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GORENSTEIN BALANCE OF HOM AND TENSOR

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By Auslander and Bridger ([1]), Proposition 3.8c)), a finitely generated left R-module C has Gorenstein dimension 0 if and only if there is an exact sequence

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow P^{1} \longrightarrow \cdots$$

of finitely generated projective left R-modules such that $C=\ker(P^{0}\rightarrow P^{1})$ and such that the dual sequence

$$\cdots \longrightarrow (P^{1})^{*} \longrightarrow (P^{0})^{*} \longrightarrow (P^{-1})^{*} \longrightarrow \cdots$$

is also exact.

In attempting to dualize the notion of Gorenstein dimension we called such modules C Gorenstein projective modules (see [8]) and then defined Gorenstein injective modules.

Auslander and Buchweitz showed that over certain rings all finitely generated modules have Gorenstein projective precovers (over Cohen-Macauley rings these are their maximal Cohen-Macauley approximations).

An application of their argument shows that over an n-Gorenstein ring all modules have Gorenstein injective preenvelopes.

Over a ring where these precovers and preenvelopes exist, we can apply methods of relative homological algebra (see Eilenberg and Moore [5]) and compute derived functors.

We can then raise the question of balance in the sense of Enochs and Jenda [6]. We can now show that Hom$(-, -)$ is right balanced by Gorenstein projective and injective modules on a suitable category. Similarly we show that $-\otimes-$ is left balanced by finitely generated Gorenstein projective modules (left and right).

1. Gorenstein Injective and Projective Resolutions.

In the following, module will mean left R-module for some ring R (unless otherwise specified).

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DEFINITION 1.1. A left R-module G is said to be Gorenstein injective if there is an exact sequence

$$\cdots \longrightarrow E^{-1} \longrightarrow E^{0} \longrightarrow E^{1} \longrightarrow E^{2} \longrightarrow \cdots$$

of injective left R-modules such that $G=\ker(E^{0}\rightarrow E^{1})$ and such that for every injective left R-module E, the sequence

$$\cdots \operatorname{Hom}(E, E^{-1}) \longrightarrow \operatorname{Hom}(E, E^{0}) \longrightarrow \operatorname{Hom}(E\text{ `}E^{1}) \longrightarrow \cdots$$

is also exact.

REMARK 1.2. In [8] it was argued that if R is left noetherian, every direct summand of a Gorenstein injective module is Gorenstein injective. In ([1], 3.11(b)) it was shown that a similar result for finitely generated Gorenstein projective modules holds.

LEMMA 1.3. *If G is a Gorenstein injective left R-module and L is a left R-module of finite projective dimension then* $\operatorname{Ext}^{i}(L, G)=0$ *for all* $i\geqq 1$.

PROOF. Letting $\cdots E^{-1}\rightarrow E^{0}\rightarrow E^{1}\rightarrow\cdots$ be as above and letting $G^{-j}=\ker(E^{-j}\rightarrow E^{-j+1})$ for $j\geqq 0$ we see that

$$\operatorname{Ext}^{i}(L, G)=\operatorname{Ext}^{i+j}(L, G^{-j})$$

for each such j. But for $j\geqq \text{proj. dim}\, L$, $\operatorname{Ext}^{i+j}(L, G^{-j})=0$ when $i\geqq 1$. A similar argument gives

LEMMA 1.4 (Auslander, Bridger [1]). *If C is a finitely generated Gorenstein projective module and L has finite injective dimension, then* $\operatorname{Ext}^{i}(C, L)=0$ *for all* $i\geqq 1$.

DEFINITION 1.5 (Iwanaga [10]). A ring R is said to be i.e. n-Gorenstein if it is left and right noetherian and if it has self-injective dimension at most n on both sides.

Over an n-Gorenstein ring R there are a plentiful supply of Gorenstein injective modules. This follows from

LEMMA 1.6 ([8], Theorem 4.2). *If R is an n-Gorenstein ring and*

$$0 \longrightarrow N \longrightarrow E^{0} \longrightarrow E^{1} \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow G \longrightarrow 0$$

is a partial injective resolution of N, then G is Gorenstein injective.

