

A GENERALIZATION OF A RESULT OF K. R. JOHNSON

By

Don REDMOND

In [2], Grytczuk showed that

$$\sum_{d|k} |c_d(n)| = z^{\omega(c_k/(k,n))}(k, n),$$

where $c_k(n)$ denotes Ramanujan's trigonometric sum and $\omega(m)$ counts the number of distinct prime divisors of m . In [3], Johnson evaluated the sum

$$\sum_{d|n} |c_k(d)|.$$

In [4], I generalized the result of Grytczuk to a larger class of functions. In this paper I generalize the result of Johnson.

If h is an arithmetic function, we define the arithmetic function H_k by

$$(1) \quad H_k(n) = \sum_{d|(k,n)} \mu(k/d)h(d).$$

In [4] it is shown that $H_1(n) = h(1)$, if $a \geq 1$,

$$(2) \quad H_{p^a}(n) = \begin{cases} h(p^a) - h(p^{a-1}) & \text{if } p^a | n \\ -h(p^{a-1}) & \text{if } p^{a-1} \parallel n \\ 0 & \text{if } p^{a-1} \nmid n \end{cases}$$

and that $H_k(n)$ is a multiplicative function of k . In [4], we investigated the sum

$$\sum_{d|k} |H_d(n)|.$$

and in this paper we shall investigate the sum

$$\sum_{d|n} |H_k(d)|.$$

Since $H_k(n)$ is not a multiplicative function of n , this task is a little more difficult. We shall assume throughout this paper that h is a multiplicative function.

To make our generalization of Johnson's result as clear as possible we shall follow his notation as closely as possible. In particular, for a given positive integer k we denote by \bar{k} the core of k , that is, the largest square-free divisor of k , and we denote by k^* the integer k/\bar{k} .

LEMMA 1. *Let k be a square-free integer. Then*

$$(3) \quad \mu(k)H_k(n) = \sum_{d_1|(n, k)} h(d)\mu(d).$$

PROOF. Since h and μ are multiplicative functions we see that the right-hand side of (3) is a multiplicative function of n . Suppose $(m, n)=1$. Then

$$\begin{aligned} & \mu(k)H_k(m)\mu(k)H_k(n) \\ &= \mu^2(k) \sum_{d_1|(m, k)} h(d)\mu(k/d) \sum_{d_1|(n, k)} h(d)\mu(k/d) \\ &= \mu^2(k) \sum_{\substack{d_1|(m, k) \\ d_2|(n, k)}} h(d_1)h(d_2)\mu(k/d_1)\mu(k/d_2). \end{aligned}$$

If $(m, n)=1$, then $(d_1, d_2)=1$. Thus $h(d_1)h(d_2)=h(d_1d_2)\mu(k/d_1)\mu(k/d_2)=\mu(k)\mu(k/d_1d_2)$. (The latter result is easily proved from the definition of the Möbius function, μ .) Also $(m, n)=1$ implies that $(m, k)(n, k)=(mn, k)$, and so if $d|(mn, k)$, we can write $d=d_1d_2$, $(d_1, d_2)=1$, so that $d_1|(m, k)$ and $d_2|(n, k)$. Conversely, if $d_1|(m, k)$ and $d_2|(n, k)$, then $d_1d_2|(mn, k)$. Thus

$$(4) \quad \begin{aligned} \mu(k)H_k(m)\mu(k)H_k(n) &= \mu^2(k) \sum_{d_1|(mn, k)} h(d)\mu(k)\mu(k/d) \\ &= \mu^2(k)\mu(k)H_k(mn). \end{aligned}$$

If $\mu(k)=0$, then both sides of (4) equal zero and so are equal to each other. If $\mu(k)\neq 0$, then $\mu^2(k)=1$ and (4) can be written

$$\mu(k)H_k(m)\mu(k)H_k(n) = \mu(k)H_k(mn),$$

that is, $\mu(k)H_k(n)$ is a multiplicative function of n , whether k is square-free or not.

Thus, to prove (3) we need only show that (3) holds when $n=p^r$, a prime power. Since k is square-free we have $(p^r, k)=(p, k)$. Thus we need only consider the case when $n=p$, a prime. We have, for the left-hand side of (3)

$$\begin{aligned} \mu(k)H_k(p) &= \mu(k) \sum_{d_1|(p, k)} \mu(k/d)h(d) \\ &= \begin{cases} \mu^2(k)h(1) & \text{if } p \nmid k \\ \mu^2(k)h(1) + \mu(k)\mu(k/p)h(p) & \text{if } p \mid k \end{cases} \\ &= \begin{cases} 1 & \text{if } p \nmid k \\ 1 - h(p) & \text{if } p \mid k, \end{cases} \end{aligned}$$

since $k=pp_1 \cdots p_r$ being square-free implies that $\mu(k)\mu(k/p)=(-1)^{r+1}(-1)^r=-1$. The right-hand side of (3) is

$$\begin{aligned} \sum_{d|(\overline{p}, k)} h(d)\mu(d) &= \begin{cases} h(1)\mu(1) & \text{if } p \nmid k \\ h(1)\mu(1) + \mu(p)h(p) & \text{if } p \mid k \end{cases} \\ &= \begin{cases} 1 & \text{if } p \nmid k \\ 1 - h(p) & \text{if } p \mid k \end{cases} \end{aligned}$$

Thus, both sides of (3) are equal in this case and the proof is complete.

