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ON THE GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS OF SECOND ORDER FULLY NONLINEAR HYPERBOLIC EQUATIONS WITH FIRST ORDER DISSIPATION IN THE EXTERIOR DOMAIN

By

Yoshihiro SHIBATA*

Abstract

In this paper, we establish global existence and uniqueness theorems of solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain, in the case that data are sufficiently small and smooth and that the space dimension $n\geqq 3$. Furthermore, we investigate some decay theorem

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Introduction

Let Ω be an unbounded domain in an n dimensional Euclidean space R^n with compact and C^∞ boundary $\partial\Omega$. Let us denote time variable by t or x_0 and space variables by $x=(x_1, \cdots, x_n)$, respectively. For differentiation, we use the symbols $\partial_t=\partial_0=\partial/\partial t$ and $\partial_x=(\partial_1, \cdots, \partial_n)$ with $\partial_j=\partial/\partial x_j$, $j=1, \cdots, n$. In this paper, we consider the following initial-boundary value problem:

$$
\text{(P)} \qquad \begin{cases} \Phi(u)=\partial_t^2 u+\partial_t u-\Delta u+F(t, x, \Lambda u)=f & \text{in } \mathcal{D}=[0, \infty)\times\Omega\,, \\ u=0 & \text{on } \mathcal{D}'=[0, \infty)\times\partial\Omega\,, \\ u(0, x)=\phi_0(x)\,, \quad (\partial_t u)(0, x)=\phi_1(x) & \text{in } \Omega\,, \end{cases}
$$

where $\Delta=\sum_{j=1}^n \partial_j^2$, $\Lambda u=(u, \tilde{\Lambda} u)$, $\tilde{\Lambda} u=(D_x^1 u, D_x^2 u, \partial_t u, \partial_t D_x^1 u, \partial_t^2 u)$, $D_x^1 u=(\partial_1 u, \cdots, \partial_n u)$, $D_x^2 u=(\partial_i\partial_j u\,;\ 1\leqq i,\ j\leqq n)$.

When the equation is quasilinear or semilinear with dissipation and Ω is R^n or bounded, global existence and uniqueness theorems and the study of the properties of solutions of problem (P) and so on have been treated by many authors (cf. [3], [7], [12], [14], [16], [20], [21] and further references in these papers). In the case $\Omega=R^n$, it is well-known that by the method due to Dionne [2], we can reduce fully nonlinear equations to quasilinear systems and hence we need not consider essentially fully nonlinear equations in R^n. Applying results due to Matsumura [7], we can thus show global existence and uniqueness theorems of Cauchy problem for $\Phi(u)$ with sufficiently small and smooth data. But, when Ω has boundary, we can not use such a method due to Dionne. In the case that Ω has boundary and $\Phi(u)$ is fully nonlinear, we must treat essentially phenomena what is called "*derivative loss*". In order to overcome such difficulties, we can use a well-known excellent method due to Nash [13] (also Moser [9]). In fact, Rabinowitz [15] established a global existence theorem of periodic solutions of fully nonlinear dissipative wave equation of the form: $\partial_t^2 u-\partial_x^2 u+\alpha\partial_t u=\varepsilon f(t, x, \Lambda u)$ $(\alpha>0)$, when $n=1$, Ω is bounded and ε is sufficiently small, by applying what is called the Nash-Moser technique. Recently, Shibata [18] showed global existence and uniqueness theorems and decay theorems of solutions of mixed problem for general second order fully nonlinear hyperbolic operators with dissipation of the form: $\sum_{j=0}^n a_j(t, x)\partial_j\partial_t u-\sum_{i,j=1}^n a_{ij}(t, x)\partial_i\partial_j u+\sum_{j=0}^n b_j(t, x)\partial_j u+c(t, x)u+F(t, x, \Lambda u)$, when Ω is bounded and data are sufficiently small and smooth, by also the Nash-Moser technique. And also, applying the Nash-Moser technique to bifurcation theorem, Craig [1] showed the existence of non-trivial branches of periodic solutions for fully nonlinear dissipative wave equation of the form: $\partial_t^2 u+\alpha\partial_t u-\Delta u-mu+F(t, x, \Lambda u)=0$ $(\alpha>0, m\in \boldsymbol{R}^1)$, when Ω is bounded, which is an extension of Rabinowitz's result [15].

The purpose of this paper is to show global existence and uniqueness theorems and some decay theorem of solutions of mixed problem (P), in the case that data are sufficiently small and smooth and that Ω is unbounded with compact and $\boldsymbol{C}^\infty$ boundary. We introduce the following assumptions, which will be assumed throughout this paper.

ASSUMPTIONS. 1° The space dimension $n\geqq 3$.

2° The nonlinear function $F(t, x, \lambda)$, $\lambda=(\mu, \tilde{\Lambda})$, is real valued and belongs to the space $\mathscr{B}^\infty([0, \infty)\times\bar{\Omega}\times\{\lambda\in\boldsymbol{R}^{n+2+(n+1)^2};\ |\lambda|\leqq 1\})$ (1).

3° In the case: $n\geqq 5$, F satisfies the following conditions: $F(t, x, 0)=0$, $(d_\lambda F)(t, x, 0)=0$ (2).

4° In the case: $3\leqq n\leqq 4$, F is of the form: $F(t, x, \lambda)=F_1(t, x, \tilde{\lambda})+F_2(t, x, \lambda)$, $\lambda=(\mu, \tilde{\lambda})$, where F_1 and F_2 satisfy the conditions: $F_1(t, x, 0)=0$, $(d_{\tilde{\lambda}}F_1)(t, x, 0)=0$, $F_2(t, x, 0)=0$, $(d_\lambda F_2)(t, x, 0)=0$, $(d_\lambda^2 F_2)(t, x, 0)=0$.

In this paper, we shall show the following

MAIN THEOREM (3). *Let Assumptions hold.* (I) *(Existence). Let m be an integer $\geqq 2$. Then there exist some positive constant c and a sufficiently small positive constant δ_0 depending essentially only on n, m and F having the following properties: if data ϕ_0, ϕ_1 and f for problem* (P) *satisfy the $\tilde{m}$-th order compati-*

(1) $\mathscr{B}^k(\bar{\omega})$, ω being an open set, is the set of all functions defined in some open set $\tilde{\omega}\supsetneqq\omega$ such that their partial derivatives of order $\leqq k$ all exists and are continuous and bounded. We denote the norm of $\mathscr{B}^k$ by $\|\cdot\|_{\mathscr{B}^k}$.

(2) Here, we have written $d_\nu G=(\partial G/\partial\nu_1, \cdots, \partial G/\partial\nu_k)$ and $d_\nu^2 G=(\partial^2 G/\partial\nu_i\partial\nu_j;\ 1\leqq i, j\leqq k)$, when $\nu=(\nu_1, \cdots, \nu_k)$.

(3) Notations used in MAIN THEOREM will be defined in the part of Notations below.

bility condition defined in § 10 *below and an inequality*:

$$\|\phi_0\|_{2,2\tilde{m}+3+[n/2]}+\|\phi_1\|_{2,2\tilde{m}+2+[n/2]}+|f|_{2,q(n),2\tilde{m}+1+[n/2]}$$
$$+\sigma(n)(\|\phi_0\|_{1,\tilde{m}+2}+\|\phi_1\|_{1,\tilde{m}+1}+|f|_{1,1,\tilde{m}})\leqq\delta\,,$$

then there exists a solution $u\in C^m([0,\infty)\times\bar{\Omega})$ *of problem* (P) *with*

$$|u|_{\infty,p(n),m}+|\Delta u|_{2,0,m-2}+|\tilde{\Lambda}u|_{2,1/2,m-2}\leqq c\delta\,,$$

for any δ *with* $0<\delta\leqq\delta_0$. *Here,* $\tilde{m}=2\max(m-1,\, 2[n/2]+7)+9+2[n/2]$. (II) (*Uniqueness*). *There exists a small number* δ_1 *with* $0<\delta_1\leqq 1$ *having the following properties*: *if* u *and* v *are* $C^3([0,\infty)\times\Omega)$ *solutions of problem* (P) *for the same data and satisfy the conditions*: $|u|_{\infty,0,0}\leqq\delta_1$, $|v|_{\infty,0,0}\leqq 1$ *then* $u\equiv v$.

Now, we give some examples of $\Phi(u)$.

EXAMPLE. (i) $\Phi(u)=\partial_t^2u+\partial_t u-\dfrac{\Delta u}{\sqrt{1+\sum_{j=1}^n(\partial_j u)^2}}$.

(ii) $\Phi(u)=\partial_t^2u+\partial_t u-\Delta u+g(u)$, where $g(u)\in C^\infty(\boldsymbol{R}^1)$ is real-valued and satisfies the condition: $(d/du)^j g(0)=0$, $j=0, 1, 2$ (for example, $g(u)=u^3$).

(iii) $\Phi(u)=\partial_t^2u+\partial_t u-\Delta u+\sum_{j=0}^n(\partial_j u)^2+\sum_{i,j=0}^n(\partial_i\partial_j u)^2$.

Our proof of (I) in MAIN THEOREM is a straight forward adaptation of a quadratic iteration scheme with a process of "*smoothing*" because of "*loss of derivatives*" at each step in the iteration. This technique is well-known as the Nash-Moser technique ([9], [13]). In order to show the convergence of our iteration scheme, it is important to show L^2 and uniform decay estimate for linearized problem, which will be shown in Part 2. In particular, the results of decay estimates are new and will be shown mainly by means of technique known in the field of perturbation theory in Part 1 and § 8 and § 9 in Part 2.

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Notations

For any multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$ where α_j and β_j are non-negative integers, we put

$$\partial_x^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n},\ |\alpha|=\alpha_1+\cdots+\alpha_n,\ \alpha!=\alpha_1!\cdots\alpha_n!,$$

$$\alpha-\beta=(\alpha_1-\beta_1,\cdots,\alpha_n-\beta_n),\ \binom{\alpha}{\beta}=\alpha!/(\alpha-\beta)!\,\alpha!,$$

and $\alpha\geqq\beta$ means $\alpha_j\geqq\beta_j$ for any j. For a non-negative integer N, we put

$$\bar{D}_x^N u=(\partial_x^\alpha u\,;\,|\alpha|\leqq N),\quad \bar{D}^N u=(\partial_t^j\partial_x^\alpha u\,;\,j+|\alpha|\leqq N),$$

$$D_x^N u=(\partial_x^\alpha u\,;\,|\alpha|=N),\quad D^N u=(\partial_t^j\partial_x^\alpha u\,;\,j+|\alpha|=N).$$

For any open set $\mathcal{O}$ in $\boldsymbol{R}^n$ and p with $1\leqq p\leqq\infty$, we denote the usual L^p and locally L^p spaces defined on $\mathcal{O}$ by $L^p(\mathcal{O})$ and $L^p_{loc}(\mathcal{O})$, respectively. For functions f and g, we put

$$(f,g)_{\mathcal{O}}=\int_{\mathcal{O}}f\cdot\bar{g}\,dx,\ \|f\|_{\mathcal{O},p}=\left(\int_{\mathcal{O}}|f|^p dx\right)^{1/p},\quad 1\leqq p<\infty,\quad \|f\|_{\mathcal{O},\infty}=\operatorname*{ess\,sup}_{\mathcal{O}}|f|.$$

For a vector valued function $h=(h_1,\cdots,h_s)$, we put

$$h^\alpha=h_1^{\alpha_1}\cdots h_n^{\alpha_n},\quad |h|^2=|h_1|^2+\cdots+|h_n|^2,\quad \|h\|_{\mathcal{O},p}=\textstyle\sum_{j=1}^s\|h_j\|_{\mathcal{O},p}.$$

Further, we put

$$\|f\|_{\mathcal{O},p,N}\equiv\|\bar{D}_x^N f\|_{\mathcal{O},p},\quad \|h\|_{\mathcal{O},p,N}\equiv\textstyle\sum_{j=1}^s\|h_j\|_{\mathcal{O},p,N},$$

$$H_p^N(\mathcal{O})\equiv\{f\in L^p(\mathcal{O})\,;\,\|f\|_{\mathcal{O},p,N}<\infty\}.$$

By $H_D(\mathcal{O})$ we denote the closure in Dirichlet norm of smooth scalar valued functions with compact support in $\mathcal{O}$, where Dirichlet norm is defined by

$$|||f|||_{\mathcal{O}}^2\equiv\int_{\mathcal{O}}|D_x^1 f|^2 dx\,.$$

When $\mathcal{O}=\Omega$ or $=R^n$, for simplicity we use the following abbreviations:

$$(f,g)=(f,g)_\Omega,\quad \|f\|_p=\|f\|_{\Omega,p},\quad \|f\|_{p,N}=\|f\|_{\Omega,p,N},\quad |||f|||=|||f|||_\Omega,$$

$$(f,g)'=(f,g)_{R^n},\quad \|f\|'_p=\|f\|_{R^n,p},\quad \|f\|'_{p,N}=\|f\|_{R^n,p,N},\quad |||f|||'=|||f|||_{R^n}.$$

For Banach spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\boldsymbol{B}(\mathcal{H}_1,\mathcal{H}_2)$ denotes the Banach space of all bounded linear operators on $\mathcal{H}_1$ to $\mathcal{H}_2$ and we denote its operator norm by $\|\cdot\|_{B(\mathcal{H}_1,\mathcal{H}_2)}$. In particular, when $\mathcal{H}=\mathcal{H}_1=\mathcal{H}_2$, for simplicity we write $\boldsymbol{B}(\mathcal{H})=\boldsymbol{B}(\mathcal{H},\mathcal{H})$ and $\|\cdot\|_{B(\mathcal{H})}=\|\cdot\|_{B(\mathcal{H},\mathcal{H})}$. For an open set ω in one dimensional complex plane $\boldsymbol{C}$, a Banach space $\mathcal{H}$ and an open set $\mathcal{O}$ in $\boldsymbol{R}^n$, by $\mathrm{Anal}(\omega;\mathcal{H})$ (resp. $\boldsymbol{C}^N(\mathcal{O};\mathcal{H})$) we denote the space of all $\mathcal{H}$-valued functions which are holomorphic in ω (resp. N-times continuously differentiable in $\mathcal{O}$). We put $C_0^N(\mathcal{O};\mathcal{H})=\{u\in\boldsymbol{C}^N(\mathcal{O};\mathcal{H})\,;$ supp u is compact and contained in $\mathcal{O}\}$. In particular, we put $\boldsymbol{C}^N(\mathcal{O})=\boldsymbol{C}^N(\mathcal{O};\boldsymbol{C})$ and $\boldsymbol{C}_0^N(\mathcal{O})=\boldsymbol{C}_0^N(\mathcal{O};\boldsymbol{C})$. For p with $1\leqq p\leqq\infty$ and a non negative integer N, we put

$$\mathcal{E}^{p,N}(\mathcal{O})=\textstyle\bigcap_{j=0}^N C^j([0,\infty)\,;\,H_p^{N-j}(\mathcal{O}))\,,$$

where $C^j([0,\infty); H_p^{N-j}(\mathcal{O}))=\{u\,;\ u=v \text{ in } (0,\infty) \text{ for some } v\in C^j((-\infty,\infty); H_p^{N-j}(\mathcal{O}))\}$. For $u\in\mathcal{E}^{p,N}(\mathcal{O})$ and a non negative real number k, we put

$$|u|_{\mathcal{O},p,k,N}=\sup_{t>0}(1+t)^k\|\bar{D}^N u(t,\ \cdot)\|_{\mathcal{O},p}\,.$$

For any closed interval $[a,b]$ $(-\infty\leqq a<b\leqq\infty)$ in R^1 and $u\in\bigcap_{j=0}^N C^j([a,b]; H_p^{N-j}(\mathcal{O}))$, where $C^j([a,b]; H_p^i(\mathcal{O}))=\{u\,;\ u=v \text{ in } (a,b) \text{ for some } v\in C^j((-\infty,\infty); H_p^i(\mathcal{O}))\}$ (i is a non negative integer), we put

$$|u|_{\mathcal{O},p,[a,b],N}=\sup_{a\leqq t\leqq b}\|\bar{D}^N u(t,\ \cdot)\|_{\mathcal{O},p}\,.$$

For simplicity, when $\mathcal{O}=\Omega$ or R^n, we use the following abbreviation:

$$|u|_{p,k,N}\equiv|u|_{\Omega,p,k,N}\,,\quad |u|_{p,[a,b],N}\equiv|u|_{\Omega,p,[a,b],N}\,,$$

$$|u|'_{p,k,N}\equiv|u|_{R^n,p,k,N}\,,\quad |u|'_{p,[a,b],N}\equiv|u|_{R^n,p,[a,b],N}\,.$$

For any positive integer L, we put

$$E^L\equiv\{u\in\mathcal{E}^{2,L}(\mathcal{O})\,;\ \partial_t^j u(0,\ x)=0,\ j=0,\ 1,\ \cdots,\ L-1\},$$

$$\tilde{E}^L\equiv\{u\in E^L\cap C^{L-1}([0,\ \infty); H_D(\mathcal{O})),\ \partial_t^L u(0,\ x)=0\}.$$

For any $r>0$ and open set $\mathcal{O}$ in R^n, we put

$$\mathcal{O}_r\equiv\{x\in\mathcal{O};\ |x|<r\}\ \ (\mathcal{O}\neq R^n),\ B_r\equiv\{x\in R^n\,;\ |x|<r\},$$

$$L_r^2(\mathcal{O})\equiv\{f\in L^2(R^n)\,;\ \operatorname{supp} f\subset\mathcal{O}_r\}.$$

Throughout this paper, r_0 denotes a fixed positive real number with $B_{r_0}\supset R^n-\Omega$. Choose a $C^\infty(R^n)$-function $\pi(x)$ so that $\pi(x)=1$ in B_{r_0} and $\pi(x)=\exp(-|x|^2)$ in $R^n-B_{r_0+1}$. Using $\pi(x)$, we define weighted L^2 norms $\langle\cdot\rangle$ and $\langle\cdot\rangle'$ as follows:

$$\langle f\rangle\equiv\int_\Omega\pi(x)|f|^2dx\,,\quad \langle f\rangle'\equiv\int_{R^n}\pi(x)|f|^2dx\,.$$

We define Banach spaces $\mathcal{J}$ and $\mathcal{J}'$ as follows:

$$\mathcal{J}\equiv\{f\in L^2_{\text{loc}}(\Omega)\,;\ \langle\bar{D}_x^2 f\rangle<\infty,\ f=0 \text{ on } \partial\Omega\},\quad \mathcal{J}'\equiv\{f\in L^2_{\text{loc}}(R^n)\,;\ \langle\bar{D}_x^2 f\rangle'<\infty\}.$$

We denote the Fourier transform of $f(x)$ by $\hat{f}(\xi)=(\mathcal{F}f)(\xi)\equiv\int\exp(-\sqrt{-1}x\xi)f(x)dx$, and also the inverse Fourier transform of $g(\xi)$ by $\tilde{g}(x)=(\mathcal{F}^{-1}g)(x)=(2\pi)^{-n}\times\int\exp(\sqrt{-1}x\xi)g(\xi)d\xi$, where $x\xi\equiv x_1\xi_1+\cdots+x_n\xi_n$, $x=(x_1,\cdots,x_n)$, $\xi=(\xi_1,\cdots,\xi_n)$. For positive integers s, i, a function $H=H(t,\ x,\ \nu)$, $\nu=(\nu_1,\ \cdots,\ \nu_s)$, defined on $[0,\ \infty)\times\bar{\Omega}\times R^s$, vectors $u=(u_1,\ \cdots,\ u_s)$, $v_j=(v_1^j,\ \cdots,\ v_s^j)\in R^s$, we put

$$(d_\nu^i H)(t,\ x,\ u)(v_1,\ \cdots,\ v_i)\equiv(\partial^i H/\partial\eta_1\cdots\partial\eta_i)(t,\ x,\ u+\textstyle\sum_{j=1}^i\eta_j v_j)|_{\eta_1=\cdots=\eta_i=0}\,.$$

For the space dimension n, we define functions $p(n)$, $q(n)$, $\sigma(n)$ by

$$\sigma(n)=\begin{cases}1 & \text{if } 3\leqq n\leqq 4\\ 0 & \text{if } n\geqq 5\end{cases},\quad p(n)=\begin{cases}5/4 & \text{if } n=3\\ 3/2 & \text{if } n=4,\\ n/4 & \text{if } n\geqq 5\end{cases}\quad q(n)=\begin{cases}3/2 & \text{if } n=3\\ 7/4 & \text{if } n=4.\\ n/4 & \text{if } n\geqq 5\end{cases}$$

Finally, throughout the paper, the letter C without subscripts will be used to denote various constants. Further, if a constant depends on A, B, $\cdots$ and we need to emphasize this fact, we shall write $C(A, B, \cdots)$.

PART 1

Local energy decay of solutions of the mixed problem for the operator $\partial_t^2+\partial_t-\Delta$

§1. Statement of main results of part 1.

In this part, under the assumption: $n\geqq 3$, we shall determine the rate of local energy decay of solutions of the following mixed problem:

$$\text{(1.1)}\qquad \begin{cases}(\partial_t^2+\partial_t-\Delta)u=0 & \text{in } \mathscr{D},\\ u=0 & \text{on } \mathscr{D}',\\ u(0, x)=\phi_0(x),\ (\partial_t u)(0, x)=\phi_1(x) & \text{in } \Omega.\end{cases}$$

We shall show the following in this part.

THEOREM 1.1. *Assume that* $n\geqq 3$. *Let* r_0 *be a fixed large number with* $B_{r_0}\supset \boldsymbol{R}^n-\Omega$, M *and* N *any non negative integers, and* r *and* r' *any real numbers with* $r'\geqq r\geqq r_0+3$. *Then, there exists a positive constant* $C=C(r, r', M, N)$ *such that if* ϕ_1, $\phi_1\in \boldsymbol{C}^\infty(\Omega)$ *have* $\operatorname{supp}\phi_0$, $\operatorname{supp}\phi_1\subset\Omega_r$ *and satisfy the compatibility condition defined in* (6.3) *in* § 6 *below and* $u\in \boldsymbol{C}^\infty(\Omega)$ *is the unique solution of* (1.1), *then*

$$\int_{\Omega_{r'}}|D_x^M\partial_t^N u(t, x)|^2dx\leqq C(1+t)^{-(n+2N)}[\|\phi_0\|_{2, M+N+3}+\|\phi_1\|_{2, M+N+2}].$$

Under the assumption that the domain Ω is non trapping, it is well-known that the rate of local energy decay of solutions of the mixed problem of the wave operator is n ([6], [22], [23], [24]). But, in our case, we don't need the assumption that Ω is non trapping by virtue of the dissipative term: ∂_t.

Since the proof of Theorem 1.1 is somewhat long, we here give a sketch of the proof. The strategy follows Murata [11]. For any $f\in L_r^2(\Omega)$, by $R(\tau)f$ we denote the solution of Dirichlet problem: $(\Delta+\tau^2-\sqrt{-1}\tau)R(\tau)f=f$ in Ω, $R(\tau)f=0$

on $\partial\Omega$. In § 2, we show that $R(\tau)f\in \text{Anal}(\{\tau\in \boldsymbol{C};\ \text{Im}\,\tau<0\};\ H_2^2(\Omega))\cap C^\infty(\boldsymbol{R}^1-\{0\};\ H_2^2(\Omega))$. Further, by virtue of dissipation we can show that $\|R(\tau)f\|_2\leqq C|\tau|^{-1}$ as $|\tau|\to\infty$, $\tau\in \boldsymbol{R}^1$. In § 3, we define the fractional power of derivatives by means of the terminology of Besov spaces. In § 4, for $f\in L_r^2(\boldsymbol{R}^n)$, by $\boldsymbol{R}_0(\tau)f$ denoting the solution of the equation: $(\Delta+\tau^2-\sqrt{-1}\tau)u=f$ in $\boldsymbol{R}^n$, we show that $\tau^N R_0(\tau)f$ is $\left(\frac{n}{2}+N\right)$-times differentiable near $\tau=0$ in the sense defined in § 3. In § 5, we show that $R(\tau)$ is a small perturbation to $R_0(\tau)$ near $\tau=0$. It follows from this result that $\tau^N R(\tau)$ is also $\left(\frac{n}{2}+N\right)$-times differentiable near $\tau=0$. In § 6, roughly speaking, we put $u(t,x)=(1/2\pi)\int R(\tau)fe^{\sqrt{-1}\tau t}d\tau$. Then, by using a theorem that the decay order with respect to t of the Fourier transformations of $\left(\frac{n}{2}+N\right)$-times differentiable functions is $\frac{n}{2}+N$, we can show that the assertion of Theorem 1.1 holds for this function $u(t,x)$.

§ 2. Definition and some properties of an inverse operator $R'(\tau)$ in κ.

Throughout part 1, we write

$$\|\cdot\|=\|\cdot\|_2,\quad \|\cdot\|'=\|\cdot\|_2',\quad \kappa_1=\{\tau\in \boldsymbol{C};\ \text{Re}\,\tau\neq 0,\ \text{Im}\,\tau<1/2\},\tag{2.1}$$
$$\kappa_2=\{\tau\in \boldsymbol{C};\ \text{Re}\,\tau=0,\ \text{Im}\,\tau<0\},\quad \kappa=\kappa_1\cup\kappa_2\,.$$

By integration by parts, we have

$$((\Delta+\tau^2-\sqrt{-1}\tau)u,\ u)=-|||u|||^2+(\tau^2-\sqrt{-1}\tau)\|u\|^2,\tag{2.2}$$
$$\text{for any } u\in H_D(\Omega)\cap H_2^2(\Omega)\,.$$

The following two lemmas are well-known (see Lax-Phillips [6, Lemma 1.1 of § 1 in Chapter IV and (1.11) of § 1 in Chapter V]).

LEMMA 2.1. *Let $u\in H_D(\Omega)$ and $v\in H_D(\boldsymbol{R}^n)$. Then there exists a positive constant $C=C(n)$ such that*

$$\int_{\Omega_r}|u|^2dx\leqq Cr^2|||u|||^2 \text{ for any } r\geqq r_0,\quad \int_{B_r}|v|^2dx\leqq Cr^2(|||v|||')^2 \text{ for any } r>0\,.$$

LEMMA 2.2. *If $u\in H_D(\Omega)$ and $\Delta u\in L^2(\Omega)$, then all second derivatives of u are square integrable and*

$$\|D_x^2u\|\leqq C\{\|\Delta u\|+|||u|||\}\,.$$

By using (2.2), the Cauchy-Schwartz inequality, Lemmas 2.1 and 2.2 and well-known Riesz's theorem, we have

THEOREM 2.3. *For each $\tau\in\kappa$, there exists a bounded linear operator $R'(\tau)$ on $L^2(\Omega)$ to $H_2^2(\Omega)\cap H_D(\Omega)$ such that for any $f\in L^2(\Omega)$, $R'(\tau)f$ satisfies the equation: $(\Delta+\tau^2-\sqrt{-1}\tau)R'(\tau)f=f$ in Ω, and the estimates:*

$$\|R'(\tau)f\|\leqq C\zeta_0(\tau)\|f\|,\quad |||R'(\tau)f|||\leqq C\zeta_1(\tau)\|f\|,\quad \|D_x^2R'(\tau)f\|\leqq C\zeta_2(\tau)\|f\|, \tag{2.3}$$

where

$$\zeta_0(\tau)=(|\mathrm{Re}\,\tau|\,|2\,\mathrm{Im}\,\tau-1|)^{-1} \text{ if } \tau\in\kappa_1 \text{ and } =(|\tau|+|\tau|^2)^{-1} \text{ if } \tau\in\kappa_2,$$

$$\zeta_1(\tau)=\begin{cases}(\zeta_0(\tau)[1+|(\mathrm{Re}\,\tau)^2-(\mathrm{Im}\,\tau)^2+\mathrm{Im}\,\tau|])^{1/2} & \text{if } \tau\in\kappa_1,\\ (|\tau|+|\tau|^2)^{-1} & \text{if } \tau\in\kappa_2,\end{cases} \tag{2.4}$$

$$\zeta_2(\tau)=1+|\tau^2-\sqrt{-1}\tau|\zeta_0(\tau)+\zeta_1(\tau) \quad \text{if } \tau\in\kappa.$$

Let τ and τ_1 be any points in κ. Formally we put

$$C(\tau)=\textstyle\sum_{j=0}^{\infty}\{[(\tau_1^2-\sqrt{-1}\tau_1)-(\tau^2-\sqrt{-1}\tau)]R'(\tau_1)\}^j. \tag{2.5}$$

In view of Theorem 2.3, if τ is close to τ_1, the series on the right of (2.5) converges in $\boldsymbol{B}(L^2(\Omega))$ norm and it is easily checked that for such τ, its limit is indeed the inverse of the operator: $1+[(\tau^2-\sqrt{-1}\tau)-(\tau_1^2-\sqrt{-1}\tau_1)]R'(\tau_1)$. It follows from this that for such τ, the equation: $(\Delta+\tau^2-\sqrt{-1}\tau)R'(\tau_1)C(\tau)f=f$ in Ω, holds. So, we see easily from Theorem 2.3 that $R'(\tau)=R'(\tau_1)C(\tau)$ for such τ. In particular, we have that $R'(\tau)\in\mathrm{Anal}(\kappa;\boldsymbol{B}(L^2(\Omega);H_2^2(\Omega)\cap H_D(\Omega)))$. So, differentiating the both side of the equation: $(\Delta+\tau^2-\sqrt{-1}\tau)R'(\tau)f=f$ in Ω with respect to τ, we have by Theorem 2.3

THEOREM 2.4. *Let $R'(\tau)$ be the same as in Theorem 2.3. Then $R'(\tau)$ belongs to* $\mathrm{Anal}(\kappa;\boldsymbol{B}(L^2(\Omega);H_2^2(\Omega)\cap H_D(\Omega)))$.

Furthemore, the following estimates hold:

$$\|(\partial/\partial\tau)^N R'(\tau)f\|\leqq C(N,\nu)|\tau|^{-1}\|f\|,$$

$$|||(\partial/\partial\tau)^N R'(\tau)f|||\leqq C(N,\nu)\|f\|,$$

$$\|D_x^2(\partial/\partial\tau)^N R'(\tau)f\|\leqq C(N,\nu)|\tau|\cdot\|f\|$$

for any $\nu>0$, integer $N\geqq 0$ and $\tau\in\boldsymbol{R}^1$ with $|\tau|>\nu$.

§ 3. Definition and some properties of the space C^k.

In order to investigate the regularity of $R'(\tau)$ near $\tau=0$, we shall define the fractional power derivatives by means of the terminology of Besov spaces and show some their properties in this section. Throughout § 3, $\mathscr{H}$ is a Banach space with norm $|\cdot|_{\mathscr{H}}$. First, we introduce the following space.

DEFINITION 3.1. Let N be a positive integer and $k=N+\sigma$ with $0<\sigma\leqq 1$. Put

$$C^k=C^k(\boldsymbol{R}^1;\mathscr{H})=\{u\in \boldsymbol{C}^{N-1}(\boldsymbol{R}^1;\mathscr{H})\cap \boldsymbol{C}^\infty(\boldsymbol{R}^1-\{0\};\mathscr{H});\langle\!\langle u\rangle\!\rangle_{k,\mathscr{H}}<\infty\},$$

where

$$\langle\!\langle u\rangle\!\rangle_{k,\mathscr{H}}=\sum_{j=0}^N\int_{R^1}\left|\left(\frac{d}{d\tau}\right)^j u(\tau)\right|_{\mathscr{H}}d\tau+\sup_{h\neq 0}|h|^{-\sigma}$$

$$\times\int_{R^1}\left|\left(\frac{d}{d\tau}\right)^N u(\tau+h)-\left(\frac{d}{d\tau}\right)^N u(\tau)\right|_{\mathscr{H}}d\tau \quad \text{if } 0<\sigma<1,$$

$$\langle\!\langle u\rangle\!\rangle_{k,\mathscr{H}}=\sum_{j=0}^N\int_{R^1}\left|\left(\frac{d}{d\tau}\right)^j u(\tau)\right|_{\mathscr{H}}d\tau+\sup_{h\neq 0}|h|^{-1}$$

$$\times\int_{R^1}\left|\left(\frac{d}{d\tau}\right)^N u(\tau+2h)-2\left(\frac{d}{d\tau}\right)^N u(\tau+h)+\left(\frac{d}{d\tau}\right)^N u(\tau)\right|_{\mathscr{H}}d\tau.$$

C^k is a subspace of the usual Besov space $B^k_{1,\infty}(\boldsymbol{R}^1;\mathscr{H})$ (see Muramatsu [10]). In the following theorem, we shall give a sufficient condition in order that f belongs to C^k.

