}=A$ for an ideal A of S, then A is called a v-ideal of S. If S satisfies the ascending chain condition for v-ideals, then S is called a Mori-semigroup. If S is a Mori-semigroup and and if each ideal of S generated by two elements is a v-ideal, then S is called an M-semigroup ([2]). If each ideal of S is a v-ideal, then S is called a reflexive semigroup. The maximal number n such that there exists a chain $P_1\supsetneqq P_2\supsetneqq\cdots\supsetneqq P_n$ of prime ideals of S is denoted by $\dim S$. If $\dim S\geqq 1$, then S has a unique maximal ideal.

In this paper we study a semigroup version of a result ([1, Théorème 3]) of Querre. Our result is the following.

MAIN THEOREM. *Let S be a Mori-semigroup. Then the following conditions are equivalent*:

(1) $\dim S=1$ *and* M^{-1} *is generated by two elements for the maximal ideal* M *of* S.

(2) *S is a reflexive semigroup.*

(3) *Each ideal of S generated by two elements is a v-ideal.*

In [4] it is shown that the conditions (2) and (3) are equivalent and (2) implies (1). Therefore it is sufficient for us to show that (1) implies (2).

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1. Notations and Preliminaries.

Let $\boldsymbol{Z}$ be the set of integers and let $\boldsymbol{N}$ be the set of natural numbers. If for $v\in M$ and $u\in S$, there exists $n\in\boldsymbol{N}$ such that $nv\in(u)$, S is called a weakly

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archimedean semigroup.

PROPOSITION 1.1. *The following are equivalent*:
(1) $\dim S=1$.
(2) *S is a weakly archimedean semigroup.*

PROOF. $(1)\Rightarrow(2)$: Let u and v be elements of M. We set $V=\{nv: n\in \boldsymbol{N}\}$. Then it is sufficient for us to show that $V\cap(u)\neq\phi$. Suppose, to the contrary, that $V\cap(u)=\phi$. Let F be the set of ideals of S such that do not intersect with V. Then F is not empty and contains a maximal element P. It can be shown that P is a prime ideal. This is a contradiction. Suppose P is not prime. Let x and y be elements of S such that $x\notin P$, $y\notin P$ and $x+y\in P$. Then we have

$$x+s=lv, \qquad y+t=mv$$

for some elements $s, t\in S$ and for some positive integers l, m. Hence we get

$$x+y+s+t=(l+m)v\in P\cap V.$$

This is a contradiction.
The implication $(2)\Rightarrow(1)$ is obvious. q. e. d.

In the remainder of this paper we assume S is a weakly archimedean semigroup. For $d\in M$ we set

$$D=\boldsymbol{Z}d+H, \qquad G/D=\Gamma,$$

where H is the unit group of S. We assume $\Gamma\neq\{\varepsilon\}$, where ε is the zero element of Γ. For each $\gamma\in\Gamma$, the coset of γ is denoted by D_γ. For each $\gamma\in\Gamma$ we choose $x_\gamma\in D_\gamma$ and set

$$p_\gamma=\min\{n\in\boldsymbol{Z}: nd+x_\gamma\in S\}, \qquad s_\gamma=p_\gamma d+x_\gamma.$$

Let β and γ be elements of Γ. Then there exist $h(\beta,\gamma)\in H$ and $I(\beta,\gamma)(\geqq 0)\in\boldsymbol{Z}$ uniquely such that

$$s_\beta+s_\gamma=h(\beta,\gamma)+I(\beta,\gamma)d+s_{\beta+\gamma}.$$

Then we obtain

PROPOSITION 1.2. [4, Proposition 5]
(1) $I(\beta,\gamma)=I(\gamma,\beta))$ *for each* $\beta,\gamma\in\Gamma$.
(2) $I(\alpha,\beta)+I(\alpha+\beta,\gamma)=I(\alpha,\beta+\gamma)+I(\beta,\gamma)$ *for each* $\alpha,\beta,\gamma\in\Gamma$.
(3) $I(\varepsilon,\gamma)=0$ *for each* $\gamma\in\Gamma$.
(4) $I(\beta,\gamma)\leqq I(\beta,-\beta)$ *for each* $\beta,\gamma\in\Gamma$.

(5) *If* $\gamma \neq \varepsilon$, *there exists* $n \in \boldsymbol{N}$ *such that* $I(n\gamma, \gamma)>0$.

(6) *If* $\gamma \neq \varepsilon$, *then* $I(-\gamma, \gamma)>0$.