LEMMA 2. *If h is a completely multiplicative function, we have*

$$H_k(nk^*) = h(k^*)H_{\bar{k}}(n),$$

where \bar{k} and k^* are defined as above.

PROOF. Since $H_k(n)$ is a multiplicative function of k we have, by (2), that

$$H_k(n) = 0 \quad \text{if } k^* \nmid n.$$

Since $k^*\bar{k} = k$ we have

$$\begin{aligned} (5) \quad H_k(nk^*) &= \sum_{d|(\overline{nk^*}, k)} h(d)\mu(k/d) \\ &= \sum_{d|k^*(\overline{n}, \bar{k})} h(d)\mu(k/d) \\ &= \sum_{d|k^*(\overline{n}, \bar{k})} h(d)\mu(k^*\bar{k}/d). \end{aligned}$$

Now $\mu(k/d) = 0$ if $k^* \nmid d$. Thus, from (5), we have

$$\begin{aligned} H_k(nk^*) &= \sum_{d|(\overline{n}, \bar{k})} h(k^*d)\mu(\bar{k}/d) \\ &= h(k^*) \sum_{d|(\overline{n}, \bar{k})} h(d)\mu(\bar{k}/d) \\ &= h(k^*)H_{\bar{k}}(n), \end{aligned}$$

which was to be proved.

LEMMA 3. *If k and n are square-free, then*

$$\mu(k)H_k(n) = \mu(n)H_n(k).$$

PROOF. By Lemma 1 we have

$$\begin{aligned} \mu(k)H_k(n) &= \sum_{d|(k, n)} h(d)\mu(d) \\ &= \sum_{d|(\overline{n}, k)} h(d)\mu(d) \\ &= \mu(n)H_n(k), \end{aligned}$$

which was to be proved.

LEMMA 4. *If h is a completely multiplicative function, $h(n) \neq 0$ and $h(k) \neq 0$ then for all k and n we have*

$$\frac{\mu(\bar{k})}{h(k^*)} H_k(nk^*) = \frac{\mu(\bar{n})}{h(n^*)} H_n(kn^*).$$

PROOF. Since \bar{k} and \bar{n} are square-free, we have, by Lemma 3,

$$(6) \quad \mu(\bar{k}) H_{\bar{k}}(\bar{n}) = \mu(\bar{n}) H_{\bar{n}}(\bar{k}).$$

By (1), we have

$$\begin{aligned} H_{\bar{k}}(\bar{n}) &= \sum_{d | (\bar{k}, \bar{n})} h(d) \mu(\bar{k}/d) \\ &= \sum_{d | (\bar{k}, n)} h(d) \mu(\bar{k}/d) \\ &= H_{\bar{k}}(n), \end{aligned}$$

Since \bar{k} being square-free implies that $(\bar{k}, \bar{n}) = (\bar{k}, n)$. Thus, by (6), we have

$$(7) \quad \mu(\bar{k}) H_{\bar{k}}(n) = \mu(\bar{n}) H_{\bar{n}}(k).$$

By Lemma 2 and (7), we have

$$\begin{aligned} \frac{\mu(\bar{k})}{h(k^*)} H_k(nk^*) &= \frac{\mu(\bar{k})}{h(k^*)} h(k^*) H_{\bar{k}}(n) \\ &= \mu(\bar{k}) H_{\bar{k}}(n) \\ &= \mu(\bar{n}) H_{\bar{n}}(k) \\ &= \frac{\mu(\bar{n})}{h(n^*)} h(n^*) H_{\bar{n}}(k) \\ &= \frac{\mu(\bar{n})}{h(n^*)} H_n(kn^*), \end{aligned}$$

which was to be proved.

LEMMA 5. *Let*

$$F_k(n) = \sum_{d | n} |H_{\bar{d}}(k)|.$$

Then $F_k(n)$ is a multiplicative function of n for any fixed k

PROOF. Let $(m, n) = 1$. Then

$$\begin{aligned} F_k(mn) &= \sum_{d | mn} |H_{\bar{d}}(k)| \\ &= \sum_{\substack{d_1 | m \\ d_2 | n}} |H_{\bar{d}_1 \bar{d}_2}(k)| \end{aligned}$$

$$\begin{aligned}
 &= \sum_{d_1|m} |H_{d_1}(k)| \sum_{d_2|n} |H_{d_2}(k)| \\
 &= F_k(m)F_k(n),
 \end{aligned}$$

since $H_{\bar{d}}(k)$ is a multiplicative function of d for k fixed. This proves the result.