REMARK. If G is a finite group, it is easy to see that the group ring ZG

is 1-Gorenstein, for

$$0 \longrightarrow ZG \longrightarrow QG \longrightarrow Q/ZG \longrightarrow 0$$

is an exact sequence of left or right ZG-modules.

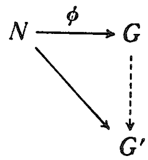
But $QG = Q \otimes_Z ZG \equiv \mathrm{Hom}_Z(ZG, Q)$ (see [3]) is an injective ZG-module. Similarly Q/ZG is injective.

From this we can show that a ZG-module M is Gorenstein injective if and only if it is divisible as a Z-module.

The condition is necessary since every injective ZG-module is divisible and M is a quotient of an injective module. Conversely, if M is divisible, $\mathrm{Hom}_Z(ZG, M)$ is injective. But $\mathrm{Hom}_Z(ZG, M) \equiv ZG \otimes_Z M$ and there is surjection $ZG \otimes_Z M \to M$. Hence by the preceeding lemma, G is Gorenstein injective.

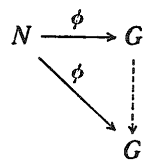
A similar argument shows that a finitely generated ZG-module C is Gorenstein projective if and only if it is a free Z-module.

DEFINITION 1.7 (Enochs [7]). If N and G are left R-modules and G is Gorenstein injective, then a linear map $\phi : N \to G$ is called a Gorenstein injective preenvelope of N if the diagram



can be completed to a commutative diagram whenever $N \to G'$ is a map into a Gorenstein injective module G'.

If furthermore,



can only be completed by automorphisms of G, then $\phi : N \to G$ is called a Gorenstein injective envelope of N.

Dually, a Gorenstein projective precover and cover $\phi : C \to M$ is defined (with C and M finitely generated and C Gorenstein projective).

We note that since injective modules are Gorenstein injective and every modules N is contained in an injective module, every Gorenstein injective preenvelope $N\to G$ is an injection. Similarly, Gorenstein projective covers $C\to M$ must be surjective.

PROPOSITION 1.8. *If R is n-Gorenstein, every R-module N has a Gorenstein injective preenvelope.*

PROOF. We use an argument dual to that in the proof of Theorem 1.1 of [2]. Let X be the class of Gorenstein injective modules and let w be the class of injective modules. Note that by [10] every injective module has finite projective dimension. So in the language of [2], w is a generator for X. Lemma 1.6 above then gives all that is needed for the dual argument to carry through. Hence for each R-module N there is an exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow L \longrightarrow 0$$

with G Gorenstein injective and L of finite projective dimension.

Then if G' is Gorenstein injective, $\mathrm{Ext}^1(L, G')=0$ so $\mathrm{Hom}(G, G')\to \mathrm{Hom}(N, G')\to 0$ is exact. This means that $N\to G$ is a Gorenstein injective preenvelope.

REMARK 1.9. There is an alternate proof of this result in [8] (see Theorem 7.2). We note that there, under our hypotheses, we get a somewhat stronger uniqueness result than that in [2] for the X-approximations.

Now we use Theorem 1.8 of [2] and get that if R is n-Gorenstein, then for every finitely generated module M there is an exact sequence

$$0 \longrightarrow L \longrightarrow C \longrightarrow M \longrightarrow 0$$

with C finitely generated and Gorenstein projective and L having finite projective dimension (and so finite injective dimension by [10]).

Since then by Lemma 1.4 $\mathrm{Ext}^1(C, L)=0$ whenever C' is finitely generated and Gorenstein projective, we get that $C\to M$ is a Gorenstein projective precover of M.