Let A be an ideal of S and set $V=\{z \in G: A \subset (z)\}$. Then we have $A^v=\bigcap_{z \in V}(z)$. For each $\gamma \in \Gamma$ we set

$$p_\gamma(A)=\min\{n \in \boldsymbol{Z}: nd+x_\gamma \in A\},$$

$$t_\gamma(A)=\max\{n \in \boldsymbol{Z}: A \subset (nd-x_\gamma)\}.$$

Then we have the following ([**4**, §1]):

$$A^v=\bigcap_{\gamma \in \Gamma}(t_\gamma(A)d-x_\gamma),$$

$$t_\gamma(A)=\min\{p_\beta(A)+I(\beta, \gamma)-p_\beta-p_\gamma: \beta \in \Gamma\},$$

$$p_\gamma(A^v)=\max\{t_\beta(A)+p_\beta+p_\gamma-I(\beta, \gamma): \beta \in \Gamma\}.$$

If $p_\gamma(A^v) \geqq p_\gamma(A)$ for each $\gamma \in \Gamma$, then $A=A^v$ holds.

PROPOSITION 1.3. ([**4**, Proposition 9]). *Let A be an ideal of S. Then the following are equivalent:*

(1) *A is a v-ideal.*

(2) *For each $\gamma \in \Gamma$ there exists $\gamma^* \in \Gamma$ such that*

$$p_\gamma(A) \leqq p_\gamma+p_\beta(A)+I(\beta, \gamma^*)-I(\gamma, \gamma^*)-p_\beta$$

for each $\beta \in \Gamma$.

For each $\theta \in \Gamma$ we set

$$N(\theta)=\{\beta \in \Gamma: I(\theta, \beta)=0, \beta \neq \varepsilon\}.$$

Let M be the unique maximal ideal of S. Then we get

$$p_\varepsilon(M)=p_\varepsilon+1, \qquad p_\gamma(M)=p_\gamma \quad (\gamma \neq \varepsilon)$$

and

$$t_\gamma(M)=\begin{cases} 1-p_\gamma & (\text{if } N(\gamma)=\phi) \\ -p_\gamma & (\text{if } N(\gamma) \neq \phi) \end{cases}$$

Therefore we obtain

$$p_\varepsilon(M^v)=\max\{t_\beta(M)+p_\beta: \beta \in \Gamma\}+p_\varepsilon(M)-1,$$

$$p_\gamma(M^v)=\max\{t_\beta(M)+p_\beta-I(\beta, \gamma): \beta \in \Gamma\}+p_\gamma(M) \quad (\gamma \neq \varepsilon).$$

Since $t_\varepsilon(M)+p_\varepsilon-I(\varepsilon, \gamma)=0$, we have

$$p_\gamma(M^v) \geqq p_\gamma(M) \quad (\gamma \neq \varepsilon).$$

Then we get the following.

PROPOSITION 1.4. *The following are equivalent*:

(1) *M is a v-ideal.*

(2) *There exists $\theta\in\Gamma$ such that $N(\theta)=\phi$.*

(3) $M^{-1}\supsetneqq S$.

PROOF. (1)$\Rightarrow$(2): Since $p_{\varepsilon}(M^{v})=p_{\varepsilon}(M)$, (2) holds. (2)$\Rightarrow$(3): If $N(\theta)=\phi$, we get $-d+s_{\theta}\in M^{-1}-S$. (3)$\Rightarrow$(1): There exists $\theta\in\Gamma$ such that $-d+s_{\theta}\in M^{-1}$ and hence $N(\theta)=\phi$. Thus we obtain $p_{\varepsilon}(M^{v})=p_{\varepsilon}(M)$ and hence $M^{v}=M$.

q. e. d.

PROPOSITION 1.5. *If there exists $\theta\in\Gamma$ such that $N(\theta)=\phi$ and $I(\gamma, \theta-\gamma)=0$ for all $\gamma\in\Gamma$, then S is a reflexive semigroup.*

PROOF. We set $\gamma^{*}=\theta-\gamma$ for each $\gamma\in\Gamma$. Since $\beta+\gamma^{*}=(\gamma-\beta)^{*}$, we obtain

$$I(\gamma-\beta, \beta)=I(\gamma-\beta, \beta)+I(\gamma, \gamma^{*})=I(\gamma-\beta, \beta+\gamma^{*})+I(\beta, \gamma^{*})=I(\beta, \gamma^{*}).$$

Let A be an ideal of S. Then we get

$$\begin{aligned}&\{p_{\gamma}+p_{\beta}(A)+I(\beta, \gamma^{*})-p_{\beta}\}d+x_{\gamma}\\&=a_{\beta}+\{I(\beta, \gamma^{*})-I(\gamma-\beta, \beta)\}d+s_{\gamma-\beta}-h(\gamma-\beta, \beta)\\&=a_{\beta}+s_{\gamma-\beta}-h(\gamma-\beta, \beta)\in A\end{aligned}$$

for each $\beta\in\Gamma$, where we set $p_{\beta}(A)d+x_{\beta}=a_{\beta}\in A$. Therefore we obtain

$$p_{\gamma}(A)\leqq p_{\gamma}+p_{\beta}(A)+I(\beta, \gamma^{*})-I(\gamma, \gamma^{*})-p_{\beta}$$

for each $\beta\in\Gamma$. By Proposition 1.3, $A^{v}=A$. q. e. d.