TEHOREM 3.2. *Let N be a positive integer and $\mathscr{H}$ be a Banach space with norm $|\cdot|_{\mathscr{H}}$. Assume that $f\in C^\infty(\boldsymbol{R}^1-\{0\};\mathscr{H})\cap C_0^{N-1}(I;\mathscr{H})$ where $I=(-2,2)$.*

(i) *Let $k=N+\sigma$ with $0<\sigma<1$ and f satisfy the following condition* (a).

(a) *For any $\tau\in I-\{0\}$,*

$$\left|\left(\frac{d}{d\tau}\right)^j f(\tau)\right|_{\mathscr{H}}\leqq C(f) \quad \textit{for any integer } j\in[0, N-1],$$

$$\left|\left(\frac{d}{d\tau}\right)^N f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-1},\quad \left|\left(\frac{d}{d\tau}\right)^{N+1} f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-2}.$$

Then, $f\in C^k(\boldsymbol{R}^1;\mathscr{H})$ and f satisfies the following:

$$\langle\!\langle f\rangle\!\rangle_{k,\mathscr{H}}\leqq C(\sigma,N)C(f),\quad |f(\tau)-f(0)|_{\mathscr{H}}\leqq C(\sigma,N)C(f)|\tau|^\sigma.$$

(ii) *Let $k=N+1$ and f satisfies the following conditions* (a) *and* (b).

(a) *There exist $f_0\in\mathscr{H}$ and a $\mathscr{H}$-valued function $f_1(\tau)$ defined on I such that*

$$\left(\frac{d}{d\tau}\right)^N f(\tau)=f_0\log|\tau|+f_1(\tau) \quad \textit{for } \tau\in I-\{0\}.$$

(b) *For any $\tau\in I-\{0\}$,*

$$\left|\left(\frac{d}{d\tau}\right)^j f(\tau)|_{\mathscr{H}}\leqq C(f) \quad \textit{for any integer } j\in[0, N-1],\right.$$

$$|f_0|_{\mathscr{H}}\leqq C(f),\quad |f_1(\tau)|_{\mathscr{H}}\leqq C(f),\quad \left|\left(\frac{d}{d\tau}\right)^{N+1} f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{-1},$$

$$\left|\left(\frac{d}{d\tau}\right)^{N+2} f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{-2}.$$

Then, $f\in C^k(\boldsymbol{R}^1;\mathscr{H})$ *and* f *satisfies the following*:

$$\langle\!\langle f\rangle\!\rangle_{k,\mathscr{H}}\leqq C(N,\sigma)C(f),\quad |f(\tau)-f(0)|_{\mathscr{H}}\leqq C(N,\sigma)C(f)|\tau|^{1/2}.$$

Theorem 3.2 follows immediately from the following four elementary lemmas.

LEMMA 3.3. *Let* $f(\tau)\in C^1(\boldsymbol{R}^1-\{0\};\mathscr{H})$. *If there exists a positive number* $\sigma<1$ *such that* $|f(\tau)|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-1}$, $\left|\frac{d}{d\tau}f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-2}$, *then for any* $h\in\boldsymbol{R}^1-\{0\}$,

$$\int_{-\infty}^{\infty}|f(\tau+h)-f(\tau)|_{\mathscr{H}}d\tau\leqq C(\sigma)C(f)|h|^{\sigma}.$$

LEMMA 3.4. *Let* $f(\tau)\in C^2(\boldsymbol{R}^1-\{0\};\mathscr{H})$. *If there exist* $f_0\in\mathscr{H}$ *and a* $\mathscr{H}$-*valued function* $f_1(\tau)$ *such that* $f(\tau)=f_0\log|\tau|+f_1(\tau)$ *and if for any* $\tau\in\boldsymbol{R}^1-\{0\}$ *the estimates*: $|f_0|_{\mathscr{H}}\leqq C(f)$, $|f_1(\tau)|_{\mathscr{H}}\leqq C(f)$, $\left|\frac{d}{d\tau}f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{-1}$ *and* $\left|\left(\frac{d}{d\tau}\right)^2f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{-2}$ *hold, then for any* $h\in\boldsymbol{R}^1-\{0\}$

$$\int_{-\infty}^{\infty}|f(\tau+2h)-2f(\tau+h)+f(\tau)|_{\mathscr{H}}d\tau\leqq(4+8\log 2)C(f)|h|.$$

LEMMA 3.5. *Let* $f(\tau)\in C^1(\boldsymbol{R}^1-\{0\};\mathscr{H})\cap C^0(\boldsymbol{R}^1;\mathscr{H})$. *If there exists a positive number* $\sigma<1$ *such that for any* $\tau\in\boldsymbol{R}^1-\{0\}$ *the eslimates*: $|f(\tau)|_{\mathscr{H}}\leqq C(f)$ *and* $\left|\frac{d}{d\tau}f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-1}$ *hold, then for any* $h,\tau\in\boldsymbol{R}^1-\{0\}$

$$|f(\tau+h)-f(\tau)|_{\mathscr{H}}\leqq C(\sigma)C(f)|h|^{\sigma}.$$

LEMMA 3.6. *Let* $f(\tau)\in C^2(\boldsymbol{R}^1-\{0\};\mathscr{H})$. *If there exists a positive number* $\sigma<1$ *such that the estimates*: $|f(\tau)|_{\mathscr{H}}\leqq C(f)$, $\left|\frac{d}{d\tau}f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-1}$, $\left|\left(\frac{d}{d\tau}\right)^2f(\tau)\right|_{\mathscr{H}}\leqq C(f)|\tau|^{\sigma-2}$ *hold, then*

$$\int_{-\infty}^{\infty}\left|h^{-1}[f(\tau+h)-f(\tau)]-\frac{d}{d\tau}f(\tau)\right|_{\mathscr{H}}d\tau\leqq C(\sigma)C(f)|h|^{\sigma}.$$

PROOF of Lemmas 3.3-3.6. Lemma 3.5 follows immediately from elementary calculus, so we omit the proof. Lemma 3.6 follows from Lemma 3.3 and Taylor series expansions. Since the proof of Lemma 3.3 is essentially the same as that of Lemma 3.4, we shall prove only Lemma 3.4. First, we shall treat the case that $h>0$. Putting $J(h,\tau)=|f(\tau+2h)-2f(\tau+h)+f(\tau)|_{\mathscr{H}}$, we divide the integral $I(h)=\int_{-\infty}^{\infty}J(h,\tau)d\tau$ into four parts, that is, $I(h)=\sum_{j=1}^4 I_j(h)$ where $I_1(h)=\int_0^{\infty}J(h,\tau)d\tau$, $I_2(h)=\int_{-h}^{0}J(h,\tau)d\tau$, $I_3(h)=\int_{-2h}^{h}J(h,\tau)d\tau$, $I_4(h)=\int_{-\infty}^{-2h}J(h,\tau)d\tau$. First, we estimate $I_1(h)$. By Taylor series expansions we have that for $\tau>0$,

$$J(h, \tau)\leqq\int_0^h\int_0^h\left|\left(\frac{d}{d\tau}\right)^2 f(\tau+\theta+\eta)\right|_{\mathscr{H}} d\theta d\eta\leqq C(f)\int_0^h\int_0^h(\tau+\theta+\eta)^{-2}d\theta d\eta .$$

Thus, by Fubini's theorem we have

$$I_1(h)\leqq C(f)\int_0^h\int_0^h\left[\int_0^\infty(\tau+\theta+\eta)^{-2}d\tau\right]d\theta d\eta=(2\log 2)C(f)h . \tag{3.1}$$

Next, we estimate $I_2(h)$. By the assumption, we have

$$\begin{aligned} I_2(h)&\leqq\int_0^h|f(\tau+h)-f(\tau)|_{\mathscr{H}}d\tau+\int_{-h}^0|f(\tau+h)-f(\tau)|_{\mathscr{H}}d\tau \\ &\leqq 4C(f)h+C(f)\left[\int_0^h|\log|\tau+h|-\log|\tau||d\tau+\int_{-h}^0|\log|\tau+h|-\log|\tau||d\tau\right. \end{aligned} \tag{3.2}$$

By elementary calculus, we have

$$\int_0^h|\log|\tau+h|-\log|\tau||d\tau=\int_0^h[\log(\tau+h)-\log\tau]d\tau=(2\log 2)h , \tag{3.3}$$

$$\begin{aligned} \int_{-h}^0|\log|\tau+h|-\log|\tau||d\tau&=\int_0^{h/2}[\log(h-\tau)-\log\tau]d\tau \\ &\quad+\int_{h/2}^h[\log\tau-\log(h-\tau)]d\tau=(2\log 2)h . \end{aligned} \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we have

$$I_2(h)\leqq(4+4\log 2)C(f)h . \tag{3.5}$$

In the way similar to the proof of (3.5), we have

$$I_3(h)\leqq(4+4\log 2)C(f)h . \tag{3.6}$$

Finally, we estimate $I_4(h)$. By the assumption, Fubini's theorem and Taylor series expansions, we have

$$\begin{aligned} I_4(h)&\leqq C(f)\int_{-\infty}^{-2h}\left[\int_0^h\int_0^h|\tau+\theta+\eta|^{-2}d\theta d\eta\right]d\tau \\ &=C(f)\int_0^h d\theta\int_0^h d\eta\int_{2h}^\infty(\tau-\theta-\eta)^{-2}d\tau=(2\log 2)C(f)h . \end{aligned} \tag{3.7}$$

Combining (3.1), (3.5), (3.6) and (3.7), we have $I(h)\leqq(4+8\log 2)C(f)h$ if $h>0$. Since $f(-\tau)$ satisfies the same assumption as that imposed on $f(\tau)$, we have also $I(h)\leqq(4+8\log 2)C(f)|h|$ if $h<0$, which completes the proof of Lemma 3.4.

Q. E. D.

Finally, we shall give a result which shows the decay order of the Fourier transformation of functions belonging to C^k. When $\mathscr{H}$ is a Hilbert space, such a result is proved, for example, in Murata [11]. He showed it by using the interpolation theory. Here, we give an elementary proof due to Muramatsu.

Although we can show such a theorem for more general Besov space defined on $\boldsymbol{R}^n$ without changing the essential part of the proof given below, for simplicity we shall show only a theorem which we need in this paper.

THEOREM 3.7. *Let $\mathcal{H}$ be a Banach space with norm $|\cdot|_{\mathcal{H}}$. Let N be a positive integer and σ be a positive number $\leqq 1$. Assume that $f\in C^{N+\sigma}(\boldsymbol{R}^1;\mathcal{H})$. Put $g(t)=(1/2\pi)\int_{-\infty}^{\infty}f(\tau)\exp(\sqrt{-1}t\tau)d\tau$. Then,*

$$|g(t)|_{\mathcal{H}}\leqq C(1+|t|)^{-(N+\sigma)}\langle\!\langle f\rangle\!\rangle_{N+\sigma,\mathcal{H}}.$$

PROOF. Since $\left(\frac{d}{d\tau}\right)^j f(\tau)\in L^1(\boldsymbol{R}^1;\mathcal{H})$, $j=0,\cdots,N$, noting the identity: $(\sqrt{-1}t)^{-j}\left(\frac{d}{d\tau}\right)^j\exp(\sqrt{-1}t\tau)=\exp(\sqrt{-1}t\tau)$, we have by integration by parts that

$$g(t)=(\sqrt{-1}t)^{-N}(1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{d}{d\tau}\right)^N f(\tau)\right]\exp(\sqrt{-1}t\tau)d\tau. \tag{3.8}$$

For simplicity, we write $h(\tau)=(1/2\pi)(d/d\tau)^N f(\tau)$. Since $f(\tau)\in L^1(\boldsymbol{R}^1;\mathcal{H})$ and $f(-\tau)$ also belongs to $C^{N+\sigma}$, we may show the theorem in the case that $t\geqq 1$. So, we assume that $t\geqq 1$ below. Choosing $\tilde{\chi}(s)\in C_0^\infty(\boldsymbol{R}^1)$ so that $\tilde{\chi}(s)=0$ if $|s|\leqq 1/4$, $|s|\geqq 2$ and $=1$ if $1/2\leqq|s|\leqq 1$, we put $\chi(s)=\tilde{\chi}(|s|)$. Choose an integer i for t so that $2^{i-1}\leqq t<2^i$. We have

$$\begin{aligned}t^\sigma\left|\int_{\boldsymbol{R}^1}h(\tau)e^{\sqrt{-1}t\tau}d\tau\right|_{\mathcal{H}}&\leqq 2^{i\sigma}\left|\int_{\boldsymbol{R}^1}h(\tau)e^{\sqrt{-1}t\tau}d\tau\right|_{\mathcal{H}}\\&\leqq 2^{i\sigma}\sup_{s>0}\left|\chi(2^{-i}s)\int_{\boldsymbol{R}^1}h(\tau)e^{\sqrt{-1}s\tau}d\tau\right|_{\mathcal{H}}.\end{aligned} \tag{3.9}$$

By the Fourier transform, we have

$$\begin{aligned}I(s)&\equiv\chi(2^{-i}s)\int e^{\sqrt{-1}\tau s}h(\tau)d\tau=\mathcal{F}^{-1}[\mathcal{F}(\chi(2^{-i}\cdot)\mathcal{F}^{-1}(h)(\cdot))](s)\\&=(1/2\pi)\int e^{\sqrt{-1}\tau s}\left[\int\mathcal{F}(\chi(2^{-i}\cdot))(\tau')h(\tau-\tau')d\tau'\right]d\tau.\end{aligned} \tag{3.10}$$

On the other hand, we have

$$\mathcal{F}(\chi(2^{-i}\cdot))(\tau')=2^i\hat{\chi}(2^i\tau'),\quad 2^i\int\hat{\chi}(2^i\tau')d\tau'=2\pi\chi(0)=0. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$I(s)=2^i\int e^{\sqrt{-1}s\tau}\left[\int\hat{\chi}(2^i\tau')[h(\tau-\tau')-h(\tau)]d\tau'\right]d\tau. \tag{3.12}$$

First, we shall consider the case that $0<\sigma<1$. By Fubini's theorem, the definition of $C^{N+\sigma}$ and (3.12) we have

$$(3.13)\qquad |I(s)|_{\mathscr{K}}\leqq 2^{i}\int|\hat{\chi}(2^{i}\tau')||\tau'|^{\sigma}d\tau'\left[\int|\tau'|^{-\sigma}|h(\tau-\tau')-h(\tau)|_{\mathscr{K}}d\tau\right]$$

$$\leqq 2^{i}(2\pi)^{-1}\langle\!\langle f\rangle\!\rangle_{N+\sigma,\mathscr{K}}\int|\hat{\chi}(2^{i}\tau)||\tau|^{\sigma}d\tau$$

$$=2^{-i\sigma}((2\pi)^{-1}\int|\hat{\chi}(\tau)||\tau|^{\sigma}d\tau)\langle\!\langle f\rangle\!\rangle_{N+\sigma,\mathscr{K}}\,.$$

Combining (3.9), (3.10) and (3.13), we have the theorem in the case that $0<\sigma<1$.

Next, we shall consider the case that $\sigma=1$. Since $\chi(s)=\chi(-s)$,

$$(3.14)\qquad \hat{\chi}(2^{i}\tau')=\hat{\chi}(-2^{i}\tau')\,.$$

From (3.12) and (3.14) it follows that

$$(3.15)\qquad I(s)=2^{i-1}\int e^{\sqrt{-1}s\tau}\left[\int\hat{\chi}(2^{i}\tau')[h(\tau+\tau')-2h(\tau)+h(\tau-\tau')]d\tau'\right]d\tau\,.$$

By Fubini's theorem, the definition of C^{N+1} and (3.15) we have

$$(3.16)\qquad |I(s)|_{\mathscr{K}}\leqq 2^{i-1}\int|\hat{\chi}(2^{i}\tau')||\tau'|d\tau'\int|h(\tau+\tau')-2h(\tau)+h(\tau-\tau')|_{\mathscr{K}}|\tau'|^{-1}d\tau$$

$$\leqq(4\pi)^{-1}2^{i}\langle\!\langle f\rangle\!\rangle_{N+1,\mathscr{K}}\int|\tau'||\hat{\chi}(2^{i}\tau')|d\tau'$$

$$=2^{-i}((4\pi)^{-1}\int|\tau'||\hat{\chi}(\tau')|d\tau')\langle\!\langle f\rangle\!\rangle_{N+1,\mathscr{K}}\,.$$

Combining (3.9), (3.10) and (3.16), we have the theorem in the case that $\sigma=1$, which completes the proof of the theorem.

§ 4. Behavior of $R_0(\tau)$ near $\tau=0$.

For $f\in L^2_r(\boldsymbol{R}^n)$, $r>0$, we can put

$$(4.1)\qquad R_0(\tau)f=(2\pi)^{-n}\int e^{\sqrt{-1}x\xi}\hat{f}(\xi)(\tau^2-\sqrt{-1}\tau-|\xi|^2)^{-1}d\tau\,,\quad \tau\in\boldsymbol{R}^1.$$

In fact, since $|\hat{f}(\xi)|\leqq C\|f\|'_1\leqq C(r)\|f\|'$ because of the fact that $f\in L^2_r(\boldsymbol{R}^n)$ and since $n\geqq 3$, $R_0(\tau)f$ is well-defined for any $\tau\in\boldsymbol{R}^1$. It follows immediately from the definition of $R_0(\tau)f$ that

$$(4.2)\qquad (\Delta+\tau^2-\sqrt{-1}\tau)R_0(\tau)f=f\qquad\text{in } \boldsymbol{R}^n.$$

The following theorem is the main result of this section.

THEOREM 4.1. *Assume that* $n\geqq 3$. *Let* $f\in L^2_r(\boldsymbol{R}^n)$, α *be a multi-index with* $0\leqq|\alpha|\leqq 2$ *and* $R_0(\tau)$ *be the same as in* (4.1). *Put* $I=\left[-\frac{1}{2},\frac{1}{2}\right]-\{0\}$. *Then, the following four assertions hold.*

1° *$R_0(\tau)f \in C^\infty(I\,;\mathcal{S}')$, where $\mathcal{S}'$ is the same as in Notations.*

2° *For any integer $j \in \left[0, \frac{n+|\alpha|-3}{2}\right]$,*

$$\lim_{\tau\to 0+}(\partial/\partial\tau)^j(\partial/\partial x)^\alpha R_0(\tau)f = \lim_{\tau\to 0-}(\partial/\partial\tau)^j(\partial/\partial x)^\alpha R_0(\tau)f\,.$$

3° *If $n+|\alpha|$ is odd, for any $\tau \in I$*

$$\langle(\partial/\partial\tau)^j(\partial/\partial x)^\alpha R_0(\tau)f\rangle' \leqq C(j,\alpha,r)\|f\|' \quad \text{for any integer } j\in[0,(n+|\alpha|-3)/2],$$

$$\langle(\partial/\partial\tau)^j(\partial/\partial x)^\alpha R_0(\tau)f\rangle' \leqq C(j,\alpha,r)\|f\|'|\tau|^{-(j+1)+\frac{n+|\alpha|}{2}} \quad \text{for any integer } j\geqq\frac{n+|\alpha|-1}{2}.$$

4° *If $n+|\alpha|$ is even, there exist $f_0 \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ and an $L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$-valued function $f_1(\tau)$ such that*

$$(\partial/\partial\tau)^{(n+|\alpha|-2)/2}(\partial/\partial x)^\alpha R_0(\tau)f = f_0\log|\tau| + f_1(\tau)$$

and the following estimates hold for any $\tau\in I$:

$$\langle f_0\rangle' \leqq C\|f\|', \quad \langle f_1(\tau)\rangle' \leqq C\|f\|',$$

$$\langle(\partial/\partial\tau)^j(\partial/\partial x)^\alpha R_0(\tau)f\rangle' \leqq C(r,j,\alpha)\|f\|' \quad \text{for any integer } j\in[0,(n+|\alpha|-4)/2],$$

$$\langle(\partial/\partial\tau)^j(\partial/\partial x)^\alpha R_0(\tau)f\rangle' \leqq C(r,j,\alpha)\|f\|'|\tau|^{-(j+1)+\frac{n-|\alpha|}{2}} \quad \text{for any integer } j\geqq\frac{n+|\alpha|}{2}.$$

In particular, combining Theorems 3.2 and 4.1, we have

COROLLARY 4.2. *Assume that $n\geqq 3$. Let $\rho(\tau)$ be a $C_0^\infty(\boldsymbol{R}^1)$ function such that $\rho(\tau)=1$ if $|\tau|\leqq 1/4$ and $=0$ if $|\tau|\geqq 1/2$. Let $R_0(\tau)$ be the same as in* (4.1). *Then, for any multi-index α with $|\alpha|\leqq 2$, integer $N\geqq 0$ and $f\in L^2_r(\boldsymbol{R}^n)$,*

$$\rho(\tau)\tau^N\partial_x^\alpha R_0(\tau)f \in C^{\frac{n+|\alpha|}{2}+N}(\boldsymbol{R}^1\,;\mathcal{S}')\,.$$

Furthermore, the following estimates hold for any $\tau\in\boldsymbol{R}^1-\{0\}$:

$$\langle\!\langle \rho(\tau)\tau^N\partial_x^\alpha R_0(\tau)f\rangle\!\rangle_{\frac{n+|\alpha|}{2}+N,\mathcal{S}'} \leqq C(n,\alpha,N,\rho,r)\|f\|',$$

$$\langle\partial_x^\alpha[R_0(\tau)f - R_0(0)f]\rangle' \leqq C(\alpha,r)|\tau|^{1/2}\|f\|'.$$

Now, we are going to show Theorem 4.1. Choosing $\chi(\xi)\in C_0^\infty(\boldsymbol{R}^n)$ so that $\chi(\xi)=1$ if $|\xi|\leqq 1$ and $=0$ if $|\xi|\geqq 2$, we put: $R_0(\tau)f = S(\tau)f + T(\tau)f$ where

$$S(\tau)f=(2\pi)^{-n}\int\chi(\xi)e^{\sqrt{-1}x\xi}\hat{f}(\xi)(\tau^2-\sqrt{-1}\tau-|\xi|^2)^{-1}d\xi\,, \tag{4.3}$$

$$T(\tau)f=(2\pi)^{-n}\int[1-\chi(\xi)]e^{\sqrt{-1}x\xi}\hat{f}(\xi)(\tau^2-\sqrt{-1}\tau-|\xi|^2)^{-1}d\xi\,.$$

Since $|(\partial/\partial\tau)^j(\tau^2-\sqrt{-1}\tau-|\xi|^2)|\geqq C(j)(1+|\xi|^2)$ for $\xi\in\mathrm{supp}(1-\chi(\xi))$ and $\tau\in I\cup\{0\}$, it follows from Parseval's equality and differentiation under integral sign that

$$\langle(\partial/\partial\tau)^j(\partial/\partial x)^\alpha T(\tau)f\rangle'\leqq C(j,\alpha)\|f\|'$$

for any multi-index α with $|\alpha|\leqq 2$, integer j and $\tau\in I\cup\{0\}$. This shows that all the assertions in Theorem 4.1 hold for $T(\tau)f$. So, we shall show that all the assertions in Theorem 4.1 also hold for $S(\tau)f$. First, we note the following two facts:

$$|s^N/(\tau^2-\sqrt{-1}\tau-s^2)^M|\leqq 2^M\cdot s^{N-2M} \tag{4.4}$$

for any N and M with $N\geqq 2M\geqq 0$, $\tau\in I$ and $s\geqq 0$,

$$|\partial_\xi^\alpha\hat{f}(\xi)|\leqq C(\alpha)\int(1+|x|)^{|\alpha|}|f(x)|dx\leqq C(\alpha,r)\|f\|' \tag{4.5}$$

for any multi-index α and $f\in L_r^2(R^n)$. Let $\rho_k^j(\tau)$, $k=0, 1, \cdots, j$, be polynomials defined by the following formula

$$(\partial/\partial\tau)^j(\tau^2-\sqrt{-1}\tau-|\xi|^2)^{-1}=\sum_{k=0}^j\rho_k^j(\tau)(\tau^2-\sqrt{-1}\tau-|\xi|^2)^{-k-1}. \tag{4.6}$$

Here, it follows immediately that $\rho_j^j(\tau)=(-1)^j j!\,(2\tau-\sqrt{-1})^j$. Since $|\tau^2-\sqrt{-1}\tau-|\xi|^2|\geqq|\tau|$, it follows from differentiation under integral sign, (4.5) and (4.6) that

$$\begin{aligned}(\partial/\partial\tau)^j(\partial/\partial x)^\alpha S(\tau)f&=\sum_{k=0}^j\rho_k^j(\tau)(2\pi)^{-n}\int\frac{e^{\sqrt{-1}x\xi}\xi^\alpha\chi(\xi)\hat{f}(\xi)}{(\tau^2+\sqrt{-1}\tau-|\xi|^2)^{k+1}}d\xi\\&=\sum_{k=0}^j\rho_k^j(\tau)(2\pi)^{-n}\int_0^2\frac{s^{|\alpha|+n-1}}{(\tau^2-\sqrt{-1}\tau-s^2)^{k+1}}ds\int_{|\omega|=1}e^{(\sqrt{-1}x\omega)s}\omega^\alpha\chi(s\omega)\hat{f}(s\omega)dS_\omega\end{aligned} \tag{4.7}$$

if $\tau\neq 0$. Here, by dS_ω we have denoted the surface element on the unit sphere: $|\omega|=1$. Since $|\tau^2-\sqrt{-1}\tau-s^2|\geqq|\tau|>0$ if $\tau\neq 0$, it follows from (4.7) that the assertion 1° in Theorem 4.1 holds for $S(\tau)f$. If $|\alpha|+n-1\geqq 2(j+1)$, noting (4.4), (4.5) and (4.7), we have by Lebesgue's dominated convergence theorem that

$$\lim_{\tau\to 0+}(\partial/\partial\tau)^j(\partial/\partial x)^\alpha S(\tau)f=\lim_{\tau\to 0-}(\partial/\partial\tau)^j(\partial/\partial x)^\alpha S(\tau)f\,,$$

$$|(\partial/\partial\tau)^j(\partial/\partial x)^\alpha S(\tau)f|\leqq C(r,\alpha,j)\|f\|',\quad \tau\in I.$$

This shows that the assertion 2° and the estimates for any integer $j\in\left[0, \dfrac{n+|\alpha|-3}{2}\right]$ in the assertions 3° and 4° of Theorem 4.1 hold for $S(\tau)f$.

Now, we are going to show that the rest of assertions in Theorem 4.1 hold also for $S(\tau)f$. For this purpose, we need the following lemma. Since we can show it by elementary calculus, we omit its proof.

LEMMA 4.3. *Let M and N be integers with $0\leqq N<2M$. Put*

$$I_{N,M}(\tau)=\int_0^2 s^N/(\tau^2-\sqrt{-1}\tau-s^2)^M ds\,,\quad \tau\in I=\left[-\frac{1}{2},\frac{1}{2}\right]-\{0\}\,.$$

Then, the following facts hold.

1° *If N is odd and $M-\dfrac{N-1}{2}=1$, then there exist complex constants $d_{N,M}$ depending only on N and M and $C^\infty(\{\tau;\ |\tau|\leqq 1/2\})$ functions $\mu_{N,M}(\tau)$ depending also essentially only on N and M such that*

$$I_{N,M}(\tau)=d_{N,M}(\log|\tau|-\sqrt{-1}\tan^{-1}2/\tau(1-\tau^2))+\mu_{N,M}(\tau)\,.$$

2° *If N is odd with $M-\dfrac{N-1}{2}\geqq 2$ or N is even there exist complex constants $d^\pm_{N,M}$ depending only on N and M and $C^\infty(\{\tau\in\boldsymbol{R}^1;\ |\tau|\leqq 1/2\})$ functions $\mu^j_{N,M}(\tau)$, $j=1,2$, depending also essentially only on N and M such that*

$$I_{N,M}(\tau)=\begin{cases} d^+_{N,M}\mu^1_{N,M}(\tau)\tau^{-M+\frac{N+1}{2}}+\mu^2_{N,M}(\tau) & \text{if } \tau>0\,,\\ d^-_{N,M}\mu^1_{N,M}(\tau)|\tau|^{-M+\frac{N+1}{2}}+\mu^2_{N,M}(\tau) & \text{if } \tau<0\,.\end{cases}$$

Now, we return to the proof of Theorem 4.1. We may show that the following two assertions hold for $J(j;\,x,\tau)=\int e^{\sqrt{-1}x\xi}\xi^\alpha\chi(\xi)\hat{f}(\xi)(\tau^2-\sqrt{-1}\tau-|\xi|^2)^{-j-1}d\xi$, in order to show the rest of assertions of Theorem 4.1.

Assertion 1°. Let j be an integer$\geqq\dfrac{n+|\alpha|-1}{2}$. Then,

$$|J(j;\,x,\tau)|\leqq C(r,j,\alpha)\|f\|'|\tau|^{-(j+1)+\frac{n+|\alpha|}{2}}\quad\text{for any } \tau\in I. \tag{4.8}$$

Assertion 2°. If $n+|\alpha|$ is even and $j=\dfrac{n+|\alpha|-2}{2}$, there exist $f_0\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ and $L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$-valued functions $f_1(\tau)$ such that

$$J(j;\,x,\tau)=f_0\log|\tau|+f_1(\tau)\,, \tag{4.9}$$

$$\langle f_0\rangle'\leqq C(r,\alpha,j)\|f\|',\quad \langle f_1(\tau)\rangle'\leqq C(r,\alpha,j)\|f\|'\quad\text{for any } \tau\in I. \tag{4.10}$$

First, we shall show Assertion 1° just mentioned. Using a polar coodinate system, we can write

$$J(j;\,x,\tau)=\int_{|\omega|=1}\omega^\alpha dS_\omega\int_0^2\frac{e^{(\sqrt{-1}x\omega)s}s^{|\alpha|+n-1}\chi(s\omega)\hat{f}(s\omega)}{(\tau^2-\sqrt{-1}\tau-s^2)^{j+1}}ds\,. \tag{4.11}$$

By Taylor series expansion, we have

$$e^{(\sqrt{-1}x\omega)s}\chi(s\omega)\hat{f}(s\omega)=\hat{f}(0)+\sum_{i=1}^{k-1}g_i(x,\ \omega)s^i+\int_0^1 H_k(x,\ \omega,\ \rho,\ s)d\rho s^k \tag{4.12}$$

where

$$g_i(x,\ \omega)=(i\,!)^{-1}(\partial/\partial s)^i e^{(\sqrt{-1}x\omega)s}\chi(s\omega)\hat{f}(s\omega)|_{s=0},\quad i\geqq 1,$$

$$H_i(x,\ \omega,\ \rho,\ s)=(1-\rho)^{i-1}((i-1)\,!)^{-1}(\partial/\partial\sigma)^i e^{(\sqrt{-1}x\omega)\sigma}\chi(\sigma\omega)\hat{f}(\sigma\omega)|_{\sigma=\rho s},\quad i\geqq 1, \tag{4.13}$$

$$k=2j+3-(n+|\alpha|).$$

Using (4.12) and $I_{N,M}(\tau)$ defined in Lemma 4.3, we can rewrite $J(j\,;\,x,\,\tau)$ as follows:

$$\begin{aligned}J(j\,;\,x,\,\tau)=\int_{|\omega|=1}\Big[&\hat{f}(0)I_{|\alpha|+n-1,\,j+1}(\tau)+\sum_{i=1}^{k-1}g_i(x,\ \tau)I_{|\alpha|+n-1+i,\,j+1}(\tau)\\&+\int_0^2\Big[\int_0^1 H_k(x,\ \omega,\ \rho,\ s)d\rho\Big]\frac{s^{k+n+|\alpha|-1}}{(\tau^2-\sqrt{-1}\,\tau-s^2)^{j+1}}ds\Big]\omega^\alpha dS_\omega.\end{aligned} \tag{4.14}$$

Since $f\in L_r^2(R^n)$, we have

$$\begin{aligned}&|\hat{f}(0)|\leqq C(r)\|f\|',\\&|g_i(x,\ \omega)|\leqq C(r,\ i)(1+|x|)^i\|f\|',\\&\int_0^1|H_k(x,\ \omega,\ \rho,\ s)|d\rho\leqq C(k,\ r)(1+|x|)^k\|f\|'.\end{aligned} \tag{4.15}$$

Noting the facts that $j+1-\dfrac{|\alpha|+n-2}{2}=\dfrac{3}{2}+\Big(j-\dfrac{n+|\alpha|-1}{2}\Big)$ and that $j\geqq\dfrac{n+|\alpha|-1}{2}$, and applying Lemma 4.3 to (4.14), we have by (4.4) and (4.15) that Assertion 1° holds.