LEMMA 6. *If k is a fixed integer, then*

$$F_k(n) = \prod_{\substack{p^a|n \\ p \nmid k}} (a+1) \prod_{\substack{p^a|n \\ p|k}} (a|h(p)-1|+1).$$

PROOF. By (2), we have

$$H_p(k) = \begin{cases} h(p)-h(1) & \text{if } p \mid k \\ -h(1) & \text{if } p \nmid k \end{cases}$$

Since $F_k(n)$ is a multiplicative function of n , for fixed k , we need only evaluate $F_k(p^a)$. We have

$$\begin{aligned}
 F_k(p^a) &= \sum_{d|p^a} a |H_{\bar{d}}(k)| \\
 &= |H_1(k)| + a |H_p(k)| \\
 &= \begin{cases} 1+a & \text{if } p \nmid k \\ 1+a|h(p)-1| & \text{if } p \mid k, \end{cases}
 \end{aligned}$$

since h being a multiplicative function implies that $h(1)=1$. This proves the result.

THEOREM. *If h is a completely multiplicative function such that $h(n) \neq 0$, then*

$$\sum_{d|n} |H_k(d)| = \begin{cases} 0 & \text{if } k^* \nmid n \\ |h(k^*)| \prod_{\substack{p^a|n/k^* \\ p \nmid k}} (a+1) \prod_{\substack{p^a|n/k^* \\ p|k}} (a|h(p)-1|+1) & \text{if } k^* \mid n. \end{cases}$$

PROOF. If $k^* \nmid n$, then $k^* \nmid d$ for and $d|n$, and so $H_k(d)=0$ for all $d|n$. This gives the first result. Suppose $k^*|n$. Then, by Lemma 4,

$$\begin{aligned}
 \sum_{d|n} |H_k(d)| &= \sum_{d_1|n/k^*} |H_k(d_1 k^*)| \\
 &= \sum_{d_1|n/k^*} \left| \frac{H_a(k d^*) \mu(\bar{d}) h(k^*)}{h(d^*) \mu(\bar{k})} \right| \\
 &= |h(k^*)| \sum_{d_1|n/k^*} \left| \frac{H^d(k d^*)}{h(d^*)} \right|
 \end{aligned}$$

$$\begin{aligned}
 &= |h(k^*)| \sum_{d|n/k^*} |H_d(k)| \\
 &= |h(k^*)| F_k(n/k^*)
 \end{aligned}$$

by Lemma 2. The result follows from Lemma 6 and completes the proof.

We now give some examples.

1. Let r be a real number and let $h(n)=n^r$. Then h is a completely multiplicative function, and so, by the Theorem, we have

$$\sum_{d|n} |H_k(d)| = \begin{cases} 0 & \text{if } k^* \nmid n \\ k^{*r} \prod_{p|k} (a+1) \prod_{p|n/k^*} (a|p^r-1|+1) & \text{if } k^* | n. \end{cases}$$

The case $r=1$ gives the result of Johnson, [4]. If we take $r=0$ we have $h(n)=1$ for all n and (8) becomes

$$\sum_{d|n} |H_k(d)| = \begin{cases} 0 & \text{if } k^* \nmid n \\ \prod_{p|k} (a+1) & \text{if } k^* | n. \end{cases}$$

2. Let z be a complex number and let $h(n)=z^{\Omega(n)}$, where $\Omega(n)$ counts the total number of prime divisors of n . Then it is not hard to show from the definition of Ω to show that h is a completely multiplicative function. Thus

$$(9) \quad \sum_{d|n} |H_k(d)| = \begin{cases} 0 & \text{if } k^* \nmid n \\ |z|^{\Omega(k^*)} \prod_{p|k} (a+1) \prod_{p|n/k^*} (a|z-1|+1) & \text{if } k^* | n \end{cases}$$

If we take the special case $z=2$, then (9) becomes

$$\begin{aligned}
 \sum_{d|n} |H_k(d)| &= \begin{cases} 0 & \text{if } k^* \nmid n \\ 2^{\Omega(k^*)} \prod_{p|n/k^*} (a+1) & \text{if } k^* | n \end{cases} \\
 &= \begin{cases} 0 & \text{if } k^* \nmid n \\ 2^{\Omega(k^*)} d(n/k^*) & \text{if } k^* | n, \end{cases}
 \end{aligned}$$

where $d(m)$ counts the number of divisors of m .

3. If X is a character modulo q and $h(n)=X(n)$, then $h(n)$ is a completely multiplicative function. If $(q, n)=1$, then

$$\sum_{d|n} |H_k(d)| = \begin{cases} 0 & \text{if } k^* \nmid n \\ \prod_{p|k} (a+1) \prod_{p|n/k^*} (a|X(p)-1|+1) & \text{if } k^* | n \end{cases}$$

One can further refine the sum by noting that if $(k, q) > 1$, then $p|k$ implies

that $X(p)=0$. Thus the two products combine to give $d(n/k^*)$.

Finally, we remark that like the result in [4] it seems likely that the result obtained in this paper can be further extended to the class of arithmetic functions considered by Anderson and Apostol in [1]. We hope to return to this in a later paper.

References

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Don REDMOND
Department of Mathematics
Southern Illinois University
Carbondale, IL. 62901
USA