2. Proof of the Main Theorem.

PROPOSITION 2.1. *Let θ be an element of Γ such that $N(\theta)=\phi$ and $S\subset(-d+s_{\theta})$. Then,*

(1) $I(\theta, -\theta)=1$.

(2) $I(\theta, \gamma)=1$ *for each* $\gamma(\neq\varepsilon)\in\Gamma$.

(3) $I(-\theta, \gamma)=0$ *for each* $\gamma(\neq\theta)\in\Gamma$.

(4) *If $N(\eta)=\phi$, then $\eta=\theta$.*

PROOF. (1): Since $0\in(-d+s_{\theta})$ and

$$-d+s_{\theta}+s_{-\theta}=\{I(\theta, -\theta)-1\}d+s_{\varepsilon}+h(\theta, -\theta)\in H,$$

(1) holds. (2): By Proposition 1.2(4), we get (2). (3): Since we have

$$I(\theta, -\theta)+I(\varepsilon, \gamma)=I(\theta, \gamma-\theta)+I(-\theta, \gamma),$$

(3) holds. (4): It is obvious. q. e. d.

PROPOSITION 2.2. *Assume that $M^{-1}\supsetneqq S$ and M^{-1} is generated by two elements. Then the following assertions hold.*

(1) *There exists uniquely $\theta\in\Gamma$ such that $N(\theta)=\phi$.*

(2) $M^{-1}=(0, -d+s_{\theta})$.

PROOF. By Proposition 1.4, there exists $\theta\in\Gamma$ such that $N(\theta)=\phi$. Suppose that $N(\eta)=\phi$ for some $\eta(\neq\theta)\in\Gamma$. Since

$$M^{-1}-S=\{-d+s_{\xi}+h : N(\xi)=\phi,\ h\in H\},$$

$$-d+s_{\theta}\notin(-d+s_{\eta}), \qquad -d+s_{\eta}\notin(-d+s_{\theta}),$$

and M^{-1} is generated by two elements, we obtain

$$M^{-1}=(-d+s_{\theta}, -d+s_{\eta})$$

and hence

$$0\in(-d+s_{\theta}, -d+s_{\eta})=(-d+s_{\theta})\cup(-d+s_{\eta}).$$

If $0\in(-d+s_{\theta})$, then $S\subset(-d+s_{\theta})$ and $\theta=\eta$ by Proposition 2.1. This is a contradiction. Similarly, if $0\in(-d+s_{\eta})$, then $\eta=\theta$. q. e. d.

PROPOSITION 2.3. *Let S be a Mori-semigroup and let u be an element of M. Then,*

(1) *There exists $n\in\boldsymbol{N}$ such that nM $(=M+M+\cdots+M)\subset(u)$.*

(2) *M is a v-ideal.*

PROOF. (1): Let F be the set of all finitely generated ideals of S contained in M and set $F^{v}=\{A^{v}: A\in F\}$. There exists a maximal element A^{v} in F^{v}, where $A\in F$. Then $A^{v}=M^{v}$ holds. Since A is finitely generated, there exists $n\in\boldsymbol{N}$ such that $nA\subset(u)$. Thus we get

$$(nA)^{v}=(nA^{v})^{v}=(nM^{v})^{v}\subset(u)$$

and hence $nM\subset(u)$.

(2): If $M^{v}=S$, then $S\subset(u)$. This is a contradiction. q. e. d.

COROLLARY. *Let S be a Mori-semigroup. Then there exists $n\in\boldsymbol{N}$ satisfying the following property: Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be arbitrary non-zero elements of Γ. Then we have*

$$I(\alpha_1, \alpha_2)+I(\alpha_1+\alpha_2, \alpha_3)+\cdots+I(\alpha_1+\cdots+\alpha_{n-1}, \alpha_n)>0.$$

PROOF. By Proposition 2.3 there exists $n\in N$ such that $nM\subset(d)$. Since $s_{\alpha_i}\in M$ $(i=1, \cdots, n)$, we obtain

$$s_{\alpha_1}+s_{\alpha_2}+\cdots+s_{\alpha_n}\in nM\subset(d). \qquad \text{q. e. d.}$$

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be elements of Γ. The system $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ is said to be O-system if it satisfies the following condition:

Let K and J be non-empty subsets of $\{1, 2, \cdots, n\}$ such that $K\cap J=\phi$ and set

$$\alpha_K=\sum_{k\in K}\alpha_k, \qquad \alpha_J=\sum_{j\in J}\alpha_j.$$

Then $I(\alpha_K, \alpha_J)=0$.