Finally, we shall show Assertion 2°. Since $n+|\alpha|$ is even and $j=\dfrac{n+|\alpha|-2}{2}$, using (4.11), (4.12), (4.13) with $i=1$ and Lemma 4.3-1°, we have

$$J\Big(\frac{n+|\alpha|-2}{2},\ x,\ \tau\Big)=f_0\log|\tau|+f_1(\tau) \tag{4.16}$$

where

$$f_0=\int\quad\omega^\alpha dS_\omega d_{|\alpha|+n-1,\,(n+|\alpha|)/2}\hat{f}(0),$$

$$\begin{aligned}f_1(\tau)=\Big[\int_{|\omega|=1}&\omega^\alpha dS_\omega\{(d_{|\alpha|+n-1,\,(n+|\alpha|)/2}(-\sqrt{-1}\tan^{-1}2/\tau(1-\tau^2))\\&+\mu_{|\alpha|+n-1,\,(n+|\alpha|)/2}(\tau)\}\Big]\hat{f}(0)\\&+\int_{|\omega|=1}\Big\{\int_0^2\Big[\int_0^1 H_1(x,\ \omega,\ \rho,\ s)d\rho\Big]\frac{s^{n+|\alpha|}ds}{(\tau^2-\sqrt{-1}\,\tau-s^2)^{(n+|\alpha|)/2}}\Big\}\omega^\alpha dS_\omega,\end{aligned}$$

and $d_{i,j}$ and $\mu_{i,j}(\tau)$ are the same as in Lemma 4.3-1°. It follows immediately that

(4.17) $$\langle f_0\rangle' \leqq C(r, \alpha, j)\|f\|', \quad \langle f_1(\tau)\rangle' \leqq C(r, \alpha, j)\|f\|' \quad \text{for } \tau \in I.$$

Combining (4.16) and (4.17), we have that Assertion 2° holds, which completes the proof of Theorem 4.1.

§5. Behavior of $R(\tau)$ near $\tau=0$.

Throughout §§5 and 6, by r let us denote a fixed constant $\geqq r_0+3$ (cf. Notations). By integration by parts, we have that for any $u \in C_0^\infty(\Omega)$

(5.1) $$(\Delta u, u) = -|||u|||^2.$$

Thus, by (5.1), Lemmas 2.1 and 2.2 and well-known Riesz's representation theorem, we have

THEOREM 5.1. *Assume that* $n \geqq 3$. *Then there exists an operator* $R'(0) \in \boldsymbol{B}(L_r^2(\Omega); H_D^2(\Omega))$, *where* $H_D^2(\Omega) \equiv \{v \in H_D(\Omega);\ \|D_x^2 v\| < \infty\}$ (cf. (2.1)), *such that for any* $f \in L_r^2(\Omega)$

$$R'(0)f = f \quad \text{in } \Omega,$$

$$|||R'(0)f||| \leqq C(r)\|f\|, \quad \|D_x^2 R'(0)f\| \leqq C(r)\|f\|.$$

Now, we introduce the following operator.

DEFINITION 5.2. Let $R'(\tau)$, $\tau \in \kappa$ and $R'(0)$ be the same as in Theorems 2.3 and 5.1, respectively. By $R(\tau)$ we denote the operator:

(5.2) $$R(\tau): L_r^2(\Omega) \to \mathcal{J}$$

that is obtained from $R'(\tau)$ by contracting the domain of definition of $R'(\tau)$ according to the formula (5.2) and considering its range in a wider space $\mathcal{J}$ (cf. Notations).

From Theorem 2.4 it follows that $R(\tau) \in \mathrm{Anal}(\kappa; \boldsymbol{B}(L_r^2(\Omega); \mathcal{J}))$. Below, we shall investigate the regularity of $R(\tau)$ near $\tau=0$. Choosing $\phi, \psi \in C^\infty(\boldsymbol{R}^n)$ so that

(5.3) $$\begin{aligned} &\phi = 1 \text{ if } |x| \leqq r-1 \text{ and } = 0 \text{ if } |x| \geqq r, \\ &\psi = 1 \text{ if } |x| \geqq r-2 \text{ and } = 0 \text{ if } |x| \leqq r-3. \end{aligned}$$

Using ϕ and ψ, we introduce the operators P and Q.

DEFINITION 5.3. Let $R_0(\tau)$ be the same operator as in (4.1). For $f \in L_r^2(\Omega)$, we define P and Q by the following formulae:

$$P(\tau)f=(1-\phi)R_0(\tau)(\psi f_0)+\phi R'(0)f\,,$$

$$Q(\tau)f=2\sum_{j=1}^n\partial_j\phi\cdot\partial_j(R_0(\tau)(\psi f_0))+\Delta\phi\cdot R_0(\tau)(\psi f_0)-(\tau^2-\sqrt{-1}\,\tau)\phi\cdot R'(0)f$$
$$-2\sum_{j=1}^n\partial_j\phi\cdot\partial_j(R'(0)f)-\Delta\phi\cdot R'(0)f\,.$$

Here, $f_0=f$ if $x\in\Omega$ and $=0$ if $x\in \boldsymbol{R}^n-\Omega$.

The following lemma for P and Q just defined follows immediately from Definition 5.3, Theorems 2.3 and 5.1, Lemma 2.1, Corollary 4.2 and well-known Rellich's theorem (cf. Mizohata [8, Theorem 3.3]).

LEMMA 5.4. *For* $\tau\in\left[-\frac{1}{4},\frac{1}{4}\right]$, *we have the folowing three assertions*:

(i) $P(\tau)\in \boldsymbol{B}(L^2_r(\Omega)\,;\,\mathcal{J})$,

(ii) $Q(\tau)$ *is a compact operator on* $L^2_r(\Omega)$ *to* $L^2_r(\Omega)$,

(iii) *for any* $f\in L^2_r(\Omega)$ $(\Delta+\tau^2-\sqrt{-1}\,\tau)P(\tau)f=f-Q(\tau)f$ *in* Ω.

We are going to show the following lemma, which is one of the most important lemmas in this part.

LEMMA 5.5. *Assume than* $n\geqq 3$. *There exists a positive constant* c_0 *such that* $\|1-Q(0)\|_{B(L^2_r(\Omega))}\geqq c_0$, *where* $Q(\tau)$ *is the same as in Definition* 5.3.

PROOF. Since $Q(0)$ is a compact operator on $L^2_r(\Omega)$ to $L^2_r(\Omega)$, in view of the well-known Riesz-Schauder theory, we have only to show the following.

(5.4) If $f\in L^2_r(\Omega)$ and $(1-Q(0))f=0$ in Ω, then $f\equiv 0$.

So, we assume that $f\in L^2_r(\Omega)$ and $(1-Q(0))f=0$ in Ω in the course of the proof. By Lemma 5.4-(iii), we have

$$P(0)f=0 \quad \text{in } \Omega. \tag{5.5}$$

In order to show that (5.5) implies that $f\equiv 0$, we need the following two lemmas.

LEMMA 5.6. *Assume that* $n\geqq 3$. *For any* $f\in L^2_r(\Omega)$,

(i) $|||R_0(0)f|||'\leqq C(r)\|f\|'$,

(ii) $\|D^2_x R_0(0)f\|'\leqq C(r)\|f\|'$ (cf. Notations).

LEMMA 5.7. *Assume that* $n\geqq 3$. *Let* $H^2_D(\Omega)$ *be the same as in Theorem* 5.1 *and* $H^2_D(\boldsymbol{R}^n)=\{u\in L^2_{loc}(\boldsymbol{R}^n)\,;\,|||u|||'<\infty,\ \|D^2_x u\|'<\infty\}$. *The following two assertions are valid.*

(i) *If* $u\in H^2_D(\Omega)$ *and* $\Delta u=0$ *in* Ω, *then* $u\equiv 0$.

(ii) *If* $v\in H^2_D(\boldsymbol{R}^n)$ *and* $\Delta v=0$ *in* $\boldsymbol{R}^n$, *then* $v\equiv 0$.

Deferring the proofs of Lemmas 5.6 and 5.7, we continue the proof of Lemma 5.5. Combining Lemma 5.6 and Theorem 5.1, we have that for any $f \in L^2_r(\Omega)$

$$\begin{aligned} &|||P(0)f||| \leqq C(r)\|f\|, \\ &\|D_x^2 P(0)f\| \leqq C(r)\|f\|, \end{aligned} \tag{5.6}$$

from which it follows that $P(0)f \in H^2_D(\Omega)$. Thus, combining (5.5), (5.6) and Lemma 5.7, we have

$$0 = P(0)f = (1-\phi)R_0(0)(\psi f_0) + \phi R'(0)f \quad \text{in } \Omega. \tag{5.7}$$

In particular, combining (5.3), (5.7) and the fact that $\Delta R'(0)f = f$, we have

$$R'(0)f = f = 0 \quad \text{in } \Omega_{r-1}. \tag{5.8}$$

Put

$$u = \begin{cases} R(0)f & \text{if } x \in \Omega, \\ 0 & \text{if } x \in R^n - \Omega. \end{cases}$$

From Theorem 5.1, (5.3) and (5.8), we obtain

$$u \in H^2_D(\boldsymbol{R}^n), \quad \Delta u = f_0 = \psi f_0 \quad \text{in } \boldsymbol{R}^n. \tag{5.9}$$

Thus, since $u - R_0(0)(\psi f_0) \in H^2_D(\boldsymbol{R}^n)$ and $\Delta(u - R_0(0)(\psi f_0)) = 0$ in $\boldsymbol{R}^n$ because of the fact (5.9) and Lemma 5.6, we have by Lemma 5.7-(ii) that

$$u = R_0(0)(\psi f_0) \quad \text{in } \boldsymbol{R}^n. \tag{5.10}$$

In particular, it follows from (5.10) and the definition of u that

$$R_0(0)(\psi f_0) = R(0)f \quad \text{in } \Omega. \tag{5.10$'$}$$

Combining (5.7) and (5.10)$'$, we have

$$R_0(0)(\psi f_0) = 0 \quad \text{in } \Omega. \tag{5.11}$$

Noting the fact that $\Delta R_0(0)(\psi f_0) = \psi f_0$, we have by (5.11) that

$$\psi f_0 = 0 \quad \text{in } \Omega. \tag{5.12}$$

Combining (5.3), (5.8) and (5.12), we have that $f=0$, which shows that (5.4) is valid.

In order to complete the proof of Lemma 5.5, we prove Lemmas 5.6 and 5.7.

Proof of Lemma 5.6. By (4.1), we have

$$R_0(0)f = -(2\pi)^{-n}\int e^{\sqrt{-1}x\xi}\hat{f}(\xi)|\xi|^{-2}d\xi. \tag{5.13}$$

From differentiation under the integral sign and Parseval's equality, Lemma 5.6-

(ii) follows immediately. In order to prove Lemma 5.6-(i), first, we shall show

$$\lim \rho^{-2}\int_{\rho\leq|x|\leq 2\rho}|R_0(0)f|^2dx=0 \quad \text{for any } f\in L_r^2(\boldsymbol{R}^n). \tag{5.14}$$

In fact, we note that the following formula holds (cf. Mizohata [8, p. 99]):

$$-(2\pi)^{-n}\int e^{\sqrt{-1}x\xi}|\xi|^{-2}d\xi=c_n|x|^{-n+2} \quad \text{if } n\geqq 3, \tag{5.15}$$

where $c_n=-2^{-2}\pi^{-n/2}\Gamma\left(\frac{n-2}{2}\right)$ (Γ is the Gammer function). Combining (5.13) and (5.15), we have

$$R_0(0)f=c_n\int f(y)/|x-y|^{n-2}dy\,. \tag{5.16}$$

Taking $\rho>2r$ and noting that $f\in L_r^2(\boldsymbol{R}^n)$ and that $n\geqq 3$, we have from (5.16)

$$\rho^{-2}\int_{\rho\leq|x|\leq 2\rho}|R_0(0)f|^2dx\leqq C\rho^{-2}\int_{\rho\leq|x|\leq 2\rho}|x|^{-2n+4}\int|f(y)|dy\leqq C(n, r)\|f\|'\rho^{-1}. \tag{5.17}$$

Thus, (5.14) follows from (5.17).

Now, we shall show that Lemma 5.6-(i) follows from (5.14). Choosing $\chi(x)\in C_0^\infty(\boldsymbol{R}^n)$ so that $\chi(x)=1$ if $|x|\leqq 1$ and $=0$ if $|x|\geqq 2$ and $0\leqq\chi\leqq 1$. Since $\partial_x^\alpha R_0(0)f$ belongs to $L^2(B_s)$ for any $s>0$ and multi-index α with $|\alpha|\leqq 2$ because of Corollary 4.2, we have by integration by parts that

$$\begin{aligned}\Sigma_{j=1}^n\int\chi(\rho^{-1}x)|\partial_jR_0(0)f|^2dx=&-\int\chi(\rho^{-1}x)f\cdot\overline{R_0(0)f}\,dx\\&+2^{-1}\rho^{-2}\int(\Delta\chi)(\rho^{-1}x)|R_0(0)f|^2dx\,.\end{aligned} \tag{5.18}$$

Here, we have used the fact that $\Delta R_0(0)f=f$. Since supp $\Delta\chi\subset\{x\in\boldsymbol{R}^n;\ 1\leqq|x|\leqq 2\}$, $\chi(\rho^{-1}x)=1$ on $|x|\leqq\rho$ and $f\in L_r^2(\boldsymbol{R}^n)$, it follows from Lemma 2.1, (5.14) and (5.18) that

$$\begin{aligned}|||R_0(0)f|||^2&=\lim_{\rho\to\infty}\int_{|x|<\rho}|\partial_jR_0(0)f|^2dx\leqq C\lim_{\rho\to\infty}\int|f||R_0(0)f|dx\\&\leqq C\|f\|\left(\int_{|x|\leq r}|R_0(0)f|^2dx\right)^{1/2}\leqq Cr\|f\|'\cdot|||R_0(0)f|||'\,.\end{aligned} \tag{5.19}$$

Lemma 5.6-(i) follows from (5.19), which completes the proof of Lemma 5.6.

PROOF of Lemma 5.7. First, we shall show that for any $u\in H_D(\Omega)$

$$\lim_{\rho\to\infty}\rho^{-2}\int_{\rho\leq|x|\leq 2\rho}|u|^2dx=0\,. \tag{5.20}$$

In fact, since $C_0^\infty(\Omega)$ is dense in $H_D(\Omega)$ with respect to the norm $|||\cdot|||$ and since (5.20) is valid for any element of $C_0^\infty(\Omega)$, it follows immediately from Lemma 2.1

that (5.20) is also valid for any element of $H_D(\Omega)$.

Let χ be the same as in the proof of Lemma 5.6. Noting that $u=0$ on $\partial\Omega$, we have by integration by parts that

$$\sum_{j=1}^{n}\int_{\Omega}\chi(\rho^{-1}x)|\partial_j u|^2dx=2^{-1}\rho^{-2}\int(\Delta\chi)(\rho^{-1}x)|u|^2dx\,. \tag{5.21}$$

Here, we have used the fact that $\Delta u=0$. Letting $\rho\to\infty$ in (5.21) and using (5.20), we have $|||u|||^2=0$. Combining this and Lemma 2.1, we have $u\equiv 0$, which completes the proof of Lemma 5.7-(i). By the same method as in the proof just mentioned, we can show Lemma 5.7-(ii). So, we omit the proof of Lemma 5.7-(ii). Q.E.D.

Combining Definition 5.3, Lemma 5.5 and Corollary 4.2-(ii), we have

LEMMA 5.8. *Let c_0 be the same as in Lemma 5.5. Then there exists a small positive constant $d<1/4$ such that*

$$\|1-Q(\tau)\|_{B(L_T^2(\Omega))}\geqq c_0/2 \quad \textit{if } \tau\in[-d,\ d].$$

From Lemma 5.8, it follows that the inverse operator $(1-Q(\tau))^{-1}\in \boldsymbol{B}(L_T^2(\Omega))$ of the operator $1-Q(\tau)$ exists and that

$$\|(1-Q(\tau))^{-1}\|_{B(L_T^2(\Omega))}\leqq 2/c_0 \quad \text{if } \tau\in[-d,\ d].$$

Combining this and Lemma 5.4, we have that for any $f\in L_T^2(\Omega)$ and $\tau\in[-d,\ d]$

$$(\Delta+\tau^2-\sqrt{-1}\tau)P(\tau)(1-Q(\tau))^{-1}f=f \ \text{ in } \Omega,\quad P(\tau)(1-Q(\tau))^{-1}\in \boldsymbol{B}(L_T^2(\Omega)\,;\,\mathcal{J}). \tag{5.22}$$

Since $|\tau^2-\sqrt{-1}\tau-|\xi|^2|\geqq C(\tau)(1+|\xi|^2)$ if $\tau\in[-d,\ d]-\{0\}$, we have $R_0(\tau)\in \boldsymbol{B}(L_T^2(\boldsymbol{R}^n)\,;\,H_2^2(\boldsymbol{R}^n))$. Thus, $P(\tau)(1-Q(\tau))^{-1}\in\boldsymbol{B}(L_T^2(\Omega)\,;\,H_D(\Omega)\cap H_2^2(\Omega))$, $\tau\in[-d,\ d]-\{0\}$. Combining this and (5.22), we have the facts that $(\Delta+\tau^2-\sqrt{-1}\tau)(R'(\tau)f-P(\tau)(1-Q(\tau)^{-1}f)=0$ in Ω and that $R'(\tau)f-P(\tau)(1-Q(\tau))^{-1}f\in H_2^2(\Omega)\cap H_D(\Omega)$. By integration by parts we have that $R'(\tau)f=P(\tau)(1-Q(\tau))^{-1}f$ for $\tau\in[-d,\ d]$.

Furthermore, since $P(0)(1-Q(0))^{-1}f$ satisfies the equation: $\Delta P(0)(1-Q(0))^{-1}f=f$ in Ω and the condition: $P(0)(1-Q(0))^{-1}f\in H_D^2(\Omega)$ and since it follows from Theorem 5.1 that $R'(0)f$ satisfies the same equation: $\Delta R'(0)f=f$ in Ω and the same condition: $R'(0)f\in H_D^2(\Omega)$, by Lemma 5.7-(i) we have also that $R'(0)f=P(0)(1-Q(0))^{-1}f$.

Summing up, we have showed

LEMMA 5.9. *Let d be the same as in Lemma 5.8. Then, $R(\tau)=P(\tau)(1-Q(\tau))^{-1}$ for $\tau\in[-d,\ d]$.*

Combining Lemmas 5.8 and 5.9, and Theorems 3.2, 4.1 and 5.1, we have

THEOREM 5.10. *Let r be a fixed constant $\geqq r_0+3$ mentioned at the beginning of this section. Assume that $n\geqq 3$. Let d be the same as in Lemma 5.8 and $\rho(\tau)$ a $C_0^\infty(\boldsymbol{R}^1)$-function such that $\rho(\tau)=1$ if $|\tau|\leqq d/4$ and $=0$ if $|\tau|\geqq d/2$. Then, for any integer $N\geqq 0$, multi-index α with $|\alpha|\leqq 2$ and $f\in L_r^2(\Omega)$, we have that $\rho(\tau)\tau^N\partial_x^\alpha R(\tau)f$ belongs to $C^{(n/2)+N}(\boldsymbol{R}^1; \mathcal{G})$.*

Furthermore,

$$\langle\!\langle \rho(\tau)\tau^N\partial_x^\alpha R(\alpha)f\rangle\!\rangle_{\frac{n}{2}+N,\mathcal{G}}\leqq C(n, N, \rho, \alpha, r)\|f\|.$$

§ 6. Proof of Theorem 1.1.

In this section, we consider the following problem:

$$(6.1)\qquad \begin{aligned} &(\partial_t^2+\partial_t-\Delta)u=0 && \text{in } \mathcal{D},\\ &u=0 && \text{on } \mathcal{D}',\\ &u(0, x)=\phi_0(x), \quad (\partial_t u)(0, x)=\phi_1(x), && \text{in } \Omega, \end{aligned}$$

where ϕ_0, $\phi_1\in C^\infty(\Omega)$, supp ϕ_0, supp $\phi_1\subset\Omega_r$ and $r\geqq r_0+3$. It is well-known that (6.1) is C^∞-well posed under the compatibility condition (6.3) below (cf. Ikawa [4]). We define $u_j(x)$ successively by

$$(6.2)\qquad u_0=\phi_0,\quad u_1=\phi_1,\quad u_j=-u_{j-1}+\Delta u_{j-2},\quad j\geqq 2.$$

The compatibility condition for (6.1) is the following:

$$(6.3)\qquad u_j(x)=0 \quad \text{on } \partial\Omega,\ j\geqq 0.$$

It follows immediately from the definitions of u_j that $\partial_t^j u(0, x)=u_j(x)$, $j\geqq 0$. Furthermore, we have that for any p with $1\leqq p\leqq\infty$ and non-negative integers j and N,

$$(6.4)\qquad \|u_j\|_{p,N}\leqq C(p, j, N)[\|\phi_0\|_{p,N+j}+\|\phi_1\|_{p,N+j-1}].$$

We shall represent u in terms of $R(\tau)$. For this purpose, choosing $\psi(t)\in C^\infty(\boldsymbol{R}^1)$ so that $\psi(t)=1$ if $t\leqq 1/2$ and $=0$ if $t\geqq 1$, we define $g_j(t, x)$ by

$$(6.5)\qquad g_j(t, x)\equiv(\partial_t^2+\partial_t-\Delta)[\textstyle\sum_{k=0}^{j+1}u_k(x)t^k/k!]\psi(t),\quad j\geqq 0.$$

Put

$$(6.6)\qquad h_j(\tau, x)\equiv\int^\infty e^{\sqrt{-1}\tau t}g_j(t, x)dt.$$

It follows from (6.2) and (6.5) that if $j\geqq 1$

$$(6.7)\qquad \partial_t^k g_j(0, x)=0,\quad 0\leqq k\leqq j-1.$$

Using (6.4) and (6.7), we have by integration by parts that

$$\|\partial_t^M h_j(\tau, \cdot)\|_{2,N} \leqq C(N, M)(1+|\tau|)^{-(j+1)}[\|\phi_0\|_{2,N+3+j}+\|\phi_1\|_{2,N+2+j}], \tag{6.8}$$
$$h_j \in \mathrm{Anal}(\boldsymbol{C}_-; \boldsymbol{C}^\infty_{(0)}(\Omega_r)) \cap \boldsymbol{C}^\infty(\boldsymbol{R}^1; \boldsymbol{C}^\infty_{(0)}(\Omega_r)).$$

Here we have put $\boldsymbol{C}_- = \{\tau \in \boldsymbol{C}; \operatorname{Im}\tau < 0\}$, $\boldsymbol{C}^\infty_{(0)}(\Omega_r) = \{v \in \boldsymbol{C}^\infty(\Omega); v = v' \text{ for some } v' \in \boldsymbol{C}^\infty_0(B_r)\}$. Put

$$w_j(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sqrt{-1}\tau t} R(\tau) h_j(\tau, \cdot)\, d\tau. \tag{6.9}$$

First, we shall show that $u = w_j$ for any $j \geqq 0$. It follows from Theorems 2.4 and 5.10, Definition 3.2 and (6.8) that

$$R(\tau) h_j(\tau, \cdot) \in \mathrm{Anal}(\boldsymbol{C}_-; \mathcal{S}) \cap C^0(\boldsymbol{R}^1; \mathcal{S}). \tag{6.10}$$

Noting that the resolvent equation:

$$R(\tau) = (\tau^2 - \sqrt{-1}\,\tau)^{-1} - (\tau^2 - \sqrt{-1}\,\tau)^{-1} R(\tau)\Delta, \tag{6.11}$$

holds, we have by (2.3) and (2.4) in Theorem 3.3 and (6.8) that

$$\langle \bar{D}_x^1 R(\tau) h_j(\tau, \cdot)\rangle \leqq C(r)|\operatorname{Re}\tau|^{-2}(1+|\tau|)^{-j-1}[\|\phi_0\|_{2,j+5}+\|\phi_1\|_{2,j+4}] \tag{6.12}$$

if $\tau \in \overline{\boldsymbol{C}_-}$ and $|\operatorname{Re}\tau| \geqq 1$. Using Cauchy's integral formula, we have by (6.9), (6.10) and (6.12) that

$$w_j(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sqrt{-1}(\mu+\sqrt{-1}\sigma)t} R(\mu+\sqrt{-1}\,\sigma) h_j(\mu+\sqrt{-1}\,\sigma, \cdot)\, d\mu \tag{6.13}$$

for any $\sigma \leqq 0$. Furthermore, it follows from (6.12) that

$$w_j(t, x) \in \boldsymbol{C}^{j+1}((-\infty, \infty); \mathcal{S}). \tag{6.14}$$

Since $|\tau^2 - \sqrt{-1}\,\tau| \geqq \frac{1}{2}|\operatorname{Im}\tau|^2 \geqq 2$ if $|\operatorname{Re}\tau| \leqq 1$ and $\operatorname{Im}\tau \leqq -2$, it follows from (6.12) and (6.13) that

$$\langle w_j(t, \cdot)\rangle \leqq C(r) e^{-\sigma t}[\|\phi_0\|_{2,j+5}+\|\phi_1\|_{2,j+4}]$$

for any $\sigma \leqq -2$ and $t < 0$, which shows

$$w_j(t, x) = 0 \quad \text{if } t < 0 \text{ and } x \in \Omega. \tag{6.15}$$

Noting that $R(\tau) h_j(\tau, \cdot) = 0$ on $\partial\Omega$ and combining (6.14) and (6.15), we have

$$\begin{aligned} &\partial_t^k w_j(0, x) = 0, \quad 0 \leqq k \leqq j+1, \\ &w_j = 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{6.16}$$

Using (6.5), (6.6), (6.11) and (6.13), we have

$$w_j(t, x) = \int_0^t g_j(s, x)(e^{-(t-s)} - 1)\, ds + f_j(t, x) \tag{6.17}$$

where

$$f_j(t, x)=-\frac{1}{2\pi}\int_{-\infty+\sqrt{-1}\sigma}^{\infty+\sqrt{-1}\sigma} e^{\sqrt{-1}t\tau}R(\tau)h_j(\tau, \cdot)(\tau^2-\sqrt{-1}\tau)^{-1}d\tau, \quad \sigma<0.$$

Here we have used the fact that $\frac{1}{2\pi}\int_{-\infty+\sqrt{-1}\sigma}^{\infty+\sqrt{-1}\sigma} e^{\sqrt{-1}t\tau}(\tau^2-\tau)^{-1}d\tau=e^{-t}-1$ if $t>0$ and $=0$ if $t<0$. It thus follows from (2.3), (2.4), (6.6), (6.8), (6.12) and (6.17) that

$$\begin{aligned}&(\partial_t^2+\partial_t-\Delta)w_j(t, x)=g_j(t, x) \quad \text{in } \Omega,\\ &\langle \bar{D}^2 f_j(t, \cdot)\rangle \leqq C(r)[\|\phi_0\|_{2,5+j}+\|\phi_1\|_{2,4+j}].\end{aligned} \tag{6.18}$$

In view of an usual energy method (cf. Lax-Phillips [6. the identity (1.2) in Chapter V]), we have from (6.16) and (6.18) that

$$u(t, x)=(\textstyle\sum_{k=0}^{j+1}u_k(x)t^k/k!)\psi(t)-w_j(t, x) \quad \text{for any } j\geqq 0. \tag{6.19}$$

Now, we shall estimate $w_j(t, x)$. Choose $\rho(\tau)\in C_0^\infty(\boldsymbol{R}^1)$ so that $\rho(\tau)=1$ if $|\tau|\leqq d/4$ and $=0$ if $|\tau|\geqq d/2$, where d is the same as in Theorem 5.10. Put

$$\begin{aligned}w_j^1(t, x)&=\frac{1}{2\pi}\int e^{\sqrt{-1}t\tau}\rho(\tau)R(\tau)h_j(\tau, \cdot)d\tau,\\ w_j^2(t, x)&=\frac{1}{2\pi}\int e^{\sqrt{-1}t\tau}(1-\rho(\tau))R(\tau)h_j(\tau, \cdot)d\tau.\end{aligned} \tag{6.20}$$

It follows from (6.8), (6.20) and Theorems 3.7 and 5.10 that

$$\begin{aligned}\langle \bar{D}_x^2\partial_t^N w_j(t, \cdot)\rangle &\leqq C(r, N)(1+|t|)^{-(n/2)-N}\langle\!\langle \tau^N\rho(\tau)\bar{D}_x^2 R(\tau)h_j(\tau, \cdot)\rangle\!\rangle_{\frac{n}{2}+N,\mathscr{g}}\\ &\leqq C(r, N)(1+|t|)^{-(n/2)-N}[\|\phi_0\|_{2,j+3}+\|\phi_1\|_{2,j+2}], \quad \text{for any integer } N\geqq 0.\end{aligned} \tag{6.21}$$

On the other hand, we have by (3.8), (6.8), (6.20) and Theorem 2.4 that for any integer $M\geqq 0$

$$\begin{aligned}&\langle \partial_t^N w_j^2(t, \cdot)\rangle \leqq C(M, N, r)(1+|t|)^{-M}[\|\phi_0\|_{2,j+3}+\|\phi_1\|_{2,j+2}], \quad 0\leqq N\leqq j,\\ &\langle D_x^1\partial_t^N w_j^2(t, \cdot)\rangle \leqq C(N, M, r)(1+|t|)^{-M}[\|\phi_0\|_{2,j+3}+\|\phi_1\|_{2,j+2}], \quad 0\leqq N\leqq j-1,\\ &\langle D_x^2\partial_t^N w_j^2(t, \cdot)\rangle \leqq C(N, M, r)(1+|t|)^{-M}[\|\phi_0\|_{2,j+3}+\|\phi_1\|_{2,j+2}], \quad 0\leqq N\leqq j-2.\end{aligned} \tag{6.22}$$

It thus follows from (6.19), (6.20), (6.21) and (6.22) that for any $r, r'\geqq r_0+3$ and $t>0$

$$\begin{aligned}&\|\partial_t^N u(t, \cdot)\|_{\Omega_{r'},2,0}\leqq C(N, r, r')(1+t)^{-(n/2)-N}[\|\phi_0\|_{2,N+3}+\|\phi_1\|_{2,N+2}],\\ &\|D_x^1\partial_t^N u(t, \cdot)\|_{\Omega_{r'},2,0}\leqq C(N, r, r')(1+t)^{-(n/2)-N}[\|\phi_0\|_{2,N+4}+\|\phi_1\|_{2,N+3}],\\ &\|D_x^2\partial_t^N u(t, \cdot)\|_{\Omega_{r'},2,0}\leqq C(N, r, r')(1+t)^{-(n/2)-N}[\|\phi_0\|_{2,N+5}+\|\phi_1\|_{2,N+4}].\end{aligned} \tag{6.23}$$

The assertions stated in Theorem 1.1 follows from (6.23) in the case that $M\leqq 2$.

In order to estimate higher order derivatives of u, we need the following well-known *a priori* estimate for Δ.

LEMMA 6.1. *Let r_1, r_2 be any positive numbers with $r_1>r_2\geqq r_0$, and $g\in H_2^{N+2}(\Omega_{r_1})$ such that $g=0$ on $\partial\Omega$. Then,*

$$\|D_x^{N+2}g\|_{\Omega_{r_2},2}\leqq C(r_1, r_2)[\|\bar{D}_x^N \Delta g\|_{\Omega_{r_1},2}+\|g\|_{\Omega_{r_1},2}].$$

Since it follows from (6.1) that

$$\Delta\partial_t^N u(t, x)=\partial_t^{N+2}u(t, x)+\partial_t^{N+1}u(t, x) \quad \text{in } \Omega, \tag{6.24}$$

applying Lemma 6.1 to (6.24), by induction we can show the following:

$$\|D_x^M\partial_t^N u(t, \cdot)\|_{\Omega_{r'},2}\leqq C(r, r', N, M)(1+t)^{-(n/2)-N}[\|\phi_0\|_{2, N+M+3}+\|\phi_1\|_{2, N+M+2}] \tag{6.25}$$

for any $r, r'\geqq r_0+3$ and $t>0$, which completes the proof of Theorem 1.1.

PART 2

Some estimates for linearized problems

In this part, we give L^2 and uniform decay estimates of solutions to linearized problems. These estimates will be used in order to show the convergence of our iteration scheme defined in part 3.

§ 7. L^2-estimates for some hyperbolic equation.

In this section, we shall obtain L^2-estimates of solutions of the following linear equation:

$$\begin{aligned}\mathcal{L}u\equiv(1+a^0(t, x))\partial_t^2 u+\textstyle\sum_{j=1}^n a^j(t, x)\partial_j\partial_t u-\sum_{i,j=1}^n(\delta_{ij}+a^{ij}(t, x))\partial_i\partial_j u&\\ +(1+b^0(t, x))\partial_t u+\textstyle\sum_{j=1}^n b^j(t, x)\partial_j u+c(t, x)u=f(t, x) &\quad \text{in } \mathcal{D},\\ u=0 &\quad \text{on } \mathcal{D}',\\ u(0, x)=(\partial_t u)(0, x)=0 &\quad \text{in } \Omega,\end{aligned} \tag{7.1}$$

where $\delta_{ij}=1$ if $i=j$ and $=0$ if $i\neq j$ and f satisfies the condition:

$$f\in E^{L-1}, \quad L\geqq 2 \tag{7.2}$$

(cf. Notations).

Throughout this section, without fear of confusion, for simplicity we write

$$u_j=\partial_j u, \quad u_t=\partial_t u, \quad \|v(t)\|_\infty=\|v(t, \cdot)\|_{\Omega,\infty}, \quad \|v(t)\|=\|v(t, \cdot)\|_{\Omega,2},$$

$$(v, w)=(v(t), w(t))=(v(t, \cdot), w(t, \cdot))_\Omega,$$

and all functions are real-valued.

We impose the following assumptions on the coefficients of the operator $\mathcal{L}$.

ASSUMPTION 7.1. Put $\mathcal{A}=\mathcal{A}(t, x)\equiv(a^j(t, x),\ j=0, \cdots, n;\ a^{ij}(t, x),\ i, j=1, \cdots, n;\ b^j(t, x),\ j=0, 1, \cdots, n;\ c(t, x))$.

1° Each component of $\mathcal{A}$ is a $\mathcal{B}^\infty([0, \infty)\times\bar{\Omega})$-function.

2° $a^{ij}(t, x)=a^{ji}(t, x)$ for all $(t, x)\in[0, \infty)\times\bar{\Omega}$.

3° For any $\xi=(\xi_1, \cdots, \xi_n)\in\boldsymbol{R}^n$ and $(t, x)\in[0, \infty)\times\bar{\Omega}$,

$$\sum_{i,j=1}^n(\delta_{ij}+a^{ij}(t, x))\xi_i\xi_j\geqq\frac{1}{2}|\xi|^2.$$

4° $$|a^0|_{\infty,0,0}\leqq\frac{1}{2}.$$

5° $$|b^0|_{\infty,0,0}+\sum_{j=0}^n|\partial_j a^j|_{\infty,0,0}\leqq\frac{1}{2}.$$

Under Assumption 7.1, $\mathcal{L}$ is a strictly hyperbolic operator with first order dissipative term. It thus follows that for any data f satisfying (7.2) there exists a unique solution $u\in\tilde{\boldsymbol{E}}^L$ (cf. Ikawa [4]). The following is the main result of this section.

THEOREM 7.2. *Let L be an integer $\geqq 2$ and Assumption 7.1 be fulfilled. Let d_1 be a small positive number defined in Lemma 7.3 below and η be a positive number. Assume that the estimates:*

$$|\mathcal{A}|_{\infty,0,0}\leqq d_1,\quad |\mathcal{A}|_{\infty,1+\eta,1}\leqq 1 \tag{7.3}$$

hold. Then, the solution $u\in\tilde{\boldsymbol{E}}^L$ of the equation (7.1) for a data $f\in\boldsymbol{E}^{L-1}$ satisfies the following two estimates:

$$|u|_{2,[0,T],L}\leqq C(L, T)[|f|_{2,[0,T],L-1}+|\mathcal{A}|_{\infty,[0,T],L}|f|_{2,[0,T],0}],\ \textit{for any}\ T>0, \tag{7.4}$$

$$|u|_{2,0,0}+|D^1u|_{2,1/2,L-1}\leqq C(L, \eta)[|f|_{2,1+\eta,L-1}+|\mathcal{A}|_{\infty,1+\eta,L}|f|_{2,1+\eta,0}]. \tag{7.5}$$

Since to obtain the estimate (7.4) is easier than the estimate (7.5) and since both methods are essentially the same, we shall show only the estimate (7.5) below. In order to show the estimate (7.5), we need following two lemmas.

LEMMA 7.3. *Let $d^{ij}(x)$, $i, j=1, \cdots, n$, be real-valued $\mathcal{B}^\infty(\bar{\Omega})$-functions and satisfy the following conditions:*

(i) $d^{ij}=d^{ji}$,

(ii) *for any* $\xi=(\xi_1, \cdots, \xi_n)\in\boldsymbol{R}^n$ *and* $x\in\Omega$ $\sum_{i,j=1}^n(\delta_{ij}+d^{ij}(x))\xi_i\xi_j\geqq\frac{1}{2}|\xi|^2$,

(iii) $|\boldsymbol{D}|_{\infty,1}\leqq 1$ *where* $\boldsymbol{D}=(d^{ij}(x);\ i, j=1, \cdots, n)$.

Then, there exists a small positive absolute constant d_1 *such that if* $|\boldsymbol{D}|_\infty \leqq d_1$ *then for any integer* $M \geqq 0$ *and* $\zeta \in H_2^{M+2}(\Omega) \cap H_D(\Omega)$

$$\|D_x^{M+2}\zeta\|_2 \leqq C(L)(\|\gamma\|_{2,M} + (1+|\boldsymbol{D}|_{\infty,M})(\|\gamma\|_2 + |||\zeta|||))$$

where $\gamma = (\Delta + \sum_{i,j=1}^n d^{ij}(x)\partial_i\partial_j)\zeta$.

LEMMA 7.4. *Let* $I(t)$, $J(t)$ *be non-negative continuous functions such that for some positive constants* c, μ *and* τ *the estimate*:

$$I(t) \leqq c\int_\tau^t (1+s)^{-1-\mu} I(s)ds + J(t) \quad \textit{for any } t > \tau,$$

holds and $\mu(1+\tau)^\mu \geqq 2c$. *Then*

$$\max_{\tau \leqq s \leqq t} I(s) \leqq 2 \max_{\tau \leqq s \leqq t} J(s).$$

To prove Lemma 7.3, we note the following two points.

(i) From assumptions (i), (ii) and (iii) in Lemma 7.3, the operator $\Delta + \sum_{i,j=1}^n d_{ij}\partial_i\partial_j$ is strongly elliptic in Ω.

(ii) Since $|\boldsymbol{D}|_\infty$ is sufficiently small, we can consider the operator $\Delta + \sum_{i,j=1}^n d_{ij}\partial_i\partial_j$ as a small perturbation of Δ outside a ball B_{r+1} $(r \geqq r_0)$.

Choose ϕ, $\psi \in \boldsymbol{C}^\infty(\boldsymbol{R}^n)$ so that $\phi = 1$ in B_{r+2} and $\phi = 0$ in $\boldsymbol{R}^n - B_{r+3}$, and that $\psi = 1$ in $\boldsymbol{R}^n - B_{r+1}$ and $\psi = 0$ in B_r. In view of the point (i) just mentioned, we can estimate $\phi\zeta$ by using Theorem Ap. 2 in Appendix I and the usual manner of estimating a second order strongly elliptic operator with zero Dirichlet condition in bounded domains. In view of the point (ii) just mentioned, we can estimate $\psi\zeta$ by using Theorem Ap. 2, the estimates for the Laplacian Δ in $\boldsymbol{R}^n$ and the estimates for ζ in Ω_{r+2}. From this point of view, we omit the detailed proof of Lemma 7.3. Since Lemma 7.4 follows from an easier calculation, we also omit its proof.

Now, we shall show the estimate (7.5). As is well-known, without loss of generality, we may assume that $u \in \boldsymbol{C}^\infty([0, \infty); H_2^L(\Omega) \cap H_D(\Omega))$ (cf. Ikawa [4] and also Shibata [18, the proof of Theorem 4.10]). Differentiating (7.1) N-times $(N \geqq 0)$ with respect to t, we write the resulting equation as follows:

$$\begin{aligned}
&(1+a^0)\partial_t^{N+2}u + \textstyle\sum_{j=1}^n a^j\partial_t^{N+1}\partial_j u - \sum_{i,j=1}^n(\delta_{ij}+a^{ij})\partial_i\partial_j\partial_t^N u + (1+b^0)\partial_t^{N+1}u \\
(7.6) \qquad &+ \textstyle\sum_{j=1}^n b^j\partial_j\partial_t^N u + N\sum_{j=0}^n(\partial_t a^j)\partial_t^N\partial_j u - N\sum_{i,j=1}^n(\partial_t a^{ij})\partial_i\partial_j\partial_t^{(N-1)^+}u \\
&+ \rho(N)cu = f_N \qquad \text{in } \mathcal{D},
\end{aligned}$$

where $(N-1)^+ = N-1$ if $N \geqq 1$ and $=0$ if $N=0$, $\rho(N)=1$ if $N=0$ and $=0$ if $N \geqq 1$,

$$
(7.7)\qquad f_N=\begin{cases} f & \text{if } N=0,\\ \partial_t^N f-\sum_{j=0}^n\{\partial_t^N(a^j\partial_t\partial_j u)-a^j\partial_t^{N+1}\partial_j u-N\partial_t a^j\cdot\partial_t^N\partial_j u\} & \\ \quad+\sum_{i,j=1}^n\{\partial_t^N(a^{ij}\partial_i\partial_j u)-a^{ij}\partial_i\partial_j\partial_t^N u-N\partial_t a^{ij}\cdot\partial_i\partial_j\partial_t^{N-1}u\} & \\ \quad+\sum_{j=0}^n\{\partial_t^N(b^j\partial_j u)-b^j\partial_j\partial_t^N u\}+\partial_t^N(cu) & \text{if } N\geqq 1. \end{cases}
$$

Multiplying (7.6) by $\partial_t^{N+1}u$ and integrating over Ω, we have by integration by parts

$$
\begin{aligned}
(7.8)\quad & \frac{1}{2}\frac{d}{dt}((1+a^0)\partial_t^{N+1}u,\ \partial_t^{N+1}u)+\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^n((\delta_{ij}+a^{ij})\partial_i\partial_t^N u,\ \partial_j\partial_t^N u)\\
&+\Big(\Big(1+b^0-\frac{1}{2}\sum_{j=0}^n a_j^j\Big)\partial_t^{N+1}u,\ \partial_t^{N+1}u\Big)+\sum_{i,j=1}^n(a_j^{ij}\partial_j\partial_t^N u,\ \partial_t^{N+1}u)\\
&-\frac{1}{2}\sum_{i,j=1}^n(a_t^{ij}\partial_i\partial_t^N u,\ \partial_j\partial_t^N u)+\sum_{j=1}^n(b^j\partial_j\partial_t^N u,\ \partial_t^{N+1}u)+\rho(N)(cu,\ \partial_t^{N+1}u)\\
&+N\sum_{j=0}^n(a_t^j\partial_j\partial_t^N u,\ \partial_t^{N+1}u)-N(\sum_{i,j=1}^n(a_t^{ij}\partial_i\partial_j\partial_t^{(N-1)^+}u,\ \partial_t^{N+1}u)=(f_N,\ \partial_t^{N+1}u)\,.
\end{aligned}
$$

When $N\geqq 1$, in order to estimate the terms: $a_t^{ij}\partial_i\partial_j\partial_t^{N-1}u$, differentiating (7.1) $(N-1)$-times with respect to t, we write the resulting equations as follows:

$$
(7.9)\qquad \sum_{i,j=1}^n(\delta_{ij}+a^{ij})\partial_i\partial_j\partial_t^{N-1}u=(1+a^0)\partial_t^{N+1}u+\sum_{j=1}^n a^j\partial_t^N\partial_j u+\partial_t^{N-1}f+g_{N-1}
$$

where

$$
\begin{aligned}
(7.10)\qquad g_{N-1}=&-\sum_{i,j=1}^n[\partial_t^{N-1}(a^{ij}\partial_i\partial_j u)-a^{ij}\partial_i\partial_j\partial_t^{N-1}u]\\
&+\sum_{j=0}^n[\partial_t^{N-1}(a^j\partial_j\partial_t u)-a^j\partial_j\partial_t^N u]+\partial_t^{N-1}[(1+b^0)\partial_t u)\\
&+\sum_{j=1}^n\partial_t^{N-1}(b^j\partial_j u)+\partial_t^{N-1}(cu)\,.
\end{aligned}
$$

When $N\geqq 1$, applying Lemma 7.3 to (7.9) and using (7.3), we have

$$
(7.11)\qquad \|D_x^2\partial_t^{N-1}u(t)\|\leqq c_0(\|\partial_t^{N-1}f(t)\|+\|D^1\partial_t^N u(t)\|+\|g_{N-1}(t)\|+\|D_x^1\partial_t^{N-1}u(t)\|]
$$

for some absolute constant c_0. Let τ be a large positive number determined later. Integrating (7.8) from τ to t $(>\tau)$ and using Assumption 7.1, (7.3), (7.11) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
(7.12)\qquad & \|D^1\partial_t^N u(t)\|^2+\int_\tau^t\|\partial_t^{N+1}u(s)\|^2ds\\
&\leqq c_1\Big[\|D^1\partial_t^N u(\tau)\|^2+(N+1)\int_\tau^t(1+s)^{-1-\eta}\|D^1\partial_t^N u(s)\|^2ds\\
&\quad+N^2\int_\tau^t(1+s)^{-2-2\eta}(\|\partial_t^{(N-1)^+}f(s)\|^2+\|g_{(N-1)^+}(s)\|^2+\|D^1\partial_t^{(N-1)^+}u(s)\|^2ds\\
&\quad+\rho(N)\int_\tau^t(1+s)^{-2-2\eta}\|u(s)\|^2ds+\int_\tau^t\|f_N(s)\|^2ds\Big]
\end{aligned}
$$

for some absolute constant $c_1>0$. Here, we have used the fact that

(7.13) $$|\mathcal{A}(t, x)|+|D^1\mathcal{A}(t, x)|\leqq|\mathcal{A}|_{\infty,1+\eta,1}(1+t)^{-1-\eta}\quad\text{for } t>0.$$

Next, multiplying (7.6) by $\partial_t^N u$ and integrating over Ω, we have by integration by parts that

(7.14) $$\begin{aligned}&\frac{d}{dt}((1+a^0)\partial_t^{N+1}u,\ \partial_t^N u)-(a_t^0\partial_t^{N+1}u,\ \partial_t^N u)-((1+a^0)\partial_t^{N+1}u)\\&-\textstyle\sum_{j=1}^n(a_j^j\partial_t^{N+1}u,\ \partial_t^N u)-\sum_{j=1}^n(a^j\partial_t^{N+1}u,\ \partial_j\partial_t^N u)\\&+\textstyle\sum_{i,j=1}^n((\delta_{ij}+a^{ij})\partial_i\partial_t^N u,\ \partial_j\partial_t^N u)+\sum_{i,j=1}^n(a_j^{ij}\partial_i\partial_t^N u,\ \partial_t^N u)\\&+\frac{1}{2}\frac{d}{dt}(\partial_t^N u,\ \partial_t^N u)+\textstyle\sum_{j=0}^n(b^j\partial_t^N\partial_j u,\ \partial_t^N u)+N\sum_{j=0}^n(a_t^j\partial_t^N\partial_j u,\ \partial_t^N u)\\&-N\textstyle\sum_{i,j=1}^n(a_t^{ij}\partial_i\partial_j\partial_t^{(N-1)^+}u,\ \partial_t^N u)+\rho(N)(cu,\ \partial_t^N u)=(f_N,\ \partial_t^N u).\end{aligned}$$

Integrating (7.14) from τ to t ($>\tau$) and using Assumption 7.1, (7.3), (7.12), (7.13) and the Cauchy-Schwarz inequality, we have

(7.15) $$\begin{aligned}&\|\partial_t^N u(t)\|^2+\|D^1\partial_t^N u(t)\|^2+\int_\tau^t\|D^1\partial_t^N u(s)\|^2ds\\&\leqq c_2\Big[\|\partial_t^N u(\tau)\|^2+\|D^1\partial_t^N u(\tau)\|^2+\rho(N)\int_\tau^t(1+s)^{-1-\eta}\|u(s)\|^2ds\\&\quad+N^2\int_\tau^t(1+s)^{-1-\eta}\|D^1\partial_t^{(N-1)^+}u(s)\|^2ds\\&\quad+(1+N)\int_\tau^t(1+s)^{-1-\eta}\|\bar{D}^1\partial_t^N u(s)\|^2ds\\&\quad+N^2\int_\tau^t(1+s)^{-1-\eta}[\|\partial_t^{(N-1)^+}f(s)\|^2+\|g_{(N-1)^+}(s)\|^2]ds\\&\quad+\int_\tau^t(1+s)^{1+\eta}\|f_N(s)\|^2ds\Big]\end{aligned}$$

for some absolute constant $c_2>0$. Applying Lemma 7.4 to (7.15), we have that there exists a positive large number $\tau=\tau(N)$ depending only on N such that

(7.16) $$\begin{aligned}&\|\partial_t^N u(t)\|^2+\|D^1\partial_t^N u(t)\|^2+\int_\tau^t\|D^1\partial_t^N u(s)\|^2ds\\&\leqq C(N)\Big[\|D^1\partial_t^N u(\tau)\|^2+\|\partial_t^N u(\tau)\|^2+(1-\rho(N))\int_\tau^t(1+s)^{-1-\eta}\\&\qquad\times[\|D^1\partial_t^{(N-1)^+}u(s)\|^2+\|\partial_t^{(N-1)^+}f(s)\|^2+\|g_{(N-1)^+}(s)\|^2]ds\\&\quad+\int_\tau^t(1+s)^{1+\eta}\|f_N(s)\|^2ds\Big]\qquad\text{if } t>\tau.\end{aligned}$$

Now, we shall investigate the total energy decay. Multiplying (7.6) by

$t\partial_t^{N+1}u$ and integrating over $[\tau, t]\times\Omega$, we have by integration by parts, the Cauchy-Schwarz inequality, Assumption 7.1, (7.3) and (7.16) that

$$
\begin{aligned}
(7.17)\qquad & t\|D^1\partial_t^N u(t)\|^2+\int_\tau^t s\|\partial_t^{N+1}u(s)\|^2 ds \\
& \leqq C(N, \eta)\Big[(1+\tau)\|D^1\partial_t^N u(\tau)\|^2\|\partial_t^N u(\tau)\|^2+(1-\rho(N))\int_\tau^t(1+s)^{-1-\eta} \\
& \qquad \times[\|\partial_t^{(N-1)^+}f(s)\|^2+\|g_{(N-1)^+}(s)\|^2+\|D^1\partial_t^{(N-1)^+}u(s)\|^2]ds \\
& \qquad +\int_\tau^t(1+s)^{1+\eta}\|f_N(s)\|^2 ds\Big] \qquad \text{if } t>\tau.
\end{aligned}
$$

In fact, when $N=0$, multiplying (7.6) with $N=0$ by $t\partial_t u$ and integrating over Ω, we have

$$
\begin{aligned}
(7.18)\qquad & \frac{1}{2}\frac{d}{dt}((1+a^0)tu_t, u_t)-\frac{1}{2}(a_t^0 tu_t, u_t)-\frac{1}{2}(a^0u_t, u_t)-\frac{1}{2}\textstyle\sum_{j=1}^n(a_j^j tu_t, u_t) \\
& +\frac{1}{2}\frac{d}{dt}\textstyle\sum_{i,j=1}^n((\delta_{ij}+a^{ij})tu_i, u_i)-\frac{1}{2}\sum_{i,j=1}^n(a_t^{ij}tu_i, u_j)+\sum_{i,j=1}^n(a_j^{ij}tu_i, u_t) \\
& -\frac{1}{2}\textstyle\sum_{i,j=1}^n((\delta_{ij}+a^{ij})u_i, u_j)+((1+b^0)tu_t, u_t)+\sum_{j=1}^n(b^j tu_j, u_t)+(ctu, u_t) \\
& =(f, tu_t).
\end{aligned}
$$

Since for any $\nu>0$ the inequalities:

$$
|(a_j^{ij}tu_i, u_t)|\leqq\frac{1}{4\nu}\|a_j^{ij}(t)\|_\infty^2 t\|u_i(t)\|^2+\nu t\|u_t(t)\|^2,
$$

$$
|(f, tu_t)|\leqq\frac{1}{4\nu}\|f(t)\|^2 t+\nu t\|u_t(t)\|^2,
$$

and

$$
|(ctu, u_t)|\leqq\frac{1}{4\nu}\|c(t)\|_\infty^2 t\|u(t)\|^2+\nu t\|u_t(t)\|^2
$$

hold, integrating (7.18) from τ to t $(>\tau)$ and using (7.16) with $N=0$, we have

$$
\begin{aligned}
(7.19)\qquad & t\|D^1u(t)\|^2+\int_\tau^t s\|\partial_t u(s)\|^2 ds \\
& \leqq c_3\Big[\tau\|D^1u(\tau)\|^2+\|u(\tau)\|^2+\int_\tau^t(1+s)^{1+\eta}\|f(s)\|^2 ds \\
& \quad +\int_\tau^t(\|a^0(s)\|_\infty+\textstyle\sum_{j=1}^n\|a_j^j(s)\|_\infty s)\|u_t(s)\|^2 ds \\
& \quad +\int_\tau^t\|c(s)\|_\infty^2 s\|u(s)\|^2 ds+\int_\tau^t(\textstyle\sum_{i,j=1}^n\|a_t^{ij}(s)\|_\infty s \\
& \quad +\textstyle\sum_{i,j=1}^n\|a_j^{ij}(s)\|_\infty^2 s)\|D_x^1 u(s)\|^2 ds
\end{aligned}
$$

$$+\int_\tau^t (\Sigma_{i,j=1}^n (\delta_{ij}+\|a^{ij}(s)\|_\infty)\|D_x^1 u(s)\|^2 ds$$

$$+\int_\tau^t (\Sigma_{j=0}^n \|b^j(s)\|_\infty s\|D^1 u(s)\|^2 ds$$

$$+\int_\tau^t \|c(s)\|_\infty^2 s\|u(s)\|^2 ds+\int_\tau^t (1+s)\|f(s)\|^2 ds\Big]$$

for some positive constant c_3. Since it follows from (7.3) and (7.13) that

$$\int_\tau^t \|c(s)\|_\infty^2 s\|u(s)\|^2 ds \leqq \max_{\tau\leqq s\leqq t}\|u(s)\|^2 \cdot \int_\tau^t (1+s)^{-1-2\eta} ds \leqq C(\eta) \max_{\tau\leqq s\leqq t}\|u(s)\|^2,$$

we have from (7.3), (7.13), (7.16) with $N=0$ and (7.19) that

$$t\|D^1 u(t)\|^2+\|u(t)\|^2+\int_\tau^t s\|\partial_t u(s)\|^2 ds$$

$$\leqq C(\eta)\Big[(1+\tau)\|\bar{D}^1 u(\tau)\|^2+\int_\tau^t (1+s)^{1+\eta}\|f(s)\|^2 ds\Big]$$

if $t\geqq\tau$, which shows (7.17) with $N=0$. When $N\geqq 1$, we can show (7.17) in the same manner. So, we omit the proof of (7.17) in the case that $N\geqq 1$.

Since

$$\int_\tau^t (1+s)^{1+\eta}\|f_N(s)\|^2 ds \leqq (|f_N|_{2,1+\eta,0})^2 \int_0^t (1+s)^{-1-\eta} ds \leqq C(\eta)(|f_N|_{2,1+\eta,0})^2,$$

it follows immediately from (7.4), (7.16) and (7.17) that

$$(7.20)\quad |u|_{2,0,0}+|D^1\partial_t^N u|_{2,1/2,0} \leqq C(N,\eta)[|f_N|_{2,1+\eta,0}+(1-\rho(N))(|\partial_t^{(N-1)^+} f|_{2,0,0}$$
$$+|g_{(N-1)^+}|_{2,0,0}+|D^1\partial_t^{(N-1)^+} u|_{2,0,0})+|f|_{2,0,N}+|\mathcal{A}|_{\infty,0,N+1}|f|_{2,0,0}].$$

In particular, (7.5) with $L=1$ follows from (7.20) with $N=0$, (7.3) and (7.7).

Now, using (7.20), we shall show (7.5) with $L\geqq 2$ by induction. When $L=2$, we have from (7.11) and (7.20) with $N=1$ that

$$(7.21)\quad |D^2 u|_{2,1/2,0} \leqq C(1,\eta)[|f_1|_{2,1+\eta,0}+|f|_{2,1/2,0}+|g_0|_{2,1/2,0}+|D^1 u|_{2,1/2,0}].$$

It follows from (7.3), (7.5) with $L=1$ just showed and (7.7) that

$$(7.22)\quad \begin{aligned} &|f_1|_{2,1+\eta,0} \leqq C(|\bar{D}^1 u|_{2,0,0}+|f|_{2,1+\eta,1}) \leqq C|f|_{2,1+\eta,1},\\ &|g_0|_{2,1/2,0} \leqq C(|D^1 u|_{2,1/2,0}+|u|_{2,0,0}) \leqq C|f|_{2,1+\eta,0}. \end{aligned}$$

Combining (7.21), (7.22) and (7.5) with $L=1$, we have

$$(7.23)\quad |D^2 u|_{2,1/2,0} \leqq C|f|_{2,1+\eta,1},$$

which shows (7.5) with $L=2$.

When $L\geqq 3$, assuming that $N\geqq 2$ and that the estimate (7.24) below is already proved for smaller value than $N+1$, we shall prove

$$|\partial_t^{N+1-K}D_x^K u|_{2,1/2,0}\leqq C(N)[|f|_{2,1+\eta,N}+|\mathcal{A}|_{\infty,1+\eta,N+1}|f|_{2,1+\eta,0}] \tag{7.24}$$

for any integer K with $0\leqq K\leqq N+1$.

It follows from (7.11) and (7.20) that

$$\begin{aligned}\textstyle\sum_{K=0}^2|\partial_t^{N+1-K}D_x^K u|_{2,1/2,0}\leqq C(N)[|f|_{2,1/2,N-1}+|g_{N-1}|_{2,1/2,0}\\ +|D_x^1\partial_t^{N-1}u|_{2,1/2,0}+|f_N|_{2,1+\eta,0}].\end{aligned} \tag{7.25}$$

Applying the induction hypotheses, Assumption 7.1, Leibniz's formula and Theorem Ap. 2 in Appendix I to (7.7) and (7.10), we have

$$\begin{aligned}&|f_N|_{2,1+\eta,0}\leqq C(N)[|f|_{2,1+\eta,N}+|\mathcal{A}|_{\infty,1+\eta,N+1}|f|_{2,1+\eta,0}],\\ &|g_{N-1}|_{2,1/2,0}\leqq C(N)[|f|_{2,1+\eta,N-1}+|\mathcal{A}|_{\infty,1+\eta,N}|f|_{2,1+\eta,0}].\end{aligned} \tag{7.26}$$

Combining (7.25) and (7.26), and applying the induction hypotheses to the term: $|D^1\partial_t^{N-1}u|_{2,1/2,0}$, we have

$$\textstyle\sum_{K=0}^2|\partial_t^{N+1-K}D_x^K u|_{2,1/2,0}\leqq C(N)[|f|_{2,1+\eta,N}+|\mathcal{A}|_{\infty,1+\eta,N+1}|f|_{2,1+\eta,0}], \tag{7.27}$$

which shows that (7.24) is valid for each K with $0\leqq K\leqq 2$.