Furthermore, if $\alpha_K\neq\varepsilon$ for each subset $K\neq\phi$ of $\{1, 2, \cdots, n\}$, then $[\alpha_1, \cdots, \alpha_n]$ is called a regular O-system.

PROPOSITION 2.4. *Let $[\alpha_1, \cdots, \alpha_n]$ be an O-system and let α_{n+1} be an element of Γ such that $I(\alpha_1+\cdots+\alpha_n, \alpha_{n+1})=0$. Then $[\alpha_1, \cdots, \alpha_n, \alpha_{n+1}]$ is an O-system.*

PROOF. Let $J\neq\phi$ be a proper subset of $\{1, \cdots, n\}$. Then

$$I(\alpha_{J^c}, \alpha_J)+I(\alpha_{J^c}+\alpha_J, \alpha_{n+1})=I(\alpha_{J^c}, \alpha_J+\alpha_{n+1})+I(\alpha_J, \alpha_{n+1})=0,$$

where $J^c=\{1, \cdots, n\}-J$. Therefore we obtain

$$I(\alpha_{J^c}, \alpha_J+\alpha_{n+1})=0, \qquad I(\alpha_J, \alpha_{n+1})=0.$$

Next, consider a proper subset $K\neq\phi$ of J^c. Then

$$I(\alpha_{J^c-K}, \alpha_K)+I(\alpha_{J^c}, \alpha_J+\alpha_{n+1})=I(\alpha_{J^c-K}, \alpha_K+\alpha_J+\alpha_{n+1})+I(\alpha_K, \alpha_J+\alpha_{n+1})=0.$$

Thus we get $I(\alpha_K, \alpha_J+\alpha_{n+1})=0$. q. e. d.

PROPOSITION 2.5. *Let $[\alpha_1, \cdots, \alpha_n]$ be a regular O-system and let α_{n+1} be a non-zero element of Γ such that $I(\alpha_1+\cdots+\alpha_n, \alpha_{n+1})=0$. Then $[\alpha_1, \cdots, \alpha_n, \alpha_{n+1}]$ is a regular O-system.*

PROOF. By Proposition 2.4, $[\alpha_1, \cdots, \alpha_n, \alpha_{n+1}]$ is an O-system. Let K be a non-empty subset of $\{1, 2, \cdots, n\}$ and set $K^c=\{1, \cdots, n\}-K$. Then we have

$$I(\alpha_1+\cdots+\alpha_n, -\alpha_K)+I(\alpha_{K^c}, \alpha_K)=I(\alpha_1+\cdots+\alpha_n, \varepsilon)+I(-\alpha_K, \alpha_K)$$

and hence

$$I(\alpha_1+\cdots+\alpha_n, -\alpha_K)=I(\alpha_K, -\alpha_K)>0.$$

Consequently, $\alpha_K+\alpha_{n+1}\neq\varepsilon$.

PROPOSITION 2.6. *Let S be a Mori-semigroup and assume that M^{-1} is generated by two elements. Then S is a reflexive semigroup.*

PROOF. By Proposition 2.2 and 2.3 there exists uniquely $\theta \in \Gamma$ such that $N(\theta)=\phi$. By Proposition 1.5 it is sufficient for us to show that $I(\alpha, \theta-\alpha)=0$ for all $\alpha \in \Gamma$. Let α be an element of Γ such that $\alpha \neq \theta, \varepsilon$. Since $N(\alpha) \neq \phi$, there exists $\alpha_2 (\neq \varepsilon) \in \Gamma$ such that $I(\alpha_1, \alpha_2)=0$, where we set $\alpha_1=\alpha$. By Proposition 2.5, $[\alpha_1, \alpha_2]$ is a regular O-system. If $\alpha_1+\alpha_2=\theta$, then we get

$$I(\alpha_1, \alpha_2)=I(\alpha, \theta-\alpha)=0.$$

If $\alpha_1+\alpha_2 \neq \theta$, then there exists $\alpha_3 \neq \varepsilon$ such that $I(\alpha_1+\alpha_2, \alpha_3)=0$. By Proposition 2.5, $[\alpha_1, \alpha_2, \alpha_3]$ is a regular O-system. By the corollary of Proposition 2.3 there exists a regular O-system $[\alpha_1, \alpha_2, \cdots, \alpha_m]$ $(m \leqq n)$ such that $\alpha_1+\alpha_2+\cdots+\alpha_m=\theta$, where n is as in the Corollary. Consequently, we have

$$I(\alpha, \theta-\alpha)=I(\alpha_1, \alpha_2+\cdots+\alpha_m)=0. \qquad \text{q. e. d.}$$

Thus the Main Theorem has been proved.

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