Now, we shall show (7.24) by induction on K. Assume that $K\geqq 3$ and that (7.24) is already proved for smaller value than K. Applying Lemma 7.3 to (7.9) with $M=K-2$, we have

$$\begin{aligned}|D_x^K\partial_t^{N+1-K}u|_{2,1/2,0}&\leqq C(N)\{|f|_{2,1/2,N-1}+|D_x^{K-2}((1+a^0)\partial_t^{N+3-K}u)|_{2,1/2,0}\\ &+|(1+a^0)\partial_t^{N+3-K}u|_{2,1/2,K-3}+\textstyle\sum_{j=1}^n|D_x^{K-2}(a^j\partial_j\partial_t^{N+2-K}u)|_{2,1/2,0}\\ &+\textstyle\sum_{j=1}^n|a^j\partial_j\partial_t^{N+2-K}u|_{2,1/2,K-3}+|g_{N+1-K}|_{2,1/2,K-2}\\ &+|\mathcal{A}|_{\infty,1/2,K-2}[|u|_{2,0,N+3-K}+|f|_{2,0,N+1-K}+|g_{N+1-K}|_{2,0,0}]\end{aligned} \tag{7.28}$$

By using Leibniz's formula, Theorem Ap. 2, (7.3), (7.10) and the induction hypotheses, we have

$$\begin{aligned}&|D_x^{K-2}((1+a^0)\partial_t^{N+3-K}u)|_{2,1/2,0}+\textstyle\sum_{j=1}^n|D_x^{K-2}(a^j\partial_j\partial_t^{N+2-K}u)|_{2,1/2,0}\\ &\leqq C(N)[|D_x^{K-2}\partial_t^{N+3-K}u|_{2,1/2,0}+|D_x^{K-1}\partial_t^{N+2-K}u|_{2,1/2,0}\\ &\quad+\textstyle\sum_{j=0}^n\sum_{i=0}^{K-3}|D_x^{K-2-i}a^j|_{\infty,0,0}|D_x^iD^1\partial_t^{N+2-K}u|_{2,1/2,0}]\\ &\leqq C(N)[|f|_{2,1+\eta,N}+|\mathcal{A}|_{\infty,1+\eta,N+1}|f|_{2,1+\eta,0}\\ &\quad+\textstyle\sum_{i=0}^{K-3}|\mathcal{A}|_{\infty,1+\eta,K-2-i}[|f|_{2,1+\eta,N+2+i-K}+|\mathcal{A}|_{\infty,1+\eta,N+3+i-K}|f|_{2,1+\eta,0}]]\\ &\leqq C(N)[|f|_{2,1+\eta,N}+|\mathcal{A}|_{\infty,1+\eta,N+1}|f|_{2,1+\eta,0}],\end{aligned} \tag{7.29}$$

$$(7.30)\qquad |(1+a^0)\partial_t^{N+3-K}u|_{2,1/2,K-3}+\textstyle\sum_{j=1}^n |a^j\partial_j\partial_t^{N+2-K}u|_{2,1/2,K-3}$$
$$\leqq C(N)[|f|_{2,1+\eta,N-1}+|\mathcal{A}|_{\infty,1+\eta,N}|f|_{2,1+\eta,0}],$$

$$(7.31)\qquad |g_{N+1-K}|_{2,1/2,K-2}\leqq C(N)[|f|_{2,1+\eta,N-1}+|\mathcal{A}|_{\infty,1+\eta,N}|f|_{2,1+\eta,0}],$$

$$(7.32)\qquad |\mathcal{A}|_{\infty,1/2,K-2}|\bar{D}^{N+3-K}u|_{2,0,0}\leqq C(N)[|\mathcal{A}|_{\infty,1/2,K-2}|f|_{2,1+\eta,N+2-K}$$
$$+|\mathcal{A}|_{\infty,1/2,K-2}|\mathcal{A}|_{\infty,1+\eta,N+3-K}|f|_{2,1+\eta,0}]$$
$$\leqq C(N)[|f|_{2,1+\eta,N-1}+|\mathcal{A}|_{\infty,1+\eta,N}|f|_{2,1+\eta,0}],$$

$$(7.33)\qquad |\mathcal{A}|_{\infty,1/2,K-2}|f|_{2,0,N+1-K}\leqq C(N)[|f|_{2,1+\eta,N-2}+|\mathcal{A}|_{\infty,1+\eta,N-1}|f|_{2,1+\eta,0}],$$

$$(7.34)\qquad |\mathcal{A}|_{\infty,1/2,K-2}|g_{N+1-K}|_{2,0,0}$$
$$\leqq C(N)|\mathcal{A}|_{\infty,1+\eta,K-2}[|f|_{2,1+\eta,N+1-K}+|\mathcal{A}|_{\infty,1+\eta,N+2-K}|f|_{2,1+\eta,0}]$$
$$\leqq C(N)[|f|_{2,1+\eta,N-1}+|\mathcal{A}|_{\infty,1+\eta,N}|f|_{2,1+\eta,0}].$$

Combining (7.28)-(7.34), we have

$$|D_x^K\partial_t^{N+1-K}u|_{2,1/2,0}\leqq C(N)[|f|_{2,1+\eta,N}+|\mathcal{A}|_{\infty,1+\eta,N+1}|f|_{2,1+\eta,0}],$$

which shows that (7.4) is valid for any K with $0\leqq K\leqq N+1$. This completes the proof of Theorem 7.2.

§ 8. Uniform decay estimate for the operator $\partial_t^2+\partial_t-\Delta$.

In this section, we shall investigate the rate of the uniform decay of solutions of the following mixed problem:

$$(8.1)\qquad \begin{aligned} &(\partial_t^2+\partial_t-\Delta)u=0 && \text{in } \mathcal{D},\\ &u=0 && \text{on } \mathcal{D}',\\ &u(0,x)=\phi_0(x),\ (\partial_t u)(0,x)=\phi_1(x) && \text{in } \Omega. \end{aligned}$$

By $\mathcal{G}(\phi_0,\phi_1;t,x)$ we denote the solution of the mixed problem (8.1) with initial data ϕ_0 and ϕ_1. Here, initial data ϕ_0 and ϕ_1 satisfy the conditions (8.2) and (8.3) below.

(8.2) Let us define $u_j(x)$, $j\geqq 0$, by the formulae (6.2). Then all u_j satisfy the condition (6.3) (compatibility condition for (8.1)).

(8.3) There exist $\tilde{\phi}_0, \tilde{\phi}_1\in\mathcal{S}(\boldsymbol{R}^n)$ such that $\tilde{\phi}_0=\phi_0$, $\tilde{\phi}_1=\phi_1$ in Ω and for any p with $1\leqq p\leqq\infty$ and integer $N\geqq 0$, the estimates: $\|\tilde{\phi}_j\|'_{p,N}\leqq C(p,N)\|\phi_j\|_{p,N}$, hold for $j=0, 1$.

It is well known that there exists one and only one $\boldsymbol{C}^\infty$-solution $\mathcal{G}(\phi_0,\phi_1;t,x)$ of the equation (8.1) with initial data ϕ_0, ϕ_1 satisfying the conditions (8.2) and (8.3). The following is the main theorem of this section.

THEOREM 8.1. *Assume that* $n\geqq 3$. *Let* N *be a non-negative integer and* ϕ_0, $\phi_1\in C^\infty(\bar{\Omega})$ *initial data satisfying the conditions* (8.2) *and* (8.3). *Let* $\mathcal{Q}(\phi_0, \phi_1; t, x)$ *be a solution of the equation* (8.1) *with initial data* ϕ_0, ϕ_1. *Then, the following estimates hold*:

$$|\mathcal{Q}(\phi_0, \phi_1; \cdots)|_{\infty, n/2, N}$$
$$\leqq C(N, n)[\|\phi_0\|_{2, N+2[n/2]+5}+\|\phi_1\|_{2, N+2[n/2]+4}+\|\phi_0\|_{1,0}+\|\phi_1\|_{1,0}],$$
$$|\mathcal{Q}(\phi_0, \phi_1; \cdots)|_{\infty, n/4, N}\leqq C(N, n)[\|\phi_0\|_{2, N+2[n/2]+5}+\|\phi_1\|_{2, N+2[n/2]+4}].$$

In order to prove Theorem 8.1, we shall need the following two lemmas.

LEMMA 8.2. *Assume that* $n\geqq 3$. *Let* r *be a fixed number* $\geqq r_0+3$ *and* N *a non-negative integer. Assume that* ϕ_0, $\phi_1\in C^\infty(\bar{\Omega})$ *satisfy the conditions* (8.2) *and* (8.3) *and* $\operatorname{supp}\phi_j\subset\Omega_r$, $j=0, 1$. *Let* g *be any* $C^\infty([0, \infty)\times\bar{\Omega})$ *function with* $\operatorname{supp} g\subset R^1\times\{x\in R^n; r\leqq|x|\leqq r+1\}$. *Then there exists one and only one solution* $u\in C^\infty([0, \infty)\times\Omega)$ *of the following mixed problem*:

$$(\partial_t^2+\partial_t-\Delta)u=g \qquad in\ \mathcal{D},$$
$$u=0 \qquad on\ \mathcal{D}',$$
$$u(0, x)=\phi_0(x), \quad (\partial_t u)(0, x)=\phi_1(x) \qquad in\ \Omega,$$

and the following two estimates are valid for u.

(i) *If* $|g|_{2, n/2, N+2}<\infty$, *then for any* $r'>r$

$$\|\bar{D}^N u(t, \cdot)\|_{\Omega_{r'}, 2, 0}\leqq C(r, r', N)(1+t)^{-n/2}(\|\phi_0\|_{2, N+3}+\|\phi_1\|_{2, N+2}+|g|_{2, n/2, N+2}).$$

(ii) *If* $n\geqq 5$ *and* $|g|_{2, n/4, N+2}<\infty$, *then for any* $r'>r$

$$\|\bar{D}^N u(t, \cdot)\|_{\Omega_{r'}, 2, 0}\leqq C(r', r, N)(1+t)^{-n/4}(\|\phi_0\|_{2, N+3}++\|\phi_1\|_{2, N+2}+|g|_{2, n/4, N+2}).$$

LEMMA 8.3 (A. Matsumura [7]). *Let* $v(t, x)$ *be a solution of the following Cauchy problem with initial data* $\tilde{\phi}_0$, $\tilde{\phi}_1\in\mathcal{S}(R^n)$:

$$(\partial_t^2+\partial_t-\Delta)v=0 \qquad in\ [0, \infty)\times R^n,$$
$$v(0, x)=\tilde{\phi}_0(x), \quad (\partial_t v)(0, x)=\tilde{\phi}_1(x) \quad in\ R^n.$$

Then, the following two estimates hold. For any integers $j\geqq 0$ *and* $N\geqq 0$

$$\|D_x^N\partial_t^j v(t, \cdot)\|_2'\leqq C(N, j)(1+t)^{-((n/2)+(N/2)+j)}[\|\tilde{\phi}_0\|_{0,[n/2]+1+N+j}'+\|\tilde{\phi}_1\|_{2,[n/2]+N+j}'$$
$$+\|\tilde{\phi}_0\|_1'+\|\tilde{\phi}_1\|_1'],$$

$$\|D_x^N\partial_t^j v(t,\ \cdot)\|_2'\leqq C(N,\ j)(1+t)^{-((n/4)+(N/2)+j)}[\|\tilde{\phi}_0\|_{2,[n/2]+1+N+j}'+\|\tilde{\phi}_1\|_{2,[n/2]+N+j}'].$$

Lemma 8.2 follows immediately from Theorem 1.1 and well-known Duhamel's principle. So, we omit the proof of Lemma 8.2. Representing a solution $v(t, x)$ in terms of Fourier transformations, we can prove Lemma 8.3 by direct calculations. So, we also omit the detailed proof of Lemma 8.3, which is given in Matsumura [7].

Now, we shall prove Theorem 8.1, below. For any data $\tilde{\phi}_0$ and $\tilde{\phi}_1$ in (8.3) let us denote by u_1 the solution of the following Cauchy problem:

$$\begin{aligned}(8.4)\qquad &(\partial_t^2+\partial_t-\Delta)u_1=0 && \text{in } [0,\ \infty)\times \boldsymbol{R}^n,\\ &u_1(0,\ x)=\tilde{\phi}_0(x),\quad (\partial_t u_1)(0,\ x)=\tilde{\phi}_1(x) && \text{in } \boldsymbol{R}^n.\end{aligned}$$

Choosing $r\geqq r_0+3$ and $\chi(x)\in C_0^\infty(\boldsymbol{R}^n)$ so that $\chi(x)=1$ if $|x|\leqq r$ and $=0$ if $|x|\geqq r+1$, we put

$$\begin{aligned}(8.5)\qquad &u_2(t,\ x)=u(t,\ x)-(1-\chi(x))u_1(t,\ x),\\ &g(t,\ x)=2\textstyle\sum_{j=1}^n\partial_j\chi(x)\cdot\partial_j u_1(t,\ x)+\Delta\chi(x)\cdot u_1(t,\ x).\end{aligned}$$

It follows from Lemma 8.3 and (8.3) that for any integer $N\geqq 0$

$$\begin{aligned}(8.6)\qquad &|(1-\chi)u_1|_{\infty,n/2,N}\leqq C(N,\ n)[\|\phi_0\|_{2,[n/2]+N}+\|\phi_1\|_{2,[n/2]+N}+\|\phi_0\|_1+\|\phi_1\|_1],\\ &|(1-\chi)u_1|_{\infty,n/4,N}\leqq C(N,\ n)[\|\phi_0\|_{2,[n/2]+N+1}+\|\phi_1\|_{2,[n/2]+N}],\end{aligned}$$

$$\begin{aligned}&\operatorname{supp} g\subset \boldsymbol{R}^1\times\{x\in\boldsymbol{R}^n;\ r\leqq|x|\leqq r+1\},\\ (8.7)\qquad &|g|_{\infty,n/2,N}\leqq C(N,\ n)[\|\phi_0\|_{2,[n/2]+N+2}+\|\phi_1\|_{2,[n/2]+N+1}+\|\phi_0\|_1+\|\phi_1\|_1],\\ &|g|_{\infty,n/4,N}\leqq C(N,\ n)[\|\phi_0\|_{2,[n/2]+N+2}+\|\phi_1\|_{2,[n/2]+N+1}].\end{aligned}$$

Furthermore, it follows from the definition of u_2 that

$$\begin{aligned}(8.8)\qquad &(\partial_t^2+\partial_t-\Delta)u_2=g && \text{in } \mathcal{D},\\ &u_2=0 && \text{on } \mathcal{D}',\\ &u_2(0,\ x)=\chi(x)\phi_0(x),\quad (\partial_t u_2)(0,\ x)=\chi(x)\phi_1(x) && \text{in } \Omega.\end{aligned}$$

It thus follows from Lemma 8.2 and (8.7) that

$$\begin{aligned}(8.9)\qquad &\|\bar{D}^N u_2(t,\ \cdot)\|_{\Omega_{r+2},2}\\ &\leqq C(r,\ N,\ n)(1+t)^{-n/2}[\|\phi_0\|_{2,N+4+[n/2]}+\|\phi_1\|_{2,N+3+[n/2]}+\|\phi_0\|_1+\|\phi_1\|_1],\end{aligned}$$

and that if $n\geqq 5$

$$(8.10)\qquad \|\bar{D}^N u_2(t,\ \cdot)\|_{\Omega_{r+2},2}\leqq C(r,\ N,\ n)(1+t)^{-n/4}[\|\phi_0\|_{2,N+4+[n/2]}+\|\phi_1\|_{2,N+3+[n/2]}].$$

Choosing $\psi(x)\in C^\infty(\boldsymbol{R}^n)$ so that $\psi(x)=1$ if $|x|\geqq r+2$ and $=0$ if $|x|\leqq r+1$. Since $\psi g=0$, $\psi\chi\phi_j=0$, $j=0, 1$, we have

$$(8.11)\qquad \begin{aligned} &(\partial_t^2+\partial_t-\Delta)(\psi u_2)=h(t, x) && \text{in } [0, \infty)\times \boldsymbol{R}^n,\\ &(\psi u_2)(0, x)=(\partial_t(\psi u_2))(0, x)=0 && \text{in } \boldsymbol{R}^n,\end{aligned}$$

where $h(t, x)=\sum_{j=1}^n \partial_j\psi(x)\cdot\partial_j u_2(t, x)+(\Delta\psi)(x)\cdot u_2(t, x)$. Since the fact that $\operatorname{supp}\partial_t^i u_2(0, x)\subset\Omega_{r+1}$ for any $i\geqq 0$ follows immediately from (8.7), (8.8) and the fact that $\operatorname{supp}\chi\subset B_{r+1}$, we have by the fact that $\psi=0$ in B_{r+1} that

$$(8.12)\qquad (\partial_t^i h)(0, x)=0 \quad \text{for any } i\geqq 0.$$

It thus follows from well-known Duhamel's principle that

$$(8.13)\qquad \bar{D}^N(\psi u_2)=\int_0^t \bar{D}^N \mathcal{G}_0(0, h(s, \cdot); t-s, x)ds.$$

Here, by $\mathcal{G}_0(\theta_0, \theta_1; t, x)$ we have denoted the solution of the following Cauchy problem with initial data θ_0, θ_1:

$$\begin{aligned} &(\partial_t^2+\partial_t-\Delta)w=0 && \text{in } [0, \infty)\times \boldsymbol{R}^n,\\ &w(0, x)=\theta_0(x),\quad (\partial_t w)(0, x)=\theta_1(x) && \text{in } \boldsymbol{R}^n.\end{aligned}$$

In view of Lemma 8.3, we have from (8.13) and the definition of the function $h(t, x)$ that

$$(8.14)\qquad \begin{aligned} |u_2|_{\infty, n/2, N} &\leqq C(N, n)\left[\sup_{t>0}\int_0^t (1+(t-s))^{-n/2}(1+s)^{-n/2}ds(1+t)^{n/2}\right]\\ &\qquad\times[|h|_{2, n/2, N+[n/2]}+|h|_{1, n/2, 0}]\\ &\leqq C(N, r, n)[|u_2|_{\Omega_{r+1}, 2, n/2, N+[n/2]+1}],\end{aligned}$$

and that if $n\geqq 5$

$$(8.15)\qquad \begin{aligned} |u_2|_{\infty, n/4, N} &\leqq C(N, n)\left[\sup_{t>0}\int_0^t (1+(t-s))^{-n/4}(1+s)^{-n/4}ds(1+t)^{n/4}\right]|h|_{2, n/4, N+[n/2]}\\ &\leqq C(N, r, n)|u_2|_{\Omega_{r+1}, 2, n/4, N+[n/2]+1}.\end{aligned}$$

Here, we have used the fact that $|h|_{1, n/2, 0}\leqq C(r)|u_2|_{\Omega_{r+1}, 2, n/2, 1}$, which follows from the Cauchy-Schwarz inequality. Combining (8.9), (8.10), (8.14) and (8.15), we have that

$$(8.16)\qquad |u_2|_{\infty, n/2, N}\leqq C(N, n)[\|\phi_0\|_{2, N+5+2[n/2]}+\|\phi_1\|_{2, N+4+2[n/2]}+\|\phi_0\|_1+\|\phi_1\|_1],$$

and that if $n\geqq 5$

$$(8.17)\qquad |u_2|_{\infty, n/4, N}\leqq C(N, n)[\|\phi_0\|_{2, N+5+2[n/2]}+\|\phi_1\|_{2, N+4+2[n/2]}].$$

On the other hand, we have by (8.9), (8.10) and well-known Sobolev's imbedding theorem that

(8.18)
$$|(1-\psi)u_2|_{\infty,n/2,N}\leqq C|(1-\psi)u_2|_{2,n/2,N+1+[n/2]}$$
$$\leqq C(N,\ n)[\|\phi_0\|_{2,N+5+2[n/2]}+\|\phi_1\|_{2,N+4+2[n/2]}+\|\phi_0\|_1+\|\phi_1\|_1],$$

and that if $n\geqq 5$

(8.19)
$$|(1-\psi)u_2|_{\infty,n/4,N}\leqq C|(1-\psi)u_2|_{2,n/4,N+1+[n/2]}$$
$$\leqq C(N,\ n)[\|\phi_0\|_{2,N+5+2[n/2]}+\|\phi_1\|_{2,N+4+2[n/2]}].$$

Combining (8.5), (8.6), (8.16), (8.17), (8.18) and (8.19), we get the theorem, which completes the proof of Theorem 8.1.

§ 9. Uniform decay estimate for some hyperbolic equation.

In this section, we shall investigate the rate of the uniform decay of solutions to the following mixed problem:

(9.1)
$$\begin{aligned}&(1+a^0(t,\ x))\partial_t^2u+\textstyle\sum_{j=1}^n a^j(t,\ x)\partial_j\partial_t u-\sum_{i,j=1}^n(\delta_{ij}+a^{ij}(t,\ x))\partial_i\partial_j u\\ &\quad+(1+b^0(t,\ x))\partial_t u+\textstyle\sum_{j=1}^n b^j(t,\ x)\partial_j u+c(t,\ x)u=f \quad \text{in } \mathcal{D},\\ &u=0 \quad \text{on } \mathcal{D}',\\ &u(0,\ x)=(\partial_t u)(0,\ x)=0 \quad \text{in } \Omega.\end{aligned}$$

The following is the main result of this section.

THEOREM 9.1. *Assume that* $n\geqq 3$. *Let* a^j, a^{ij}, b^j, c *be the same as in* (9.1) *and put* $\mathcal{A}=\mathcal{A}(t,\ x)=(\mathcal{A}'(t,\ x),\ c(t,\ x))$, *where* $\mathcal{A}'=\mathcal{A}'(t,\ x)=(a^j(t,\ x),\ j=0,\ \cdots,\ n;\ a^{ij}(t,\ x),\ i,\ j=1,\ \cdots,\ 0;\ b^j(t,\ x),\ j=0,\ \cdots,\ n)$. *Let* L *be a non-negative integer and* $K=L+2[n/2]+5$. *Let* $p(n)$, $q(n)$ *and* $\sigma(n)$ *be the same as in Notations. Assume that* $\mathcal{A}$ *satisfies Assumption* 7.1 *and the condition* (7.3) *with* $\eta=p(n)-1$ *that if* $n\geqq 5$

(9.2)
$$|\mathcal{A}|_{\infty,p(n),0}\leqq 1,\quad |\mathcal{A}|_{\infty,p(n),K+1}<\infty,$$

and that if $3\leqq n\leqq 4$

(9.2)′
$$|\mathcal{A}'|_{\infty,p(n),0}+|c|_{\infty,q(n),0}+|\mathcal{A}'|_{2,1/2,0}+|c|_{2,1,0}\leqq 2,$$
$$|\mathcal{A}'|_{\infty,p(n),K}+|c|_{\infty,q(n),K}<\infty.$$

If u *is a solution of the equation* (9.1) *with data* $f\in \boldsymbol{E}^K$ *satisfying the condition*:
$$|f|_{2,q(n),K}+\sigma(n)|f|_{1,1,0}<\infty,$$
then for any N *with* $0\leqq N\leqq L$
$$|u|_{\infty,p(n),N}\leqq C(N,\ n)[|f|_{2,q(n),N+2[n/2]+5}+\sigma(n)|f|_{1,1,0}$$
$$+(|\mathcal{A}|_{\infty,p(n),N+2[n/2]+6}+\sigma(n)|c|_{\infty,q(n),N+2[n/2]+4})|f|_{2,q(\ ),0}].$$

To show Theorem 9.1, we need the following two preliminary lemmas.

LEMMA 9.2. *Assume that* $n\geqq 3$. *Let* L *be a non-negative integer and* $K=L+2[n/2]+4$. *Let* $p(n)$, $q(n)$ *and* $\sigma(n)$ *be the same as in Notations. Let* u *be a solution of the equation*:

$$(9.3)\qquad \begin{aligned}&(\partial_t^2+\partial_t-\Delta)u=f && \text{in } \mathcal{D},\\ &u=0 && \text{on } \mathcal{D}',\\ &u(0, x)=(\partial_t u)(0, x)=0 && \text{in } \Omega,\end{aligned}$$

where $f\in \boldsymbol{E}^K$ *satisfies the condition*:

$$(9.4)\qquad |f|_{2,q(n),K}+\sigma(n)|f|_{1,1,0}<\infty,$$

$$(9.5)\qquad f=0 \quad \text{in } [0, \infty)\times\Omega_R$$

for some large R *with* $\Omega_R\supset \boldsymbol{R}^n-\Omega$. *Then for any integer* N *with* $0\leqq N\leqq L$

$$|u|_{\infty,p(n),N}\leqq C(R, N, n)[|f|_{2,q(n),N+2[n/2]+4}+\sigma(n)|f|_{1,1,0}].$$

LEMMA 9.3. *Let* R *be a large number with* $\Omega_R\supset \boldsymbol{R}^n-\Omega$, N *a positive integer,* $k\geqq 0$ *and* $f\in\mathcal{E}^{2,N}(\Omega_R)$ *with* $\partial_t^j f(0, x)=0$, $j=0, 1, \cdots, N-1$. *Then there exists a solution* $u\in\mathcal{E}^{2,N+1}(\Omega_R)$ *with* $\partial_t^j u(0, x)=0$, $j=0, 1, \cdots, N+1$ *of the equations*: $(\partial_t^2+\partial_t-\Delta)u=f$ *in* $[0, \infty)\times\Omega_R$ *and* $u=0$ *on* $[0, \infty)\times\partial\Omega_R$, *where* $\partial\Omega_R$ *is the boundary of* Ω_R.

Moreover, u *satisfies the estimate*:

$$|u|_{\Omega_R,2,k,N+1}\leqq C(R, k, N)|f|_{\Omega_R,2,k,N},$$

if $|f|_{\Omega_R,2,k,N}<\infty$.

PROOF of Lemmas 9.2 and 9.3. Lemma 9.3 follows immediately from Theorem 4.10 of [18], so we may show only Lemma 9.2. It follows from the fact that $f\in\boldsymbol{E}^K$ that $u\in\tilde{\boldsymbol{E}}^{K+1}$. Using this fact and (9.5), we have by Duhamel's principle

$$(9.6)\qquad \bar{D}^N u(t, x)=\int_0^t \bar{D}^N \mathcal{G}(0, f(s, \cdot); t-s, x)ds$$

where $\mathcal{G}$ is the same as in Theorem 8.1. When $3\leqq n\leqq 4$, we have by (9.6) and Theorem 8.1 that

$$\begin{aligned}\|\bar{D}^N u(t, \cdot)\|_\infty &\leqq C(N, n)\Big[\int_0^{t/2}(1+(t-s))^{-n/2}(1+s)^{-1}ds\{|f|_{2,1,N+2[n/2]+4}+|f|_{1,1,0}\}\\ &\qquad +\int_{t/2}^t(1+(t-s))^{-n/4}(1+s)^{-q(n)}ds\cdot|f|_{2,q(n),N+2[n/2]+4}\Big]\\ &\leqq C(N, n)(1+t)^{-p(n)}[|f|_{2,q(n),N+2[n/2]+4}+\sigma(n)|f|_{1,1,0}].\end{aligned}$$

When $n \geqq 5$, using the facts that $(1+|t|)^{-n/4} \in L^1(\boldsymbol{R}^1)$ and that $\frac{n}{4}=p(n)=q(n)$, we have from (9.6) and Theorem 8.1 that

$$\|\bar{D}^N u(t, \cdot)\|_\infty \leqq C(N, n)(1+t)^{-p(n)} |f|_{2, q(n), N+2[n/2]+4} . \tag{9.8}$$

Combining (9.7) and (9.8), we have

$$|u|_{\infty, p(n), N} \leqq C(N, n)[|f|_{2, q(n), N+2[n/2]+4} + \sigma(n)|f|_{1,1,0}],$$

which completes the proof.

To prove Theorem 9.1, the following lemma is essential, which we can prove by Lemmas 9.2 and 9.3.

LEMMA 9.4. *Assume that* $n \geqq 3$. *Let* L *be a non negative integer and* $K=L+2[n/2]+4$. *Let* $p(n)$, $q(n)$ *and* $\sigma(n)$ *be the same as in Notations. Let* u *be a solution of the equation* (9.3) *for* $f \in \boldsymbol{E}^K$ *which satisfies the condition* (9.4). *Then* u *satisfies the estimate*:

$$|u|_{\infty, p(n), N} \leqq C(N, n)[|f|_{2, q(n), N+2[n/2]+4} + \sigma(n)|f|_{1,1,0}]$$

for any integer N *with* $0 \leqq N \leqq L$.

PROOF. Let R be a large number with $\Omega_R \supset \boldsymbol{R}^n - \Omega$. Choose $C_0^\infty(\boldsymbol{R}^n)$-functions $\phi(x)$ and $\psi(x)$ so that $\phi(x)=1$ if $x \in \Omega_R$ and $=0$ outside Ω_{R+1} and $\psi(x)=1$ if $x \in \Omega_{R+2}$ and $=0$ outside Ω_{R+3}. Let v be a solution of the equations:

$$\begin{aligned} &(\partial_t^2+\partial_t-\Delta)v=\phi(x)f(t, x) && \text{in } [0, \infty)\times\Omega_{R+3}, \\ &v=0 && \text{on } [0, \infty)\times\partial\Omega_{R+3}, \\ &v(0, x)=(\partial_t v)(0, x)=0 && \text{in } \Omega_{R+3}, \end{aligned}$$

where $\partial\Omega_{R+3}$ is the boundary of Ω_{R+3}. It follows from Lemma 9.3 that

$$|v|_{\Omega_{R+3}, 2, q(n), N+1} \leqq C(R, n, K)|f|_{2, q(n), N} \tag{9.9}$$

for any integer N with $0 \leqq N \leqq K$. Put $u=\psi v+w$. Then, w satisfies the equations:

$$\begin{aligned} &(\partial_t^2+\partial_t-\Delta)w=g && \text{in } \mathcal{D}, \\ &w=0 && \text{on } \mathcal{D}', \\ &w(0, x)=(\partial_t w)(0, x)=0 && \text{in } \Omega, \end{aligned}$$

where $g=(1-\phi)f+2\sum_{j=0}^n \partial_j\psi \cdot \partial_j v+\Delta\psi \cdot v$. Noting that supp $g \subset \boldsymbol{R}^1 \times (\boldsymbol{R}^n - \Omega_R)$, we have by Lemma 9.2 that

$$|u|_{\infty, p(n), N} \leqq C(N, n)[|g|_{2, q(n), N+2[n/2]+4} + \sigma(n)|g|_{1,1,0}]. \tag{9.10}$$

Since supp $\psi\subset\Omega_{R+3}$, we have by the Cauchy-Schwarz inequality, (9.9) and the fact that $q(n)>1$ that

$$
(9.11)\qquad
\begin{aligned}
&|g|_{2,q(n),N+2[n/2]+4}\leqq C(N,\ n)|f|_{2,q(n),N+2[n/2]+4}\,,\\
&|g|_{1,1,0}\leqq C(n)[|f|_{1,1,0}+|f|_{2,q(n),0}]\,.
\end{aligned}
$$

Combining (9.10) and (9.11), we have the desired inequality, which completes the proof.

Now, we prove Theorem 9.1. Put $g=a^0u_t^2u+\sum_{j=1}^n a^j\partial_j\partial_t u-\sum_{i,j=1}^n a^{ij}\partial_i\partial_j u+\sum_{j=0}^n b^j\partial_j u+cu$. Then, we rewrite the equation (9.1) in terms of g as follows:

$$
\begin{aligned}
&(\partial_t^2+\partial_t-\Delta)u=f-g && \text{in } \mathcal{D},\\
&u=0 && \text{on } \mathcal{D}',\\
&u(0,\ x)=(\partial_t u)(0,\ x)=0 && \text{in } \Omega.
\end{aligned}
$$

Since it follows from Theorem 7.2 and the fact that $f\in\boldsymbol{E}^K$ that $u\in\tilde{\boldsymbol{E}}^{K+1}$, we have that $f-g\in\boldsymbol{E}^{K-1}$. Applying Lemma 9.4, we have

$$
(9.12)\qquad
\begin{aligned}
|u|_{\infty,p(n),N}\leqq C(N,\ n)[|f|_{2,q(n),N+2[n/2]+4}+|g|_{2,q(n),N+2[n/2]+4}\\
+\sigma(n)[|f|_{1,1,0}+|g|_{1,1,0}]]
\end{aligned}
$$

for any integer N with $0\leqq N\leqq L$. Put $N'=N+2[n/2]+4$. Note that $p(n)+\dfrac{1}{2}\geqq q(n)>1$ and that $p(n)=q(n)=\dfrac{n}{4}$ if $n\geqq5$. Using Theorem 7.2 with $\eta=p(n)-1$, (9.2), (9.2)$'$ and Theorem Ap. 2 in Appendix I, we have

$$
(9.13)\qquad
\begin{aligned}
|g|_{2,q(n),N'}&\leqq C(n,\ n)[|\mathcal{A}'|_{\infty,p(n),N'}|\bar{D}^1D^1u|_{2,1/2,0}+|c|_{\infty,q(n),N'}|u|_{2,0,0}\\
&\qquad+|f|_{2,p(n),N'+1}+|\mathcal{A}|_{\infty,p(n),N'+2}|f|_{2,p(n),0}]\\
&\leqq C(N,\ n)[|f|_{2,p(n),N'+1}+(|\mathcal{A}|_{\infty,p(n),N'+2}+\sigma(n)|c|_{\infty,q(n),N'})|f|_{2,p(n),0}]\,.
\end{aligned}
$$

When $3\leqq n\leqq4$, it follows from Theorem 7.2 with $\eta=p(n)-1$, (9.2)$'$ and the Cauchy-Schwarz inequality that

$$
(9.14)\qquad |g|_{1,1,0}\leqq C(|\mathcal{A}'|_{2,1/2,0}|\bar{D}^1D^1u|_{2,1/2,0}+|c|_{2,1,0}|u|_{2,0,0})\leqq C|f|_{2,p(n),1}\,.
$$

Combining (9.12), (9.13) and (9.14), we get the desired estimate, which completes the proof of Theorem 9.1.

PART 3

Proof of main theorem

Throughout part 3, we assume that the assumptions stated in Introduction are fulfilled and use notations defined in Introduction and Notations.

§ 10. Compatibility condition.

For $u \in C^\infty(\tilde{\mathscr{D}})$ with $u=0$ on $\mathscr{D}'$, we define f, ϕ_0, ϕ_1 by

$$f(t, x) \equiv (\partial_t^2 + \partial_t - \Delta)u + F(t, x, \Lambda u),$$
$$\phi_0(x) \equiv u(0, x), \quad \phi_1(x) \equiv \partial_t u(0, x). \tag{10.1}$$

Furthermore, we put

$$u_j(x) \equiv (\partial_t^j u)(0, x), \quad j \geqq 2. \tag{10.2}$$

In this section, under some smallness assumption imposed on ϕ_0, ϕ_1 and f, we shall represent u_j, $j \geqq 2$, in terms of ϕ_0, ϕ_1 and f. And then, by using such representations, we estimate u_j, $j \geqq 2$, by ϕ_0, ϕ_1, f.

$$u_2 + \phi_1 - \Delta\phi_0 + F(0, x, \bar{D}_x^2\phi_0, \bar{D}_x^1\phi_1, u_2) = f(0, x). \tag{10.3}$$

Putting $\lambda = (\lambda', \lambda'', \nu)$ where $\lambda' = (\mu, \lambda_1, \cdots, \lambda_n, \lambda_{ij}, i, j=1, \cdots, n)$, $\lambda'' = (\lambda_0, \lambda_{01}, \cdots, \lambda_{0n})$ and $\nu = \lambda_{00}$, we consider the following non-linear equation:

$$\Psi(U) = \nu + \lambda_0 - \textstyle\sum_{i=1}^n \lambda_{ii} + F(0, x, \lambda', \lambda'', \nu) - g = 0. \tag{10.4}$$

Here, we have put $U = (\lambda', \lambda'', \nu, x, g)$. Since $F(0, x, 0) = 0$, $(0, 0, 0, x, 0)$ is a solution of the equation (10.4). Since $(d_\lambda F)(0, x, 0) = 0$ and $F \in \mathscr{B}^\infty([0, \infty) \times \bar{\Omega} \times \{|\lambda| \leqq 1\})$, there exists a positive small constant c_F depending only on F such that $|\partial\Psi/\partial\nu| \geqq 1/2$ if $|\lambda| \leqq c_F$ and $x \times \bar{\Omega}$. The following lemma thus follows from the implicit function theorem.

LEMMA 10.1. *There exist a sufficiently small positive number d_2 and a $C^\infty(\bar{\Omega} \times \{(\lambda', \lambda'', g); |\lambda'| + |\lambda''| + |g| \leqq d_2\})$ function $\nu(x, \lambda', \lambda'', g)$ such that ν is uniquely determined, $\nu(x, 0, 0, 0) = 0$ and*

$$\Psi(\lambda', \lambda'', \nu(x, \lambda', \lambda'', g), x, g) = 0, \quad \text{if } x \in \bar{\Omega} \text{ and } |\lambda'| + |\lambda''| + |g| \leqq d_2.$$

From Lemma 10.1, we obtain

$$u_2(x) = \nu(x, \bar{D}_x^2\phi_0(x), \bar{D}_x^1\phi_1(x), f(0, x)) \tag{10.5}$$

if $\|\phi_0\|_{\infty,2}+\|\phi_1\|_{\infty,1}+\|f(0,\ \cdot)\|_\infty\leqq d_2$.

Differentiating the first equation of (10.1) $j-2$ times with respect to t and restricting $t=0$, we get the linear equation with respect to u_j, $j\geqq 3$. So, we can easily prove the following by induction on $j\geqq 3$.

LEMMA 10.2. *There exists a small positive constant* $d_3\leqq d_2$ *such that the equations*:

$$u_j(x)=\nu_j(x,\ \bar{D}_x^j\phi_0(x),\ \bar{D}_x^{j-1}\phi_1(x),\ (\bar{D}^{j-2}f)(0,\ x)),\quad j\geqq 2,$$

hold for some $\mathcal{B}^\infty$ *functions* ν_j *with* $\nu_j(x,\ 0)=0$ *if* $\|\phi_0\|_{\infty,2}+\|\phi_1\|_{\infty,1}+\|f(0,\ \cdot)\|_\infty\leqq d_3$.

Applying Theorem Ap. 4 in Appendix II to the representation of $u_j(x)$ given in Lemma 10.2, we can easily show the following.

LEMMA 10.3. *Let* ϕ_0, ϕ_1 *and* f *be the same as in* (10.1) u_j, $j\geqq 2$, *be the same as in* (10.2) *and* d_3 *the same as in Lemma* 10.2. *If* $\|\phi_0\|_{\infty,2}+\|\phi_1\|_{\infty,1}+\|f(0,\ \cdot)\|_\infty\leqq d_3$, *then the inequalities*:

$$\|u_j\|_{p,N}\leqq C(p,\ j,\ N)[\|\phi_0\|_{p,N+j}+\|\phi_1\|_{p,N+j-1}+\|(\bar{D}^{j+N-2}f)(0,\ \cdot)\|_p]$$

hold for any integers $N\geqq 0$ *and* $j\geqq 2$ *and* p *with* $1\leqq p\leqq\infty$.

In view of Lemmas 10.2 and 10.3, we introduce the compatibility condition for the problem (P) as follows.

DEFINITION 10.4. Let d_3 be the same as in Lemma 10.2 and N an integer $\geqq 2$. We shall say that $\phi_0(x)$, $\phi_1(x)$ and $f(t,\ x)$ satisfy the N-th order compatibility condition if ϕ_0, ϕ_1 and f satisfy the following two conditions:

(i) $\|\phi_0\|_{\infty,2}+\|\phi_1\|_{\infty,1}+\|f(0,\ \cdot)\|_\infty\leqq d_3$,

(ii) The functions ϕ_0, ϕ_1 and $u_j(x)\equiv\nu_j(x,\ \bar{D}_x^j\phi_0(x),\ \bar{D}_x^{j-1}\phi_1(x),\ (\bar{D}^{j-2}f)(0,\ x))$, $j=2$, $\cdots$, N, vanish on $\partial\Omega$, where ν_j, $j\geqq 2$, are the same as in Lemma 10.2.

§11. Smoothing operator.

In this section, we shall define a smoothing operator which will be needed to define our iteration scheme. Choosing $\phi(x)\in\mathcal{S}(\boldsymbol{R}^n)$, $\phi(t)\in\mathcal{S}(\boldsymbol{R}^1)$ so that

$$\int_{R^n}\phi(x)dx=1,\quad \int_{R^n}x^\alpha\phi(x)dx=0,\quad |\alpha|\geqq 1,$$

(11.1)

$$\phi(t)=0\quad\text{if } t<0,\quad \int_{R^1}\phi(t)dt=1,\quad \int_{R^1}t^j\phi(t)dt=0,\quad j\geqq 0.$$

If ϕ is the inverse Fourier transformation of a $C_0^\infty(\boldsymbol{R}^n)$ function which is 1 near the origin, ϕ satisfies all the conditions (11.1) about ϕ. The existence of a function such as ψ follows immediately from Boas' theorem (cf. [19]).

Since the boundary of Ω is compact, via local map, by using Seeley's extension theorem ([17]), we can show the existence of a function u' defined on $[0, \infty)\times \boldsymbol{R}^n$ for any function u defined on $[0, \infty)\times\bar{\Omega}$ satisfying the following properties:

$$
(11.2)\qquad \begin{aligned} &u'(t, x)=u(t, x) \quad \text{on } [0, \infty)\times\Omega, \\ &|u'|'_{p,k,N}\leqq C(p, k, N)|u|_{p,k,N}, \end{aligned}
$$

for any p with $1\leqq p\leqq\infty$, non-negative real number k and non-negative integer N. Furthermore, if $\partial_t^j u(0, x)=0$, $j=0, 1, \cdots, N$, we can construct an extension u' of u such that u' satisfies (11.2) and the conditions: $\partial_t^j u'(0, x)=0$, $j=0, 1, \cdots, N$ (N is a non-negative integer).

Using such an extension u' of u, we define the smoothing operator $S(\theta)u$, $\theta\geqq 1$, by

$$
(11.3)\qquad S(\theta)u\equiv\int_0^\infty\int_{R^n}\theta^{n+1}\phi(\theta(x-y))\psi(\theta(t-s))u'(s, y)dsdy\,.
$$

Of course, $S(\theta)$ depends on the manner of extensions of functions, but the manner of extensions is independent of functions. So, when we define $S(\theta)$, we fix the manner of the extension of functions. The following facts are valid.

LEMMA 11.1. *Let $\theta\geqq 1$, $k\geqq 0$, p be a real number with $1\leqq p\leqq\infty$ and N, M non-negative integers. Let $S(\theta)$ be a smoothing operator defined by* (11.3). *Then, following three assertions are valid.*

(i) *For any u with $|u|_{p,k,N}\leqq\infty$*

$$
|S(\theta)u|_{p,k,N}\leqq C(p, k, N)|u|_{p,k,N}, \quad (\partial_t^i S(\theta)u)(0, x)=0 \quad \textit{for any } i\geqq 0.
$$

(ii) *For any $u\in \boldsymbol{C}^N([0, \infty)\times\bar{\Omega})$ with $|u|_{p,k,N}<\infty$,*

$$
|(1-S(\theta))u|_{p,k,0}\leqq C(p, k, N)\theta^{-N}|u|_{p,k,N}\,.
$$

(iii) *If $M>N\geqq 0$, for any $u\in\boldsymbol{C}^N([0, \infty)\times\bar{\Omega})$ satisfying the conditions: $|u|_{p,k,N}<\infty$ and $(\partial_t^j u)(0, x)=0$, $j=0, 1, \cdots, N-1$*

$$
|S(\theta)u|_{p,k,M}\leqq C(p, k, N, M)\theta^{M-N}|u|_{p,k,N}\,.
$$

PROOF. (i) The assertion (i) follows immediately from differentiation under the integral sign and the fact that $\psi(t)=0$ when $t<0$.
(ii) When $(t, x)\in[0, \infty)\times\Omega$, from Taylor series expansion, (11.1) and (11.2), we obtain

$$(1-S(\theta))u=\left[\int_{-\infty}^{\theta t/2}+\int_{\theta t/2}^{\theta t}\right]ds\int_{R^n}\int_0^1(1-\rho)^{N-1}((N-1)!)^{-1}\left[\sum_{|\alpha|+i=N}(-s)^i(-y)^\alpha\theta^{-N}\right.$$
$$\left.\times\frac{N!}{i!\,\alpha!}(\partial_t^i\partial_x^\alpha u')(t-\rho s\theta^{-1},\ x-\rho y\theta^{-1})\psi(s)\phi(y)\right]d\rho dy\equiv I_1+I_2\,.$$

Since the inequality:

$$(1+t)/(1+t-\rho s\theta^{-1})\leqq 2 \tag{11.4}$$

holds if $s\leqq\theta t/2$ and $0\leqq\rho\leqq 1$, applying (11.4) to I_1, we have

$$|I_1|_{p,k,0}\leqq C(p,\ k,\ L)\theta^{-N}|u|'_{p,k,N}\,. \tag{11.5}$$

Next, since $\theta\geqq 1$ and $\psi(s)\in\mathcal{S}(\boldsymbol{R}^1)$, we have that

$$|\partial_s^i\psi(s)|\leqq C(k,\ N,\ i)(1+t)^{-k}(1+|s|)^{-(N+2)} \tag{11.6}$$

if $\theta t/2\leqq s\leqq\theta t$. So, applying (11.6) with $i=0$ to I_2,

$$|I_2|_{p,k,0}\leqq C(p,\ k,\ N)\theta^{-N}|u'|'_{p,0,N}\,. \tag{11.7}$$

Combining (11.2), (11.5) and (11.7), we obtain the assertion (ii).

(iii) By differentiation under the integral sign and integration by parts, we obtain

$$\partial_t^j\partial_x^\alpha S(\theta)u=\theta^{M-N}(-1)^N\int_{-\infty}^{\theta t}\int_{R^n}(\partial_t^{j-l}\psi)(s)(\partial_x^{\alpha-\beta}\phi)(y) \tag{11.8}$$
$$\times(\partial_t^l\partial_x^\beta u')(t-s\theta^{-1},\ x-y\theta^{-1})dsdy$$

for any j and multi-index α with $j+|\alpha|=M$, where l and β are some number and multi-index satisfying the conditions: $l+|\beta|=N$, $0\leqq\beta\leqq\alpha$ and $0\leqq l\leqq j$. Applying (11.4) with $\rho=1$ and (11.6) with $i=j-l$ to (11.8) in the manner similar to the proof of (ii), we obtain the assertion (iii) from (11.2), which completes the proof.

§ 12. Construction of an iteration scheme.

Let $\tilde{m}$ be a positive large integer given in MAIN THEOREM, data ϕ_0, ϕ_1 and f for (P) satisfy $\tilde{m}$-th order compatibility condition. Let $u_j(x)$, $j\geqq 2$, be functions defined in Definition 10.4 for ϕ_0, ϕ_1 and f. Choosing $\rho(t)\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^1)$ so that $\rho(t)=1$ if $|t|\leqq 1/2$ and $=0$ if $|t|\geqq 1$, we put

$$v(t,\ x)\equiv\left(\textstyle\sum_{j=0}^{\tilde{m}}u_j(x)t^j/j!\right)\rho(t)\,,\quad\text{where } u_0=\phi_0,\ u_1=\phi_1. \tag{12.1}$$

From Lemma 10.2, Definition 10.4 and (10.2) we obtain

$$\partial_t^j[f-((\partial_t^2+\partial_t-\Delta)v+F(t,\ x,\ \Lambda v))]|_{t=0}=0\,,\quad\text{for } j=0,\ 1,\ \cdots,\ \tilde{m}-2. \tag{12.2}$$

If u is a solution of (P), putting

$$u(t, x)=v(t, x)+w(t, x), \tag{12.3}$$

we have that $w(t, x)$ satisfies the following equation:

$$\begin{aligned} &\mathcal{L}w+G(t, x, \Lambda w)=g && \text{in } \mathcal{D}, \\ &w=0 && \text{on } \mathcal{D}', \\ &w(0, x)=(\partial_t w)(0, x)=0 && \text{in } \Omega, \end{aligned} \tag{12.4}$$

where

$$\begin{aligned} &G(t, x, \Lambda w)=\int_0^1(1-r)(d_\lambda^2 F)(t, x, \Lambda v+r\Lambda w)(\Lambda w, \Lambda w)dr, \\ &\mathcal{L}w=(\partial_t^2+\partial_t-\Delta)w+(d_\lambda F)(t, x, \Lambda v)\Lambda w, \\ &g=f-((\partial_t^2+\partial_t-\Delta)v+F(t, x, \Lambda v)). \end{aligned} \tag{12.5}$$

In particular, it follows from (12.2) that

$$g\in \boldsymbol{E}^{\tilde{m}-1} \tag{12.6}$$

if ϕ_0, ϕ_1 and f satisfy the conditions stated in MAIN THEOREM. Therefore, we may solve the equation (12.4) under the assumption that g satisfies the condition (12.6), in order to show the existence of solutions of (P).

To end this section, we give an iteration scheme to solve the equation (12.4), following Klainerman [5]. First, by w_0 we denote the solution of the following equation:

$$\begin{aligned} &\mathcal{L}w_0=g && \text{in } \mathcal{D}, \\ &w_0=0 && \text{on } \mathcal{D}', \\ &w_0(0, x)=(\partial_t w_0)(0, x)=0 && \text{in } \Omega. \end{aligned} \tag{12.7}$$

We define w_p, $p\geqq 1$, successively by

$$w_p=w_{p-1}+\dot{w}_{p-1}=\textstyle\sum_{j=0}^{p-1}\dot{w}_j+w_0. \tag{12.8}$$

We must define $\dot{w}_j$, $j\geqq 0$. For this, first of all, we introduce some notations. Let $S(\cdot)$ be a smoothing operator defined in § 11, and θ a large positive fixed number >1. Put

$$S_p w\equiv S(\theta_p)w, \quad \theta_p\equiv\theta^p. \tag{12.9}$$

We define linear operators L_p, $p\geqq 0$, by

$$L_p w=\mathcal{L}w+(d_\lambda G)(t, x, S_p\Lambda w_p)\Lambda w, \tag{12.10}$$

and error terms e'_p, e''_p and e_p by

$$
\begin{aligned}
& e'_p \equiv (d_\lambda G)(t,\ x,\ \Lambda w_p)\Lambda \dot{w}_p - (d_\lambda G)(t,\ x,\ S_p \Lambda w_p)\Lambda \dot{w}_p\,, \\
(12.11) \qquad & e''_p \equiv G(t,\ x,\ \Lambda w_{p+1}) - G(t,\ x,\ \Lambda w_p) - (d_\lambda G)(t,\ x,\ \Lambda w_p)\Lambda \dot{w}_p\,, \\
& e_p \equiv e'_p + e''_p\,.
\end{aligned}
$$

We define the summation E_p, $p \geqq 0$, of error terms by

$$
(12.12) \qquad E_p \equiv \sum_{j=0}^{p-1} e_j\,, \quad p \geqq 1\,, \quad E_0 = 0\,.
$$

Finally, we put

$$
\begin{aligned}
& g_0 \equiv -S_0[G(t,\ x,\ \Lambda w_0)]\,, \\
(12.13) \qquad & g_p \equiv -(S_p - S_{p-1})E_{p-1} - S_p e_{p-1} - (S_p - S_{p-1})G(t,\ x,\ \Lambda w_0)\,, \quad p \geqq 1.
\end{aligned}
$$

Now, let us define $\dot{w}_p$, $p \geqq 0$, by the solution of the following linear equation:

$$
\begin{aligned}
& L_p \dot{w}_p = g_p & & \text{in } \mathcal{D}, \\
(12.14) \qquad & \dot{w}_p = 0 & & \text{on } \mathcal{D}', \\
& \dot{w}_p(0,\ x) = (\partial_t \dot{w}_p)(0,\ x) = 0 & & \text{in } \Omega.
\end{aligned}
$$

In particular, we have from (12.10)-(12.14) that

$$
(12.15) \qquad \mathcal{L} w_{p+1} + G(t,\ x,\ \Lambda w_{p+1}) = g + (1 - S_p)G(t,\ x,\ \Lambda w_0) + (1 - S_p)E_p + e_p\,.
$$

We shall show the following in § 14 below.

LEMMA 12.1. *Let m be an integer $\geqq 2$. Put $\beta = \max[2[n/2]+7,\ m-1]$, $\tilde{L} = 2\beta + 1$ and $\tilde{m} = \tilde{L} + 2[n/2] + 8$. Assume that the assumptions 1°-4° stated in Introduction hold. Then there exist a sufficiently small positive constant δ_0 and a large positive constant $d(\tilde{m})$ depending on $\tilde{m}$ having the following properties: for any δ with $0 \leqq \delta \leqq \delta_0$, if ϕ_0, ϕ_1 and f are data for (P) satisfying the $\tilde{m}$-th order compatibility condition and the condition:*

$$
\begin{aligned}
& \|\phi_0\|_{2,\,2\tilde{m}+3+[n/2]} + \|\phi_1\|_{2,\,2\tilde{m}+2+[n/2]} + |f|_{2,\,q(n),\,2\tilde{m}+1+[n/2]} \\
& \qquad + \sigma(n)[\|\phi_0\|_{1,\,\tilde{m}+2} + \|\phi_1\|_{1,\,\tilde{m}+1} + |f|_{1,\,1,\,\tilde{m}}] \leqq \delta / d(\tilde{m})\,,
\end{aligned}
$$

then there exists a solution $\dot{w}_j \in \tilde{\boldsymbol{E}}^{\tilde{m}}$ of the equation (12.14) *such that*

$$
\text{(i)} \qquad |\Lambda \dot{w}_j|_{2,\,0,\,L} + |\tilde{\Lambda} \dot{w}_j|_{2,\,1/2,\,L} \leqq \delta \theta_j^{-\beta+L} \quad \text{if } 0 \leqq L \leqq \tilde{L},
$$

$$
\text{(ii)} \qquad |\Lambda \dot{w}_j|_{\infty,\,p(n),\,L} \leqq \delta \theta_j^{-\beta+L} \quad \text{if } 0 \leqq L \leqq \tilde{L}.
$$

Here $d(\tilde{m})$ and δ_0 will be defined in § 14.

§ 13. Some lemmas to estimate non-linear term.

Throughout this section, F, G and v are the same as in § 12. In this section, we shall give some lemmas which are needed in order to estimate $G(t, x, \Lambda w)$, the error terms e'_p, e''_p and the coefficients of the operator L_p. First, we shall give a lemma needed to estimate the coefficients of the operator L_p. For this purpose, we define the coefficients vectors $\mathcal{A}'(t, x, \Lambda U)$ and $\mathcal{A}(t, x, \Lambda U)$ by

$$\begin{aligned}\mathcal{A}'(t, x, \Lambda U) &\equiv (a^j(t, x, \Lambda U),\ j=0, 1, \cdots, n; \\ &\quad a^{ij}(t, x, \Lambda U),\ i, j=1, \cdots, n;\ b^j(t, x, \Lambda U),\ j=0, \cdots, n), \\ \mathcal{A}(t, x, \Lambda U) &\equiv (\mathcal{A}'(t, x, \Lambda U), c(t, x, \Lambda U)),\end{aligned} \tag{13.1}$$

where a^j, a^{ij}, b^j, c are defined by the following formulae:

$$\begin{aligned}(d_\lambda G)(t, x, \Lambda U)\Lambda W &= \textstyle\sum_{j=0}^n a^j(t, x, \Lambda U)\partial_j\partial_t W - \sum_{i,j=1}^n a^{ij}(t, x, \Lambda U)\partial_i\partial_j W \\ &\quad + \textstyle\sum_{j=0}^n b^j(t, x, \Lambda U)\partial_j W + c(t, x, \Lambda U)W.\end{aligned} \tag{13.2}$$

Of course, without loss of generality, we may assume that $a^{ij}=a^{ji}$. We have

LEMMA 13.1. *Assume that the assumption* 1°-4° *in Introduction are fulfilled, that all semi-norms appearing below are finite and that* $|\Lambda v|_{\infty,0,0}+|\Lambda U|_{\infty,p(n),0}\leqq 1$. *Then, the following five assertions are valid.*

(i) $|\mathcal{A}(\cdots, \Lambda U)|_{\infty,p(n),L}\leqq C(L)[|\Lambda U|_{\infty,p(n),L}+|\Lambda v|_{\infty,0,L}]$.

(ii) *If* $3\leqq n\leqq 4$, $|c(\cdots, \Lambda U)|_{\infty,q(n),L}\leqq C(L)[|\Lambda U|_{\infty,p(n),L}+|\Lambda v|_{\infty,0,L}]$.

(iii) *If* $3\leqq n\leqq 4$, $|\mathcal{A}'(\cdots, \Lambda U)|_{2,1/2,0}+|c(\cdots, \Lambda U)|_{2,1,0}\leqq C[|\Lambda U|_{2,0,0}+|\tilde{\Lambda} U|_{2,1/2,0}]$.

(iv) $|(d_\lambda F)(\cdots, \Lambda v)|_{\infty,q(n),L}\leqq C(L)|\Lambda v|_{\infty,0,L}$.

(v) $|(d_\lambda F)(\cdots, \Lambda v)|_{2,1,0}\leqq C|\Lambda v|_{2,0,0}$.

PROOF. Since $d_\lambda F(t, x, 0)=0$, (iv) and (v) follow immediately from Theorem Ap. 3 in Appendix II. We may write symbolically

$$\begin{aligned}(d_\lambda G)(t, x, \Lambda U)\Lambda W &= \int_0^1 r(1-r)(d_\lambda^3 F)(t, x, \Lambda v+r\Lambda U)dr(\Lambda U, \Lambda U, \Lambda W) \\ &\quad +2\int_0^1 (1-r)(d_\lambda^2 F)(t, x, \Lambda v+r\Lambda U)dr(\Lambda U, \Lambda W).\end{aligned} \tag{13.3}$$

In view of (13.3), we may write symbolically

$$\begin{aligned}\mathcal{A}(t, x, \Lambda U) &= \int_0^1 r(1-r)(d_\lambda^3 F)(t, x, \Lambda v+r\Lambda U)dr(\Lambda U, \Lambda U) \\ &\quad +2\int_0^1 (1-r)(d_\lambda^2 F)(t, x, \Lambda v+r\Lambda U)dr\Lambda U.\end{aligned} \tag{13.4}$$

Applying Theorem Ap. 2 in Appendix I and Leibniz's formula to (13.4), we have the assertion (i). If $3\leqq n\leqq 4$, we may write symbolically

$$(13.5)\quad (d_\lambda G)(t, x, \Lambda U)\Lambda W=\int_0^1 r(1-r)(d_\lambda^3 F)(t, x, \Lambda v+r\Lambda U)dr(\Lambda U, \Lambda U, \Lambda W)$$

$$+2\int_0^1(1-r)(d_\lambda^2 F_1)(t, x, \tilde{\Lambda}v+r\tilde{\Lambda}U)dr(\tilde{\Lambda}U, \tilde{\Lambda}W)+(d_\lambda^2 F_2)(t, x, \Lambda v)(\Lambda U, \Lambda W)$$

$$+\int_0^1(1-r)^2(d_\lambda^3 F_2)(t, x, \Lambda v+r\Lambda U)(\Lambda U, \Lambda U, \Lambda W)dr.$$

It thus follows from (13.2) and (13.5) that

$$(13.6)\quad |c(\cdots, \Lambda U)|_{\infty, q(n), L}\leqq\int_0^1|(d_\lambda^3 F)(\cdots, \Lambda v+r\Lambda U)(\Lambda U, \Lambda U)|_{\infty, q(n), L}dr$$

$$+|d_\lambda^2 F_2(\cdots, \Lambda v)\Lambda U|_{\infty, q(n), L}.$$

Since the fact that $(d_\lambda^2 F_2)(t, x, \Lambda v)=0$ if $t>1$ follows from the facts that $(d_\lambda^2 F_2)(t, x, 0)=0$ and that $v=0$ (cf. (12.1)), noting that $2p(n)\geqq q(n)$ and that $|\Lambda v|_{\infty, 0, 0}+|\Lambda U|_{\infty, p(n), 0}\leqq 1$ and applying Theorem Ap. 2 and Leibniz's formula to (13.6), we have the assertion (ii). Finally, it follows from (13.5) that

$$(13.7)\quad |\mathcal{A}'(\cdots, \Lambda U)|_{2, 1/2, 0}+|c(\cdots, \Lambda U)|_{2, 1, 0}$$

$$\leqq C\Big[\int_0^1|(d_\lambda^3 F)(\cdots, \Lambda v+r\Lambda U)(\Lambda U, \Lambda U)|_{2, 1, 0}dr$$

$$+\int_0^1|(d_\lambda^2 F_1)(\cdots, \tilde{\Lambda}v+r\tilde{\Lambda}U)\tilde{\Lambda}U|_{2, 1/2, 0}dr+|(d_\lambda^2 F_2)(\cdots, \Lambda v)\Lambda U|_{2, 1, 0}\Big].$$

Since $(d_\lambda^2 F_2)(t, x, \Lambda v)=0$ if $t>1$, noting that $p(n)>1$ and that $|\Lambda U|_{\infty, p(n), 0}\leqq 1$, we obtain the assertion (iii) easily from (13.7). This completes the proof of the lemma.

Now, we introduce the following notation which will be needed in order to estimate e'_p. We put

$$(13.8)\quad e'(\Theta, \Lambda U, \Lambda W)\equiv(d_\lambda G)(t, x, \Lambda U)\Lambda W-(d_\lambda G)(t, x, S(\Theta)\Lambda U)\Lambda W).$$

Here $S(\Theta)$, $\Theta\geqq 1$, is the smoothing operator defined by (11.3). We have

LEMMA 13.2. *Assume that all semi-norms appearing below are finite, that the assumptions* 1°-4° *in Introduction are fulfilled and that* $|\Lambda U|_{\infty, p(n), 0}+|\Lambda v|_{\infty, 0, 0}\leqq 1$. *Then the following two assertions are valid.*

(i) *For any integer* $L\geqq 0$,

$$|e'(\Theta, \Lambda U, \Lambda W)|_{2, q(n), L}$$

$$\leqq C(L)[|(1-S(\Theta))\Lambda U|_{\infty, p(n), L}(|\Lambda W|_{2, 0, 0}+|\tilde{\Lambda}W|_{2, 1/2, 0})$$

$$+|(1-S(\Theta))\Lambda U|_{\infty,p(n),0}(|\Lambda W|_{2,0,L}+|\tilde{\Lambda}W|_{2,1/2,L})$$
$$+(|\Lambda v|_{\infty,0,L}+|\Lambda U|_{\infty,p(n),L})$$
$$\times\{|(1-S(\Theta))\Lambda U|_{\infty,p(n),0}(|\Lambda W|_{2,0,0}+|\tilde{\Lambda}W|_{2,1/2,0})\}].$$

(ii) *If* $3\leqq n\leqq 4$, *for any integer* $L\geqq 0$,

$$|e'(\Theta,\ \Lambda U,\ \Lambda W)|_{1,1,L}$$
$$\leqq C(L)[|(1-S(\Theta))\Lambda U|_{2,0,L}|\Lambda W|_{2,0,0}+|(1-S(\Theta))\Lambda U|_{2,0,0}|\Lambda W|_{2,0,L}$$
$$+(|\Lambda U|_{\infty,0,L}+|\Lambda v|_{\infty,0,L})|(1-S(\Theta))\Lambda U|_{2,0,0}|\Lambda W|_{2,0,0}$$
$$+|(1-S(\Theta))\tilde{\Lambda}U|_{2,1/2,L}|\tilde{\Lambda}W|_{2,1/2,0}+|(1-S(\Theta))\tilde{\Lambda}U|_{2,1/2,0}|\tilde{\Lambda}W|_{2,1/2,L}$$
$$+(|\Lambda U|_{\infty,0,L}+|\Lambda v|_{\infty,0,L})|(1-S(\Theta))\tilde{\Lambda}U|_{2,1/2,0}|\tilde{\Lambda}W|_{2,1/2,0}].$$

PROOF. (i) First we assume that $n\geqq 5$. Applying (13.3) to (13.8), we may write symbolically

$$e'(\Theta,\ \Lambda U,\ \Lambda W) \tag{13.9}$$
$$=\int_0^1\int_0^1 r^2(1-r)(d_\lambda^4F)(t,\ x,\ \Lambda v+rS(\Theta)\Lambda U+sr(1-S(\Theta))\Lambda U)drds$$
$$\times(\Lambda U,\ \Lambda U,\ (1-S(\Theta))\Lambda U,\ \Lambda W)$$
$$+\int_0^1 r(1-r)(d_\lambda^3F)(t,\ x,\ \Lambda v+rS(\Theta)\Lambda U)dr[((1-S(\Theta))\Lambda U,\ \Lambda U,\ \Lambda W)$$
$$+(S(\Theta)\Lambda U,\ (1-S(\Theta)\Lambda U,\ \Lambda W)]$$
$$+2\int_0^1\int_0^1 r(1-r)(d_\lambda^3F)(t,\ x,\ \Lambda v+rS(\Theta)\Lambda U+sr(1-S(\Theta))\Lambda U)drds$$
$$\times(\Lambda U,\ (1-S(\Theta))\Lambda U,\ \Lambda W)$$
$$+2\int_0^1(1-r)(d^2F_\lambda)(t,\ x,\ \Lambda v+rS(\Theta)\Lambda U)dr((1-S(\Theta))\Lambda U,\ \Lambda W).$$

Noting that $p(n)=q(n)$, $n\geqq 5$, and that $|\Lambda v|_{\infty,0,0}+|\Lambda U|_{\infty,p(n),0}\leqq 1$ and applying Leibniz's formula, Theorems Ap. 2 and Ap. 3 and Lemma 11.1-(i) to (13.9), we obtain the assertion (i) when $n\geqq 5$.

Next, we assume that $3\leqq n\leqq 4$. It follows from the assumption 4° in Introduction that

$$\int_0^1(1-r)(d_\lambda^2F)(t,\ x,\ \Lambda v+rS(\Theta)\Lambda U)dr((1-S(\Theta))\Lambda U,\ \Lambda W) \tag{13.10}$$
$$=\int_0^1(1-r)(d_{\tilde{\lambda}}^2F_1)(t,\ x,\ \tilde{\Lambda}v+rS(\Theta)\tilde{\Lambda}U)dr((1-S(\Theta))\tilde{\Lambda}U+\tilde{\Lambda}W)$$
$$+(d_\lambda^2F_2)(t,\ x,\ \Lambda v)((1-S(\Theta))\Lambda U,\ \Lambda W)$$
$$+\int_0^1(1-r)^2(d_\lambda^3F_2)(t,\ x,\ \Lambda v+rS(\Theta)\Lambda U)dr(S(\Theta))\Lambda U,\ (1-S(\Theta))\Lambda U,\ \Lambda W).$$

Noting that $(d_\lambda^2 F_2)(t, x, \Lambda v)=0$ if $t \geqq 1$, that $|\Lambda v|_{\infty,0,0}+|\Lambda U|_{\infty,p(n),0} \leqq 1$ and that $p(n)+\dfrac{1}{4}=q(n)$ $(3 \leqq n \leqq 4)$, and applying Leibniz's formula, Theorems Ap. 2 and Ap. 3 to (13.9) and (13.10), we obtain the assertion (i) when $3 \leqq n \leqq 4$.

(ii) Using Leibniz's formula, Theorems Ap. 2 and Ap. 3, the Cauchy-Schwarz inequality, and the fact that $p(n)>1$, we have

$$
\begin{aligned}
&2\int_0^1\int_0^1 |r(1-r)(d_\lambda^3 F)(\cdots, \Lambda v+rS(\Theta)\Lambda U+sr(1-S(\Theta))\Lambda U)drds \\
&\qquad\qquad\qquad\qquad \times(\Lambda U, (1-S(\Theta))\Lambda U, \Lambda W)|_{1,1,L} \\
&\leqq C(L)[(1+|\Lambda U|_{\infty,0,L}+|\Lambda v|_{\infty,0,L})|\Lambda U|_{\infty,p(n),0}|(1-S(\Theta))\Lambda U|_{2,0,0}|\Lambda W|_{2,0,0} \\
&\qquad +|\Lambda U|_{\infty,p(n),L}|(1-S(\Theta))\Lambda U|_{2,0,0}|\Lambda W|_{2,0,0} \\
&\qquad +|\Lambda U|_{\infty,p(n),0}|(1-S(\Theta))\Lambda U|_{2,0,L}|\Lambda W|_{2,0,0} \\
&\qquad +|\Lambda U|_{\infty,p(n),0}|(1-S(\Theta)\Lambda U|_{2,0,0}|\Lambda W|_{2,0,L}],
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 |(1-r)d_\lambda^2 F_1)(\cdots, \tilde{\Lambda} v+rS(\Theta)\tilde{\Lambda} U)dr((1-S(\Theta))\tilde{\Lambda} U, \tilde{\Lambda} W)|_{1,1,L} \\
&\leqq C(L)[(1+|\tilde{\Lambda} v|_{\infty,0,L}+|\tilde{\Lambda} U|_{\infty,0,L})|(1-S(\Theta))\tilde{\Lambda} U|_{2,1/2,0}|\tilde{\Lambda} W|_{2,1/2,0} \\
&\qquad +|(1-S(\Theta))\tilde{\Lambda} U|_{2,1/2,L}|\tilde{\Lambda} W|_{2,1/2,0}+|(1-S(\Theta))\tilde{\Lambda} U|_{2,1/2,0}|\tilde{\Lambda} W|_{2,1/2,L}].
\end{aligned}
$$

We can estimate other parts of (13.9) and (13.10) in the same manner. So, noting that $(d_\lambda^2 F_2)(t, x, \Lambda v)=0$ if $t \geqq 1$ and that $|\Lambda U|_{\infty,p(n),0}+|\Lambda v|_{\infty,p(n),0} \leqq 1$, we obtain the assertion (ii). Q.E.D.

Finally, in order to estimate the error term e_p'', we introduce the following:

$$
e''(\Lambda U, \Lambda W)=G(t, x, \Lambda U+\Lambda W)-G(t, x, \Lambda U)-(d_\lambda G)(t, x, \Lambda U)\Lambda W. \tag{13.11}
$$

Using Taylor series expansions, we may write symbolically

$$
e''(\Lambda U, \Lambda W)=\int_0^1 \{(d_\lambda G)(t, x, \Lambda U+s\Lambda W)\Lambda W-(d_\lambda G)(t, x, \Lambda U)\Lambda W\}\,ds.
$$

Thus, in the same manner as in the proof of Lemma 13.2, we can show the following lemma. We omit the proof.

LEMMA 13.3. *Assume that all semi-norms appearing below are finite, that the assumptions* 1°-4° *in Introduction are fulfilled and that*

$$
|\Lambda v|_{\infty,0,0} \leqq 1/2, \quad |\Lambda U|_{\infty,p(n),0} \leqq 1/4, \quad |\Lambda W|_{\infty,p(n),0} \leqq 1/4.
$$

Then the following two assertions are valid.

(i) *For any integer* $L\geqq 0$,

$$|e''(\Lambda U,\ \Lambda W)|_{2,q(n),L}$$

$$\leqq C(L)(|\Lambda v|_{\infty,0,L}+|\Lambda U|_{\infty,p(n),L}+|\Lambda W|_{\infty,p(n),L})|\Lambda W|_{\infty,p(n),0}$$

$$\times(|\Lambda W|_{2,0,0}+|\tilde{\Lambda} W|_{2,1/3,0})$$

$$+|\Lambda W|_{\infty,p(n),L}(|\Lambda W|_{2,0,0}+|\tilde{\Lambda} W|_{2,1/2,0})$$

$$+|\Lambda W|_{\infty,p(n),L}(|\Lambda W|_{2,0,0}+|\tilde{\Lambda} W|_{2,1/2,0})$$

$$+|\Lambda W|_{\infty,p(n),0}(|\Lambda W|_{2,0,L}+|\tilde{\Lambda} W|_{2,1/2,L})].$$

(ii) *If* $3\leqq n\leqq 4$, *for any integer* $L\geqq 0$,

$$|e''(\Lambda U,\ \Lambda W)|_{1,1,L}$$

$$\leqq C(L)[|\Lambda W|_{2,0,L}|\Lambda W|_{2,0,0}+|\tilde{\Lambda} W|_{2,1/2,L}|\tilde{\Lambda} W|_{2,1/2,0}$$

$$+(|\Lambda v|_{\infty,0,L}+|\Lambda U|_{\infty,p(n),L}+|\Lambda W|_{\infty,p(n),L})(|\Lambda W|_{2,0,0}+|\tilde{\Lambda} W|_{2,1/2,0})^2].$$

§ 14. Proof of convergence of the iteration scheme.

In this section, we shall prove Lemma 12.1. Throughout this section, we use the notations defined in §§ 12 and 13. Since it follows from Theorem Ap. 3, Lemma 10.3 and Sobolev's inequality that

$$|(d_\lambda F)(\cdots,\ \Lambda v)|_{\infty,p(n),1}$$

$$\leqq C(F,\ n)[\|\phi_0\|_{2,m+4+[n/2]}+\|\phi_1\|_{2,m+2+[n/2]}+|f|_{2,0,m+2+[n/2]}],$$

there exists a small positive number δ_1 such that

$$|(d_\lambda F)(\cdots,\ \Lambda v)|_{\infty,0,0}\leqq d_1/2,\quad |(d_\lambda F)(\cdots,\ \Lambda v)|_{\infty,p(n),1}\leqq 1/2 \tag{14.1}$$

if $\|\phi_0\|_{2,\tilde{m}+4+[n/2]}+\|\phi_1\|_{2,\tilde{m}+3+[n/2]}+|f|_{2,0,\tilde{m}+2+[n/2]}\leqq\delta_1$. This fact guarantees the hyperbolicity of the operator $\mathcal{L}$ (cf. Assumption 7.1 and the condition (7.3)). We have

LEMMA 14.1. *Let* δ *be a positive number* $\leqq\min(1,\ \delta_1)$ *where* δ_1 *is the same as in* (14.1). *Let* $d(\tilde{m})>1$ *be a large number which will be defined by* (14.4) *below. If data* ϕ_0, ϕ_1 *and* f *for* (*P*) *satisfy the* $\tilde{m}$*-th order compatibility con dition and*

$$\|\phi_0\|_{2,2\tilde{m}+3+[n/2]}+\|\phi_1\|_{2,2\tilde{m}+2+[n/2]}+|f|_{2,q(n),2\tilde{m}+1+[n/2]}$$

$$+\sigma(n)[\|\phi_0\|_{1,\tilde{m}+2}+\|\phi_1\|_{1,\tilde{m}+1}+|f|_{1,1,\tilde{m}}]\leqq\delta/d(\tilde{m}),$$

then there exists one and only one solution $w_0\in\boldsymbol{E}^{\tilde{m}}$ *of the equation* (12.7) *satisfying* *the estimates*:

$$|\Lambda w_0|_{2,0,\tilde m-2}+|\tilde\Lambda w_0|_{2,1/2,\tilde m-2}<\delta\,,\quad |\Lambda w_0|_{\infty,p(n),\tilde m-2}<\delta\,.$$

PROOF. First, we note the following estimates: if all semi-norms appearing below are finite

$$(14.2)\qquad \begin{aligned}&|g|_{p,k,L}\leqq C(p,\ k,\ L)[|f|_{p,k,L}+\|\phi_0\|_{p,\tilde m+2+L}+\|\phi_1\|_{p,\tilde m+1+L}+|f|_{p,0,\tilde m+L}]\,,\\ &|(d_\lambda F)(\cdots,\ \Lambda v)|_{p,0,L}\leqq C(L,\ p)[\|\phi_0\|_{p,\tilde m+2+L}+\|\phi_1\|_{p,\tilde m+1+L}+|f|_{p,0,\tilde m+L}]\,,\end{aligned}$$

for any p with $1\leqq p\leqq\infty$, real number $k\geqq 0$ and integer $L\geqq 0$. In fact, (14.2) follows immediately from Theorem Ap. 5 and the facts that $F(t,\ x,\ 0)=0$ and that $(d_\lambda F)(t,\ x,\ 0)=0$. Since it follows from (14.1) and the facts that $\delta<\delta_1$ and $d(\tilde m)>1$ that the operator $\mathcal{L}$ satisfies Assumption 7.1 and the condition (7.3), we obtain from Theorems 7.2 and 9.1 and (14.2) that

$$(14.3)\qquad \begin{aligned}&|\Lambda w_0|_{2,0,\tilde m-2}+|\tilde\Lambda w_0|_{2,1/2,\tilde m-2}\\ &\leqq c_1(\tilde m)[|f|_{2,q(n),2\tilde m-1}+\|\phi_0\|_{2,2\tilde m+1}+\|\phi_1\|_{2,2\tilde m}\\ &\qquad+(\|\phi_0\|_{\infty,2\tilde m+2}+\|\phi_1\|_{\infty,2\tilde m+1}+|f|_{\infty,0,2\tilde m})\\ &\qquad\qquad\times(|f|_{2,q(n),\tilde m}+\|\phi_0\|_{2,\tilde m+2}+\|\phi_1\|_{2,\tilde m+1})]\,,\\ &|\Lambda w_0|_{\infty,p(n),\tilde L}\\ &\leqq c_2(\tilde m)[\|\phi_0\|_{2,2\tilde m+1}+\|\phi_1\|_{2,2\tilde m}+|f|_{2,q(n),2\tilde m-1}\\ &\qquad+\sigma(n)(|f|_{1,1,\tilde m}+\|\phi_0\|_{1,\tilde m+2}+\|\phi_1\|_{1,\tilde m+1})\\ &\qquad+(\|\phi_0\|_{\infty,2\tilde m+2}+\|\phi_1\|_{\infty,2\tilde m+1}+|f|_{\infty,0,\tilde m})\\ &\qquad\qquad\times(|f|_{2,q(n),\tilde m}+\|\phi_0\|_{2,\tilde m+2}+\|\phi_1\|_{2,\tilde m+1})\,,\end{aligned}$$

for some large positive constants $c_1(\tilde m)$ and $c_2(\tilde m)$. Applying Sobolev's inequality to (14.3) and noting that $\delta\leqq 1$, we obtain that there exists a large positive number $d(\tilde m)>1$ such that

$$(14.4)\qquad |\Lambda w_0|_{2,0,\tilde m-2}+|\tilde\Lambda w_0|_{2,1/2,\tilde m-2}<\delta\,,\quad |\Lambda w_0|_{\infty,p(n),L}<\delta\,,$$

if $\|\phi_0\|_{2,2\tilde m+3+[n/2]}+\|\phi_1\|_{2,2\tilde m+2+[n/2]}+|f|_{2,q(n),2\tilde m+1+[n/2]}+\sigma(n)[|f|_{1,1,\tilde m}+\|\phi_0\|_{1,\tilde m+2}+\|\phi_1\|_{1,\tilde m+1}]\leqq\delta/d(\tilde m)$. This completes the proof of the lemma.

Now, we shall prove Lemma 12.1 by induction on j. Thus, we assume that

[A.2] *for* $p\geqq 1$, $\dot w_0,\ \cdots,\ \dot w_{p-1}$, *are alreay defined and all the statements of Lemma* 12.1 *are already proved for* $\dot w_0,\ \cdots,\ \dot w_{p-1}$.

Under the assumption [A.2], we shall prove that $\dot w_p$ is also well-defined and the assertions of Lemma 12.1 also hold for $\dot w_p$. Let τ be a sufficiently small positive fixed number. In the course of the proof, all constants depending essentially on $\tilde{L}$, $\tilde{m}$, τ, n, θ and β will be simply denoted by C and all constants depending on L, for arbitrary non-negative integer L, will be denoted by C_L, respectively.

The following lemma follows from Lemmas 11.1 and 14.1, Theorem Ap. 1, the induction hypotheses [A. 2] and the fact that $\{\theta_j\}_{j=0,1,2,\dots}$ is the geometric series. The proof is essentially the same as in Klainerman [5, p. 79-p. 80] and Shibata [18, Lemmas 5.4 and 5.11]. So, we omit the proof.

LEMMA 14.2. *Let the assumptions* [A. 1] *and* [A. 2] *be fulfilled. Put* $w_{j+1}=w_0+\sum_{k=0}^{j}\dot{w}_k$, $j=0, 1, \cdots, p-1$. *Then the following seven assertions are valid for all* $j=0, 1, \cdots, p-1$.

(i) $w_j \in \tilde{\boldsymbol{E}}^{\tilde{m}}$.

(ii) $|S_j\Lambda w_j|_{2,0,L}+|S_j\tilde{\Lambda} w_j|_{2,1/2,L}\leqq C_L\delta\theta_j^{-\beta+L}$ *if* $-\beta+L\geqq\tau$.

(iii) $|S_j\Lambda w_j|_{2,0,L}+|S_j\tilde{\Lambda} w_j|_{2,1/2,L}\leqq C\delta$ *if* $-\beta+L\leqq-\tau$.

(iv) $|S_j\Lambda w_j|_{\infty,p(n),L}\leqq C_L\delta\theta_j^{-\beta+L}$ *if* $-\beta+L\geqq\tau$.

(v) $|S_j\Lambda w_j|_{\infty,p(n),L}\leqq C_L\delta\theta_j^{-\beta+L}$ *if* $-\beta+L\leqq-\tau$.

(vi) $|(1-S_j)\Lambda w_j|_{2,0,L}+|(1-S_j)\tilde{\Lambda} w_j|_{2,1/2,L}\leqq C\delta\theta_j^{-\beta+L}$ *if* $0\leqq L\leqq\tilde{L}$.

(vii) $|(1-S_j)\Lambda w_j|_{\infty,p(n),L}\leqq C\delta\theta_j^{-\beta+L}$ *if* $0\leqq L\leqq\tilde{L}$.

Now, we are going to estimate the error term e_p. Since

$$|\Lambda v|_{\infty,0,0}\leqq C[\|\phi_0\|_{2,\tilde{m}+3+[n/2]}+\|\phi_1\|_{2,\tilde{m}+2+[n/2]}+|f|_{2,0,\tilde{m}+1+[n/2]}]$$

and since

$$|\Lambda v|_{\infty,0,\tilde{m}}\leqq C[\|\phi_0\|_{2,2\tilde{m}+3+[n/2]}+\|\phi_1\|_{2,2\tilde{m}+2+[n/2]}+|f|_{2,0,2\tilde{m}+1+[n/2]}],$$

we obtain from Lemmas 14.1 and 14.2 that there exists a positive small number δ_2 such that

$$(14.5)\qquad \begin{aligned} &|\Lambda w_j|_{\infty,p(n),0}\leqq 1/4 && \text{for } 0\leqq j\leqq p-1,\\ &|\Lambda \dot{w}_j|_{\infty,p(n),0}\leqq 1/4 && \text{for } 0\leqq j\leqq p-1,\\ &|\Lambda v|_{\infty,0,0}\leqq 1/2,\\ &|\Lambda v|_{\infty,0,\tilde{m}}\leqq C\delta && \text{for some positive constant } C, \end{aligned}$$

if the assumptions [A. 1] and [A. 2] and the following assumption hold:

[A. 3] $\qquad \|\phi_0\|_{2,\tilde{m}+3+[n/2]}+\|\phi_1\|_{2,\tilde{m}+2+[n/2]}+|f|_{2,0,\tilde{m}+1+[n/2]}\leqq\delta_2$.

Let e' and e'' be the same as in (13.8) and (13.11), respectively. We have

(14.6) $\qquad e'_j = e'(\theta_j, \Lambda w_j, \Lambda \dot{w}_j), \quad e''_j = e''(\Lambda w_j, \Lambda \dot{w}_j), \quad e_j = e'_j + e''_j.$

The following lemma thus follows immediately from (14.5), (14.6) and Lemmas 13.2, 13.3 and 14.2.

LEMMA 14.3. *Let the assumptions* [A.1], [A.2] *and* [A.3] *be fulfilled. Then, the following three assertions are valid for all* $j=0, 1, \cdots, p-1$.

(i) $e_j \in \tilde{\boldsymbol{E}}^{\tilde{L}} \cap \boldsymbol{C}^{\tilde{L}}(\mathcal{D})$.

(ii) $|e_j|_{2,q(n),L} \leqq C\delta^2 \theta_j^{-2\beta+L}$ $\qquad$ *if* $0 \leqq L \leqq L$.

(iii) $|e_j|_{1,1,L} \leqq C\delta^2 \theta_j^{-2\beta+L}$ $\qquad$ *if* $0 \leqq L \leqq \tilde{L}$ *and* $3 \leqq n \leqq 4$.

Now, we are going to estimate g_p. For this purpose, we begin with

LEMMA 14.4. *Let the assumption* [A.1], [A.2] *and* [A.3] *be fulfilled. Then the following five assertions are valid.*

(i) $G(t, x, w_0) \in \tilde{\boldsymbol{E}}^{\tilde{L}} \cap \boldsymbol{C}^{\tilde{L}}(\mathcal{D})$.

(ii) $|G(\cdots, \Lambda w_0)|_{2,q(n),L} \leqq C\delta^2$ $\qquad$ *if* $0 \leqq L \leqq \dot{L}$.

(iii) $|(1-S_p)G(\cdots, \Lambda w_0)|_{2,q(n),L} \leqq C\delta^2 \theta_p^{-\beta+L}$ $\qquad$ *if* $0 \leqq L \leqq \tilde{L}$,

$|(1-S_{p-1})G(\cdots, \Lambda w_0)|_{2,q(n),L} \leqq C\delta^2 \theta_p^{-2\beta+L}$ $\qquad$ *if* $0 \leqq L \leqq \tilde{L}$.

(iv) $|(S_p - S_{p-1})G(\cdots, \Lambda w_0)|_{2,q(n),L} \leqq C\delta^2 \theta_p^{-2\beta+L}$ $\qquad$ *if* $L \geqq 0$.

(v) *If* $3 \leqq n \leqq 4$,

$|G(\cdots, \Lambda w_0)|_{1,1,L} \leqq C\delta^2$ $\qquad$ *if* $0 \leqq L \leqq \tilde{L}$,

$|(1-S_p)G(\cdots, \Lambda w_0)|_{1,1,0} \leqq C\delta^2 \theta_p^{-2\beta+L}$,

$|(S_p - S_{p-1})G(\cdots, \Lambda w_0)|_{1,1,0} \leqq C\delta^2 \theta_p^{-2\beta+L}$.

PROOF. (i) The assertion (i) follows immediately from Lemma 14.1 and the facts that $G(t, x, 0)=0$ and that $\Lambda w_0 \in \boldsymbol{E}^{\tilde{L}} \cap \boldsymbol{C}^{\tilde{L}}(\mathcal{D})$.
(ii) It follows from (12.5) that

$$G(t, x, \Lambda w_0) = \int_0^1 (1-r)(d_\lambda^2 F)(t, x, \Lambda v + r\Lambda w_0)(\Lambda w_0, \Lambda w_0)\,dr. \tag{14.7}$$

If $n \geqq 5$, noting that $p(n)=q(n)$ and applying Leibniz's formula and Theorem Ap. 2 to (14.7), we obtain from Lemma 14.1 and (14.5) that for $0 \leqq L \leqq \tilde{L}$

$$|G(\cdots, \Lambda w_0)|_{2,q(n),L} \leqq C[(1+|\Lambda w_0|_{\infty,0,L})(|\Lambda w_0|_{2,0,0}|\Lambda w_0|_{\infty,p(n),0})$$
$$+|\Lambda w_0|_{2,0,L}|\Lambda w_0|_{\infty,p(n),0} + |\Lambda w_0|_{2,0,0}|\Lambda w_0|_{\infty,p(n),L}] \leqq C\delta^2.$$

If $3 \leqq n \leqq 4$, it follows from the assumption 4° in Introduction that we may write symbolically

$$(14.8)\qquad G(t, x, \Lambda w_0) = \int_0^1 (1-r)(d_\lambda^2 F_1)(t, x, \tilde{\Lambda} v + r\tilde{\Lambda} w_0)(\tilde{\Lambda} w_0, \tilde{\Lambda} w_0) dr$$
$$+ \frac{1}{2}(d_\lambda^3 F_2)(t, x, \Lambda v)\Lambda w_0, \Lambda w_0) + \int_0^1 (1-r)^2 (d_\lambda^3 F_2)$$
$$\times (t, x, \Lambda v + r\Lambda w_0)(\Lambda w_0, \Lambda w_0, \Lambda w_0) dr.$$

Since $(d_\lambda^3 F_2)(t, x, \Lambda v) = 0$ if $t \geqq 1$, applying Leibniz's formula and Theorem Ap. 2 to (14.8) and noting that $p(n) + \frac{1}{4} = q(n)$, $p(n) > 1$ and $0 < \delta \leqq 1$, we obtain from Lemma 14.1 and (14.5) that for $0 \leqq L \leqq \tilde{L}$

$$\begin{aligned}
&|G(\cdots, \Lambda w_0)|_{2,q(n),L} \\
&\leqq C[(1 + |\Lambda w_0|_{\infty,0,L}) |\tilde{\Lambda} w_0|_{2,1/2,0} |\tilde{\Lambda} w_0|_{\infty,p(n),0} + |\tilde{\Lambda} w_0|_{2,1/2,L} |\tilde{\Lambda} w_0|_{\infty,p(n),0} \\
&\quad + |\tilde{\Lambda} w_0|_{2,1/2,0} |\tilde{\Lambda} w_0|_{\infty,p(n),L} + |\Lambda w_0|_{2,0,L} |\Lambda w_0|_{\infty,p(n),0} \\
&\quad + |\Lambda w_0|_{2,0,0} |\Lambda w_0|_{\infty,p(n),L} + (1 + |\Lambda w_0|_{\infty,0,L}) |\Lambda w_0|_{2,0,0} (|\Lambda w_0|_{\infty,p(n),0})^2 \\
&\quad + |\Lambda w_0|_{2,0,L} (|\Lambda w_0|_{\infty,p(n),0})^2 + |\Lambda w_0|_{2,0,0} |\Lambda w_0|_{\infty,p(n),L} |\Lambda w_0|_{\infty,p(n),0}] \\
&\leqq C\delta^2.
\end{aligned}$$

Thus, we have the assertion (ii).

(iii) It follows from Lemma 11.1 and the assertion (ii) just proved that

$$(14.9)\qquad \begin{aligned}
&|(1-S_p)G(\cdots, \Lambda w_0)|_{2,q(n),0} \leqq C\theta_p^{-\tilde{L}} |G(\cdots, \Lambda w_0)|_{2,q(n),\tilde{L}} \leqq C\delta^2 \theta_p^{-\tilde{L}} \leqq C\delta^2 \theta_p^{-2\beta}, \\
&|(1-S_p)G(\cdots, \Lambda w_0)|_{2,q(n),\tilde{L}} \leqq C\delta^2 \leqq C\delta^2 \theta_p^{-2\beta+\tilde{L}}.
\end{aligned}$$

Here, we have used the fact that $\tilde{L} = 2\beta + 1$ and $\theta_p \geqq 1$. It follows from Theorem Ap. 1 that for $0 \leqq L \leqq \tilde{L}$

$$(14.10)\qquad |D^L(1-S_p)G(\cdots, \Lambda w_0)|_{2,q(n),0}$$
$$\leqq C(|(1-S_p)G(\cdots, \Lambda w_0)|_{2,q(n),0})^{1-(L/\tilde{L})} (|(1-S_p)G(\cdots, \Lambda w_0)|_{2,q(n),\tilde{L}})^{L/\tilde{L}}.$$

Combining (14.9) and (14.10), we have the first assertion of (iii). Noting that $\theta_p = \theta \cdot \theta_{p-1}$, we have the second assertion in the same manner.

(iv) If $L > \tilde{L}$, we obtain the assertion (iv) from Lemma 11.1-(iii) and the second inequality of (14.9). If $0 \leqq L \leqq \tilde{L}$, noting that $S_p - S_{p-1} = 1 - S_{p-1} - (1 - S_p)$, we obtain the assertion (iv) from the assertion (iii) just proved.

(v) Applying Leibniz's formula, Theorem Ap. 2, and the Cauchy-Schwarz inequality to (14.8) and noting that $p(n) + \frac{1}{4} = q(n)$ and that $0 < \delta \leqq 1$, we obtain from Lemma 14.1 and (14.5) that for $0 \leqq L \leqq \tilde{L}$

(14.11) $$|G(\cdots, \Lambda w_0)|_{1,1,L} \leqq C[(1+|\tilde{\Lambda} w^0|_{\infty,0,L})(|\tilde{\Lambda} w_0|_{2,1/2,0})^2$$
$$+|\tilde{\Lambda} w_0|_{2,1/2,L}|\tilde{\Lambda} w_0|_{2,1/2,0}+\delta(|\Lambda w_0|_{2,0,0})^2+|\Lambda w_0|_{2,0,L}|\Lambda w_0|_{2,0,0}$$
$$+(1+|\Lambda w_0|_{\infty,0,L})|\Lambda w_0|_{\infty,p(n),0}(|\Lambda w_0|_{2,0,0})^2+|\Lambda w_0|_{\infty,p(n),L}(|\Lambda w_0|_{2,0,0})^2]$$
$$\leqq C\delta^2,$$

which shows the first assertion of (v). The other assertion of (v) are able to be proved by using the first assertion of (v) just proved, Lemma 11.1 and Theorem Ap. 2, in the same manner as before. This completes the proof.

Next, we estimate the summation E_p of error terms e_j, $j=0, \cdots, p-1$.

LEMMA 14.5. *Let the assumptions* [A. 1]-[A.3] *be fulfilled. Put* $E_p=\sum_{j=0}^{p-1} e_j$, $p\geqq 1$. *The following nine assertions are valid.*

(i) $E_p \in \tilde{\boldsymbol{E}}^{\tilde{L}} \cap C^{\tilde{L}}(\mathcal{D})$.

(ii) $|E_p|_{2,q(n),L} \leqq C\delta^2\theta_p^{-2\beta+L}$ *if* $0\leqq L\leqq \tilde{L}$, $L-2\beta>\tau$.

(iii) $|E_p|_{2,q(n),L} \leqq C\delta^2$ *if* $L-2\beta<-\tau$.

(iv) $|E_p|_{1,1,L} \leqq C\delta^2\theta_p^{-2\beta+L}$ *if* $0\leqq L\leqq \tilde{L}$, $L-2\beta>\tau$, $3\leqq n\leqq 4$.

(v) $|E_p|_{1,1,L} \leqq C\delta^2$ *if* $L-2\beta<-\tau$, $3\leqq n\leqq 4$.

(vi) $|(1-S_{p-1})E_p|_{2,q(n),L} \leqq C\delta^2\theta_p^{-2\beta+L}$, $|(1-S_p)E_p|_{2,q(n),L} \leqq C\delta^2\theta_p^{-2\beta+L}$ *if* $0\leqq L\leqq \tilde{L}$.

(vii) $|(1-S_{p-1})E_p|_{1,1,L} \leqq C\delta^2\theta_p^{-2\beta+L}$,
$|(1-S_p)E_p|_{1,1,L} \leqq C\delta^2\theta_p^{-2\beta+L}$ *if* $0\leqq L\leqq \tilde{L}$, $3\leqq n\leqq 4$.

(viii) $|(S_p-S_{p-1})E_p|_{2,q(n),L} \leqq C\delta^2\theta_p^{-2\beta+L}$ *for any* $L\geqq 0$ *if* $3\leqq n\leqq 4$.

(ix) $|(S_p-S_{p-1})E_p|_{1,1,0} \leqq C\delta^2\theta_p^{-2\beta}$ *if* $3\leqq n\leqq 4$.

PROOF. The assertion (i) follows immediately from Lemma 14.3. The assertions (ii)-(v) follows from Lemma 14.3 and the fact that $\{\theta_j\}_{j=0,1,2\cdots}$ is a geometric series. In the same manner as in the proof of Lemma 14.4, we can show the assertions (vi)-(ix). So, we omit the proof. Q.E.D.

From (12.13), Lemmas 11.1, 14.3, 14.4 and 14.5 we obtain

LEMMA 4.6. *Let the assumptions* [A. 1]-[A. 3] *be fulfilled. Then,*

(i) $g_p \in \tilde{\boldsymbol{E}}^\infty$.

(ii) $|g|_{2,q(n),L} \leqq C_L\delta^2\theta_p^{-2\beta+L}$ *for any* $L\geqq 0$.

(iii) $|g_p|_{1,1,0} \leqq C\delta^2\theta_p^{-2\beta}$ *if* $3\leqq n\leqq 4$.

(iv) $g_0 \in \tilde{E}^\infty$.

(v) $|g_0|_{2,q(n),L} \leqq C_L \delta^2 \theta_0^{-2\beta+L}$ *for any* $L \geqq 0$.

(vi) $|g_0|_{1,1,0} \leqq C\delta^2 \theta_0^{-2\beta}$ *if* $3 \leqq n \leqq 4$.

Now, we shall estimate w_p by using Theorems 7.2 and 9.1. For this purpose, first of all, we have to examine if the operator L_j satisfies Assumption 7.1 and the conditions (7.3), (9.2) and (9.2)$'$. Since

$$L_j w = (\partial_t^2 + \partial_t - \Delta) w + (d_\lambda F)(t, x, \Lambda v)\Lambda w + (d_\lambda G)(t, x, S_j \Lambda w_j)\Lambda w ,$$

we represent the last term symbolically in terms of coefficient vectors $\mathcal{A}'$ and c defined by (13.1) and (13.2) as follows:

$$(d_\lambda G)(t, x, S_j \Lambda w_j)\Lambda w = \mathcal{A}'(t, x, S_j \Lambda w_j)\Lambda w + c(t, x, S_j \Lambda w_j) w .$$

Therefore, combining Lemmas 13.1, 14.1 and 14.2 and (14.5), we obtain

LEMMA 14.7. *Let the assumptions* [A.1]-[A.3] *be fulfilled. Put*

$$\mathcal{A}_j = (\mathcal{A}'(t, x, S_j \Lambda w_j), \quad c(t, x, S_j \Lambda w_j)) .$$

Let d_1 *be the same as in Theorem* 7.1. *Then there exists a small positive constant* δ_3 *such that if* $0 \leqq \delta \leqq \delta_3$ *then the following eight assertions are valid for all* $j = 0, 1, \cdots, p$.

(i) $|d_\lambda F)(\cdots, \Lambda v)|_{\infty,p(n),L} + |\mathcal{A}_j|_{\infty,p(n),L} \leqq C_L \delta \theta_j^{-\beta+L}$ *if* $-\beta + L \geqq \tau$, $L \leqq \tilde{m}$.

(ii) $|(d_\lambda F)(\cdots, \Lambda v)|_{\infty,p(n),L} + |\mathcal{A}_j|_{\infty,p(n),L} \leqq C\delta$ *if* $-\beta + L \leqq -\tau$.

(iii) $|(d_\lambda F)(\cdots, \Lambda v)|_{\infty,p(n),1} + |\mathcal{A}_j|_{\infty,p(n),1} \leqq 1$.

(iv) $|(d_\lambda F)(\cdots, \Lambda v)|_{\infty,0,0} + |\mathcal{A}_j|_{\infty,0,0} \leqq d_1$.

(v) $|(d_\lambda F)(\cdots, \Lambda v)|_{2,1,0} + |\mathcal{A}'(\cdots, S_j \Lambda w_j)|_{2,1/2,0} + |c(\cdots, S_j \Lambda w_j)|_{2,1,0} \leqq 1$.

(vi) $|(d_\lambda F)(\cdots, \Lambda v)|_{2,q(n),0} + |\mathcal{A}'(\cdots, S_j \Lambda w_j)|_{\infty,p(n),0} + |c(\cdots, S_j \Lambda w_j)|_{\infty,q(n),0} \leqq 1$ *if* $3 \leqq n \leqq 4$,

(vii) $|(d_\lambda F)(\cdots, \Lambda v)|_{\infty,q(n),L} + |\mathcal{A}'(\cdots, S_j \Lambda w_j)|_{\infty,p(n),L} + |c(\cdots, S_j \Lambda w_j)|_{\infty,q(n),L} \leqq C_L \delta \theta_j^{-\beta+L}$ *if* $-\beta + L > \tau$, $0 \leqq L \leqq \tilde{m}$ *and* $3 \leqq n \leqq 4$.

(viii) $|(d_\lambda F)(\cdots, \Lambda v)|_{\infty,q(n),L} + |\mathcal{A}'(\cdots, S_j \Lambda w_j)|_{\infty,p(n),L} + |c(\cdots, S_j \Lambda w_j)|_{\infty,q(n),L} \leqq C\delta$ *if* $-\beta + L \leqq -\tau$, $3 \leqq n \leqq 4$.

It follows from Lemma 14.7 that the operator L_p satisfies Assumption 7.1 and conditions (7.3), (9.2) and (9.2)$'$. Thus, applying Theorem 7.2 with $\eta = p(n) - 1$ and Theorem 9.1 to (12.14), we obtain from Lemmas 14.6 and 14.7 that

$$(14.12)\qquad \begin{aligned} &|\Lambda\dot{w}_p|_{2,0,L}+|\tilde{\Lambda}\dot{w}_p|_{2,1/2,L}\leqq C_L\delta^2\theta_p^{-2\beta+L+1},\\ &|\Lambda\dot{w}_p|_{\infty,0,L}\leqq C_L\delta^2\theta_p^{-2\beta+L+2[n/2]+7}\end{aligned}$$

for $0\leqq L\leqq\tilde{L}$. Choose $\delta_4>0$ so small that

$$(14.13)\qquad \delta_4\max_{0\leqq L\leqq\tilde{L}}C_L\leqq 1$$

where C_L is the same as in (14.12). Put

$$(14.14)\qquad \delta_0=\min(\delta_1,\ \delta_2,\ \delta_3,\ \delta_4)$$

where δ_1, δ_2, δ_3 and δ_4 are the same as in (14.1), [A. 3], Lemma 14.7 and (14.13), respectively. We obtain from (14.12)-(14.14) and the fact: $\beta\geqq 2[n/2]+7$ that if $0<\delta\leqq\delta_0$ and

$$\begin{aligned}&\|\phi_0\|_{2,2\tilde{m}+3+[n/2]}+\|\phi_1\|_{2,2\tilde{m}+2+[n/2]}+|f|_{2,q(n),2\tilde{m}+1+[n/2]}\\ &\quad+\sigma(n)(\|\phi_0\|_{1,\tilde{m}+2}+\|\phi_1\|_{1,\tilde{m}+1}+|f|_{1,1,\tilde{m}})\leqq\delta/d(\tilde{m}),\end{aligned}$$

then

$$|\Lambda\dot{w}_p|_{2,0,L}+|\tilde{\Lambda}\dot{w}_p|_{2,1/2,L}\leqq\delta\theta_p^{-\beta+L},\ \ |\Lambda\dot{w}_p|_{\infty,p(n),L}\leqq\delta\theta_p^{-\beta+L}$$

for $0\leqq L\leqq\tilde{L}$, which shows that $\dot{w}_p$ also satisfies the assertion of Lemma 12.1.

To complete the induction, it remains only to verify that $\dot{w}_0$ satisfies the assertion of Lemma 12.1. In view of Lemmas 14.6 and 14.7, its proof follows exactly as before. We have completed the proof of Lemma 12.1, just now.

§15. Proof of main theorem.

First, we shall prove the existence of solutions of (P). Since $\beta=\max[m-1, 2[n/2]+7]$, it follows from Lemma 12.1 that there exists a function $w\in \boldsymbol{C}^m(\mathcal{D})\cap\boldsymbol{E}^m$ such that

$$(15.1)\qquad w=\textstyle\sum_{j=0}^{\infty}\dot{w}_j+w_0\,,$$

$$(15.2)\qquad |\Lambda w|_{2,0,m-2}+|\tilde{\Lambda}w|_{2,1/2,m-2}+|\Lambda w|_{\infty,p(n),m-2}\leqq C\delta\,,$$

where $C=2(2\theta-1)/(\theta-1)$. Furthermore, we obtain from (12.7) and (12.14) that

$$(15.3)\qquad w=0\ \text{ on }\mathcal{D}',\quad w(0,\ x)=(\partial_t w)(0,\ x)=0\ \text{ in }\Omega.$$

On the other hand, combining (12.15), Lemmas 14.3, 14.4 and 14.5, we have

$$(15.4)\qquad \begin{aligned}&|\mathcal{L}w+G(\cdots,\ \Lambda w)-g|_{2,0,0}\leqq C\Big[|w-w_{p+1}|_{2,0,2}\\ &\quad+\Big|\int_0^1(d_\lambda G)(\cdots,\ \Lambda w_{p+1}+\theta\Lambda(w-w_{p+1}))d\theta(\Lambda w-\Lambda w_{p+1})\Big|_{2,0,0}\end{aligned}$$

$$+|(1-S_p)E_p|_{2,0,0}+|e_p|_{2,0,0}+|(1-S_p)G(\cdots, \Lambda w_0)|_{2,0,0}\Big]$$
$$\leqq C(\delta\theta^{-p}+\delta^2\theta^{-2\beta p}) \qquad \text{for any } p\geqq 0.$$

It follows from (15.4) and the fact: $w\in C^2(\mathscr{D})$ $(m\geqq 2)$ that $\mathcal{L}w+G(t, x, \Lambda w)=g$ in $\mathscr{D}$. Therefore, in view of § 12, putting $u=v+w$, we obtain that u is a solution of (P), which completes the proof of the existence of solutions of (P).

Next, we shall prove the uniqueness theorem. For this purpose, we begin with

LEMMA 15.1. *Let R and T be any positive numbers with $R>r_0$ (cf. Notations) and μ be a large fixed number with $\mu\geqq 2(n+1)$. Put*

$$\Gamma=\{(x, t);\ x\in\Omega,\ |x|\leqq R+\mu(T-t),\ 0\leqq t\leqq T\}.$$

Let a_j, $j=0, \cdots, n$, a_{ij}, $i, j=1, \cdots, n$, b_j, $j=0, \cdots, n$, c be real valued $C^1(\mathscr{D})$ functions such that

$$\mathcal{A}\equiv \sup_{(t,x)\in\Gamma}|(a_j,\ j=0, \cdots, n,\ a_{ij},\ i, j=1, \cdots, n,\ b_j,\ j=0, \cdots, n,\ c)|\leqq 1/2, \tag{15.5}$$

$$a_{ij}=a_{ji}. \tag{15.6}$$

Let us define a linear operator $\mathcal{L}$ by

$$\mathcal{L}=\partial_t^2+\partial_t-\Delta+\textstyle\sum_{j=0}^n a_j\partial_j\partial_t-\sum_{i,j=0}^n a_{ij}\partial_i\partial_j+\sum_{j=0}^n b_j\partial_j+c\,.$$

If $u\in C^2(\mathscr{D})$ satisfies the equations:

$$\begin{aligned} &\mathcal{L}u=0 && \text{in } \Gamma, \\ &u=0 && \text{on } [0, T]\times\partial\Omega, \\ &u(0, x)=(\partial_t u)(0, x)=0 && \text{in } \Omega_{R+\mu T} \text{ (cf. Notations)}, \end{aligned} \tag{15.7}$$

then $u\equiv 0$ in Γ.

PROOF. We prove the lemma by well-known energy method. It follows from the fact $\mu\geqq 2(n+1)$

$$\begin{aligned} &[n_0(1+a_0)-\textstyle\sum_{j=1}^n n_j a_j](\partial_t u)^2-2[\sum_{ij-1}^n n_j(\delta_{ij}+a_{ij})\partial_i u]\partial_t u \\ &\qquad +n_0\textstyle\sum_{ij=1}^n(\delta_{ij}+a_{ij})\partial_i u\partial_j u \\ &\qquad \geqq n_0\{(1-\mathcal{A})(\partial_t u)^2-2\mu^{-1}(n+\mathcal{A})|D_x^1 u||\partial_t u|+(1-\mathcal{A})|D_x^1 u|^2\}\geqq 0 \end{aligned} \tag{15.8}$$

if $|x|=R+\mu(T-t)$, $0\leqq t\leqq T$, where we have put

$$n_0=\mu(1+\mu^2)^{-1/2}, \quad n_j=x_j(|x|(1+\mu^2)^{1/2})^{-1}, \quad j=1, \cdots, n.$$

Noting (15.6), we have the identity:

$$(15.9)\qquad \mathcal{L}u\cdot\partial_t u=\frac{1}{2}\partial_t[(1+a_0)(\partial_t u)^2+\sum_{ij=1}^n(\delta_{ij}+a_{ij})\partial_i u\partial_j u]$$
$$+\frac{1}{2}\sum_{j=1}^n\partial_j[a_j(\partial_t u)^2]-\sum_{ij=1}^n\partial_j[(\delta_{ij}+a_{ij})\partial_i u\partial_t u]$$
$$-\Big[\Big\{\frac{1}{2}\sum_{j=0}^n\partial_j a_j-(1+b_0)\Big\}(\partial_t u)^2-\sum_{i=1}^n(\sum_{j=1}^n\partial_j a_{ij}+b_i)\partial_i u\partial_t u$$
$$-\frac{1}{2}\sum_{ij=1}^n(\partial_t a_{ij})\partial_i u\partial_j u-cu\partial_t u\Big].$$

Now, we introduce the following notations:

$$\Gamma(t_0,\ t_1)=\{(t,\ x);\ x\in\Omega,\ |x|\leqq R+\mu(T-t),\ t_0\leqq t\leqq t_1\}$$
$$\text{for any } t_0,\ t_1 \text{ with } 0\leqq t_0<t_1\leqq T,$$
$$G(t)=\{x\in\Omega,\ |x|\leqq R+\mu(T-t)\}\quad \text{for any } t \text{ with } 0\leqq t\leqq T.$$

Integrating (15.9) over $\Gamma(0,\ s)$, we have by the divergence theorem and (15.5), (15.7), (15.8) and (15.9) that

$$(15.10)\qquad \int_{G(t)}\{(\partial_t u(t,\ x))^2+|D_x^1 u(t,\ x)|^2\}\,dx$$
$$\leqq 2\int_{G(t)}\{(1+a_0(t,\ x))(\partial_t u(t,\ x))^2+\sum_{ij=1}^n(\delta_{ij}+a_{ij}(t,\ x))\partial_i u(t,\ x)\partial_j u(t,\ x)\}\,dx$$
$$\leqq c\mathcal{A}_1\int_0^t\int_{G(t)}\{(\partial_s u(s,\ x))^2+|D_x^1 u(s,\ x)|^2+u(s,\ x)^2\}\,ds\,dx$$

for some constant c depending only on μ and n. Here, we have put

$$\mathcal{A}_1=\sup_{(t,\,x)\in\Gamma}|(\bar{D}^1 a_j,\ j=0,\ \cdots,\ n,\ \bar{D}^1 a_{ij},\ i,\ j=1,\ \cdots,\ n,\ b_j,\ j=0,\ \cdots\ n,\ c)|+1.$$

For any t_0, t_1 with $0\leqq t_0<t_1\leqq T$, let us put

$$E(t_0,\ t_1)=\sup_{t_0\leqq s\leqq t_1}\int_{G(s)}\{(\partial_t u(s,\ x))^2+|D_x^1 u(s,\ x)|^2\}\,dx.$$

We have by the Cauchy-Schwarz inequality that

$$(15.11)\qquad \int_{G(s)}|u(s,\ x)|^2dx\leqq\int_0^s\int_{G(s)}s|(\partial_t u)(r,\ x)|^2drdx$$
$$\leqq s\int_0^s\int_{G(r)}|(\partial_t u)(r,\ x)|^2drdx\leqq s^2E(0,\ s).$$

Here, we have used the fact that $G(r)\supset G(s)$ if $0\leqq r<s\leqq T$. Combining (15.10) and (15.11), we have

$$(15.12)\qquad E(0,\ t)\leqq c\mathcal{A}_1(1+T^2)tE(0,\ t).$$

If we choose a positive number t_0 so small that $c\mathcal{A}_1(1+T^2)t_0\leqq 1/2$, we have by

(15.12) that $E(0, t)=0$ for $0\leqq t\leqq t_0$, which implies that $u\equiv 0$ in $\Gamma(0, t_0)$. Replacing 0 by t_0 and t_0 by $2t_0$ and repeating the argument just mentioned, we have that $u\equiv 0$ in $\Gamma(t_0, 2t_0)$. A finite number of iterations of this argument implies that $u\equiv 0$ in $\Gamma(0, T)=\Gamma$. This completes the proof of the lemma.

Now, we shall show the uniqueness theorem by using Lemma 15.1. Let u, v be $C^3(\mathscr{D})$ solutions of (P) satisfying

$$|\Lambda u|_{\infty,0,0}\leqq\delta_1. \tag{15.13}$$

Here δ_1 is a positive constant determined later. Put $w=u-v$. By Taylor series expansion, we have

$$\begin{aligned} &(\partial_t^2+\partial_t-\Delta)w+\int_0^1(d_\lambda F)(t, x, \theta\Lambda u+(1-\theta)\Lambda v)d\theta\Lambda w=0 && \text{in } \mathscr{D}, \\ &w=0 && \text{on } \mathscr{D}', \\ &w(0, x)=(\partial_t w)(0, x)=0 && \text{in } \Omega. \end{aligned} \tag{15.14}$$

First, we shall prove $u=v$ in $[0, 1]\times\Omega$. For this purpose, we may show that $w=0$ in $\{(t, x);\ |x|\leqq R+\mu(1-t),\ x\in\Omega,\ 0\leqq t\leqq 1\}$ for any $R>r_0$. If we choose δ_1 so small that

$$\|u(0, \cdot)\|_{\infty,2}+\|(\partial_t u)(0, \cdot)\|_{\infty,1}+\|f(0, \cdot)\|_\infty\leqq d_2, \tag{15.15}$$

where $f=(\partial_t^2+\partial_t-\Delta)u+F(t, x, \Lambda u)$ and d_2 is the same as in (10.5), we have by (10.5) that

$$(\partial_t^2 v)(0, x)=(\partial_t^2 u)(0, x)=\nu(x, \bar{D}_x^2 u(0, x), \bar{D}_x^1 u(0, x), f(0, x)).$$

This implies

$$\begin{aligned} &v(t, x)=u(0, x)+\int_0^t(\partial_t v)(s, x)ds, \\ &(\partial_t v)(t, x)=(\partial_t u)(0, x)+\int_0^t(\partial_t^2 v)(s, x)ds, \\ &(\partial_t^2 v)(t, x)=(\partial_t^2 u)(0, x)+\int_0^t(\partial_t^3 v)(s, x)ds. \end{aligned} \tag{15.16}$$

For any t_0, t_1 with $0\leqq t_0<t_1\leqq 1$, let us put

$$\Gamma(t_0, t_1)=\{(t, x);\ x\in\Omega,\ |x|\leqq R+\mu(1-t),\ t_0\leqq t\leqq t_1\}$$

where μ is the same as in Lemma 15.1. Put

$$\rho=\sup_{(t,x)\in\Gamma(0,1)}|(\Lambda v)(t, x)|. \tag{15.17}$$

It follows from (15.13), (15.16) and (15.17) that

$$\sup_{(s,x)\in\Gamma(0,t)} |(\Lambda v)(s, x)| \leqq t\rho + c(n)\delta_1 \tag{15.18}$$

for some constant $c(n)$ depending only on n. On the other hand, since $d_\lambda F(t, x, 0) = 0$, we can choose δ_2 small so that

$$|d_\lambda F(t, x, \lambda)| \leqq 1/2 \quad \text{if } (t, x) \in \mathcal{D},\ |\lambda| \leqq \delta_2, \tag{15.19}$$

Therefore, if we choose δ_1 and t_0 so small that

$$c(n)\delta_1 \leqq \delta_2/2, \quad \rho t_0 \leqq \delta_2/2, \tag{15.20}$$

we have by (15.13), (15.18) and (15.19) that

$$\left|\int_0^1 d_\lambda F(t, x, \theta\Lambda u + (1-\theta)\Lambda v)\,d\theta\right| \leqq 1/2 \tag{15.21}$$

for $(t, x) \in \Gamma(0, t_0)$. Since $u, v \in \boldsymbol{C}^3(\mathcal{D})$, $\int_0^1 d_\lambda F(t, x, \theta\Lambda u + (1-\theta)\Lambda v)\,d\theta \in \boldsymbol{C}^1(\mathcal{D})$. By this and (15.21) we have that the linear operator $\partial_t^2 + \partial_t - \Delta + \int_0^1 d_\lambda F(t, x, \theta\Lambda u + (1-\theta)\Lambda v)\,d\theta$ satisfies all conditions in Lemma 15.1. So, applying Lemma 15.1 to (15.14), we have that $w = u - v = 0$ in $\Gamma(0, t_0)$. In particular, $u(t_0, x) = v(t_0, x)$, $(\partial_t u)(t_0, x) = (\partial_t v)(t_0, x)$, and $(\partial_t^2 u)(t_0, x) = (\partial_t^2 v)(t_0, x)$. Replacing 0 by t_0 and t_0 by $2t_0$ and repeating the argument, we have that $w = u - v = 0$ in $\Gamma(t_0, 2t_0)$ without changing the choise of δ_1. Because, δ_1 depends only on d_2, $c(n)$ and δ_2 (cf. (15.15) and (15.20)) and δ_2 depends only on $F(t, x, \lambda)$. A finite number of iterations of this argument implies that $u = v$ in $\Gamma(0, 1)$. Since we can choose R arbitrarily large, we have $u = v$ in $[0, 1] \times \Omega$. Since δ_1 depends only on d_2, $c(n)$ and δ_2, we can show by repeated use of the argument just mentioned that $u = v$ in $\mathcal{D}$ without changing the choice of δ_1. This completes the proof of the uniqueness theorem.

APPENDIX

I. Interpolation inequality

THEOREM AP. 1 (Interpolation inequality). *Let $\mathcal{O} \subset \boldsymbol{R}^n$ be a domain. Assume that the boundary of $\mathcal{O}$ is compact and C^∞, or $= \boldsymbol{R}^{n-1}$, or that $\mathcal{O} = \boldsymbol{R}^n$. Assume that all semi-norms appearing below are finite. Then, the following three assertions are valid for any integers N and M with $0 \leqq N \leqq M$ and p with $1 \leqq p \leqq \infty$.*

(i) $\|D_x^N \phi\|_{\mathcal{O},p} \leqq C(\|\phi\|_{\mathcal{O},p})^{1-(N/M)}(\|\phi\|_{\mathcal{O},p,M})^{N/M}$.

(ii) *For any closed interval $I = [a, b] \subset \boldsymbol{R}^1$ $(-\infty \leqq a < b \leqq \infty)$,*

$|D^N f|_{\mathcal{O},p,I,0} \leqq C(|f|_{\mathcal{O},p,I,0})^{1-(N/M)}(|f|_{\mathcal{O},p,I,M})^{N/M}$.

(iii) *For any non-negative real number k.*

$$|D^N g|_{\mathcal{O},p,k,0} \leqq C(|g|_{\mathcal{O},p,k,0})^{1-(N/M)}(|g|_{\mathcal{O},p,k,M})^{N/M}.$$

Here all the constants C are independent of the functions ϕ, f and g.

PROOF. First, we shall prove the theorem in the case of $\mathcal{O}=\boldsymbol{R}^n$. The assertion (i) is the well-known classical interpolation inequality. Using a representation theorem due to Muramatsu [10], we can show the following in the same manner as in the proof of Lemma 2.2.4 in Shibata [18].

(Ap. 1) $$|D^N g|''_{p,k,0} \leqq C(|g|''_{p,k,0})^{1-(N/M)}(|g|''_{p,k,M})^{N/M}$$

where $|g|''_{p,k,N}=\sup_{t\in R^1}(1+|t|)^k\|\bar{D}^N g(t,\cdot)\|'_p$. If $\mathcal{O}$ satisfies the assumption of the theorem, via local map, using an extension theorem due to Seeley [17], we can the following three assertions.

(Ap. 2) *For any ϕ defined on $\mathcal{O}$, there exists ϕ' defined of $\boldsymbol{R}^n$ such that $\phi=\phi'$ in $\mathcal{O}$ and $\|\phi'\|'_{p,N} \leqq C\|\phi\|_{\mathcal{O},p,N}$.*

(Ap. 3) *For any f defined on $I\times\mathcal{O}$, there exists f' defined on $\boldsymbol{R}^{n+1}$ such that $f=f'$ on $I\times\mathcal{O}$ and $|f'|''_{p,0,N} \leqq C|f|_{\mathcal{O},p,I,N}$.*

(Ap. 4) *For any g defined on $[0,\infty)\times\mathcal{O}$, there exists g' defined on $\boldsymbol{R}^{n+1}$ such that $g=g'$ on $[0,\infty)\times\mathcal{O}$ and $|g'|''_{p,k,N} \leqq C|g|_{\mathcal{O},p,k,N}$.*

Here, if we fix the manner of the extension of functions, all the constants C appearing in (Ap. 2)-(Ap. 4) depend essentially on only k, N, I, p and $\mathcal{O}$ but independent of ϕ, f and g. Combining (Ap. 1)-(Ap. 4), we have the theorem, which completes the proof.

The following theorem is also proved in Shibata [18, Lemma 2.2.9]. We can show it by using Theorem Ap. 1 and the following elementary inequality:

$$a\cdot b \leqq \frac{a^p}{p}+\frac{b^q}{q}, \quad \text{where } a,\ b\geqq 0,\ 1\leqq p\leqq\infty \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$

THEOREM AP. 2. *Let $\mathcal{O}$ be the same as in Theorem Ap. 1. Let the semi-norms appearing below be finite. Let p, q, k, k' be real numbers with $1\leqq p$, $q\leqq\infty$, k, $k'\geqq 0$ and I, I' be any closed intervals in $\boldsymbol{R}^1$ and M, N, i, j be non-negative integers with $i\leqq M$ and $j\leqq N$. Then the following six inequalities hold.*

1° $\|\phi\|_{p,M}\|\psi\|_{q,N} \leqq C[\|\phi\|_{p,i}\|\psi\|_{q,M+N-i}+\|\phi\|_{p,M+N-j}\|\psi\|_{q,j}]$.

2° $\|\phi\|_{p,M}|f|_{q,I,N} \leqq C[\|\phi\|_{p,i}|f|_{q,I,M+N-i}+\|\phi\|_{p,M+N-j}|f|_{q,I,j}]$.

3° $\|\phi\|_{p,M}|f|_{q,k,N} \leqq C[\|\phi\|_{p,i}|f|_{q,k,M+N-i}+\|\phi\|_{p,M+N-j}|f|_{q,k,j}]$.

4° $|f|_{p,I,M}|g|_{q,I',N} \leqq C[|f|_{p,I,i}|g|_{q,I',M+N-i}+|f|_{p,I,M+N-j}|g|_{q,I',j}]$.

5° $|f|_{p,I,M}|g|_{q,k,N} \leqq C[|f|_{p,I,i}|g|_{q,k,M+N-i}+|f|_{p,I,M+N-j}|g|_{q,k,j}]$.

6° $|f|_{p,k,M}|g|_{q,k',N} \leqq C[|f|_{p,k,i}|g|_{q,k',M+N-i}+|f|_{p,k,M+N-j}|g|_{q,k',j}]$.

Here all the constants C appearing in 1°-6° are independent of ϕ, ψ, f and g and we have omitted the index $\mathcal{O}$ in semi-norms.

II. Moser's lemma

Using Theorem Ap. 1 and the well-known technique essentially due to Moser [9], we have the following theorem (see also Klainerman [5, Lemma 5.1] and Shibata [18, Lemma 5.12]).

THEOREM AP. 3. *Let $\mathcal{O}$ be the same as in Theorem Ap. 1. Let $u=(u_1, \cdots, u_s)$ with $u_j \in C^\infty([0, \infty)\times\mathcal{O})$, $j=1, \cdots, s$, and $|u|_{\mathcal{O},\infty,0,0} \leqq 1$. If $H(t, x, w)=H(t, x, w_1, \cdots, w_s)$ is a $\mathcal{B}^\infty([0, \infty)\times\mathcal{O}\times\{w\in \boldsymbol{R}_s;\ |w|\leqq 1\}$ function, then for any integer $N\geqq 0$*

(i) $|H(\cdots, u(\cdot, \cdot))|_{\mathcal{O},\infty,0,L} \leqq C(L, H)(1+|u|_{\mathcal{O},\infty,0,L})$.

Moreover, if $H(t, x, 0)=0$, then

(ii) $|H(\cdots, u(\cdot, \cdot))|_{\mathcal{O},\infty,0,L} \leqq C(L, H)|u|_{\mathcal{O},\infty,0,L}$.

Here, we have assumed that semi-norms appearing above are all finite.

THEOREM AP. 4. *Let N be a non-negative integer and $1\leqq p\leqq\infty$. Let $\mathcal{O}$ be the same as in Theorem Ap. 1, $H(x, w)=H(x, w_1, \cdots, w_s)$ be a $\mathcal{B}^\infty(\mathcal{O}\times\{w\in \boldsymbol{R}^s;\ |w|\leqq 1\})$ function and $w(x)=(w_1(x), \cdots, w_s(x))$ with $w_j\in H_p^N(\mathcal{O})$ and $\|w\|_{\mathcal{O},\infty}\leqq 1$. Assume that $H(x, 0)=0$. Then,*

$$\|H(\cdot, w(\cdot))\|_{\mathcal{O},p,N} \leqq C(\mathcal{O}, p, N)\|w\|_{\mathcal{O},p,N}\,.$$

PROOF. Combining Theorem Ap. 1 and the well-known Nirenberg-Gagliardo inequality, we obtain

$$\|D_x^i w\|_{\mathcal{O},kp/i,0} \leqq C(p, \mathcal{O}, i, k)(\|w\|_{\mathcal{O},\infty,0})^{1-(i/k)}(\|w\|_{\mathcal{O},p,k})^{i/k}\,.$$

Thus, by means of the technique which is used to show the well-known Moser's lemma [9], we can show the theorem. Q.E.D.

THEOREM AP. 5. *Assume that all semi-norms appearing below are finite. Let ϕ_0, ϕ_1 and f be data for (P) and v a function defined by (12.1) for ϕ_0, ϕ_1 and f, and $H(t, x, \lambda)$ a $\mathcal{B}^\infty([0, \infty)\times\bar{\Omega}\times\{|\lambda|\leqq 1\})$ function satisfying the condition: $H(t, x, 0)=0$. Then for any p with $1\leqq p\leqq\infty$ and integer $L\geqq 0$*

$$|H(\cdots, \Lambda v)|_{p,0,L} \leqq C(L, p)[\|\phi_0\|_{p,\bar{m}+2+L}+\|\phi_1\|_{p,\bar{m}+1+L}+|f|_{p,0,\bar{m}+L}]\,.$$

PROOF. Since $H(t, x, 0)=0$, we may write symbolically

$$H(t, x, \Lambda v)=\tilde{H}(t, x, \Lambda v)\Lambda v$$

for some $\tilde{H}$. It thus follows that there exists a $\mathcal{B}^{\infty}([0, \infty)\times\bar{\Omega}\times\Gamma)$ function $G(t, x, \gamma)$ (Γ is some compact set) such that $G(t, x, 0)=0$ and

$$H(t, x, \Lambda v)=G(t, x, \bar{D}_x^2 u_{\tilde{m}}(x), \cdots, \bar{D}_x^2 u_0(x)), \tag{Ap. 6}$$

where $u_0=\phi_0$ and $u_1=\phi_1$. Since $\rho(t)=0$ if $t\geqq 1$, we may assume that $G(t, x, \gamma)=0$ if $t\geqq 1$. We have from (Ap. 6) that for any integer $L\geqq 0$

$$D^L H(t, x, \Lambda v)=\sum_{N=0}^{L} D_x^N(\partial_t^{L-N} G(t, x, \bar{D}_x^2 u_{\tilde{m}}(x), \cdots, \bar{D}_x^2 u_0(x)). \tag{Ap. 7}$$

Of course, it follows from the fact: $G(t, x, 0)=0$ that $\partial_t^{L-N}G(t, x, 0)=0$. Therefore, applying Theorem Ap. 4 to (Ap. 7), we have

$$\|D^L H(t, \cdot, \Lambda v)\|_p\leqq C(L, p)\sum_{j=0}^{\tilde{m}}\|\bar{D}_x^2 u_j\|_{p, L}. \tag{Ap. 8}$$

Combining (Ap. 8) and Lemma 10.3, we obtain the desired estimate, which completes the proof.

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Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki, 305 Japan