DEFINITION 1.10. An exact sequence

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \cdots$$

is called a Gorenstein injective resolution of the module N if each G^i is Gorenstein injective and if $N\to G^0$, $\ker(G^i\to G^{i+1})\to G^i$ for $i\geqq 0$ are all Gorenstein injective preenvelopes (equivalently, whenever G is a Gorenstein injective module, the sequence

$$\cdots \mathrm{Hom}(G^1, G) \longrightarrow \mathrm{Hom}(G^0, G) \longrightarrow \mathrm{Hom}(N, G) \longrightarrow 0$$

is exact).

In a similar way we define a Gorenstein projective resolution

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

of the finitely generated module M (with all the C_i finitely generated too).

From Proposition 1.8 above we see that if R is n-Gorenstein then every module N admits a Gorenstein injective resolution. Also from the above we see that every finitely generated module M admits a Gorenstein projective resolution. Such resolutions are unique up to homotopy.

PROPOSITION 1.11. *If R is n-Gorenstein and G is an R-module then the following are equivalent:*

1) *G is Gorenstein injective*
2) $\mathrm{Ext}^i(L, G)=0$ *for all modules L with* $\mathrm{pd}\, L<\infty$ *and all* $i\geq 1$
3) $\mathrm{Ext}^1(L, G)=0$ *for all modules L with* $\mathrm{pd}\, L<\infty$.

PROOF. 1)⇒2) by 2)⇒3) is trivial.

3)⇒1). By the proof of Proposition 1.8 we have an exact sequence $0\to G \to \bar{G}\to L\to 0$ with $\bar{G}$ Gorenstein injective and $\mathrm{pd}\, L<\infty$. By hypotheses this sequence splits so as a direct summand of a Gorenstein injective module G is also Gorenstein injective.

2. Balance of Hom (—, —).

For a ring R we now let $\mathcal{FG}$ Gor Proj, Gor Inj, $\mathcal{FG}$ and Mod denote (respectively) the categories of finitely generated Gorenstein projective modules, Gorenstein injective modules, finitely generated modules and all modules.

We now use the language of [6] and attempt to justify our choice of terminology (Gorenstein projective and injective) by proving:

THEOREM 2.1. *If R is n-Gorenstein then* Hom(—, —) *is right-balanced by* $\mathcal{FG}$ Gor Proj×Gor Inj on $\mathcal{FG}$×Mod.

PROOF. We only need prove that if $N\to G$ is any Gorenstein injective preenvelope of N and if $0\to N\to G\to L\to 0$ is exact, then Hom$(C, -)$ leaves the sequence exact for any finitely generated Gorenstein projective module C, and also the corresponding dual result.

Since we know there is some such preenvelope with inj. $\dim L<\infty$ we first prove the result under this assumption. By [10], inj. dim $L \leqq n$. Let $0 \to L \to E^0 \to E^1 \to \cdots \to E^n \to 0$ be an injective resolution of L. Since C is Gorenstein projective, there is an exact sequence

$$0 \longrightarrow C \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^{n-1} \longrightarrow P^n$$

with the P^i's finitely generated and projective. Let $D = \text{im}(P^{n-1} \to P^n)$. Any map $C \to L$ then gives rise to a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C & \longrightarrow & P^0 & \longrightarrow \cdots \longrightarrow & P^{n-1} & \longrightarrow & D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & E^0 & \longrightarrow \cdots \longrightarrow & E^{n-1} & \longrightarrow & E^n & \longrightarrow & 0 \end{array}$$

But $D \to E^n$ can be extended to $P^n \to E^n$ which in turn can be lifted to a map $P^n \to E^{n-1}$. Then the usual way of constructing homotopies gives an extension $P^0 \to L$. But $P^0 \to L$ can be lifted to a map $P^0 \to G$ which by restriction gives the required lifting $C \to G$.

In the general case, if $0 \to N \to G \to L \to 0$ and $0 \to N \to G' \to L' \to 0$ are both exact and given by Gorenstein injective preenvelopes of N, then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

giving a homotopy equivalence of the two complexes. But then

$$0 \longrightarrow \text{Hom}(C, N) \longrightarrow \text{Hom}(C, G) \longrightarrow \text{Hom}(C, L) \longrightarrow 0$$

and

$$0 \longrightarrow \text{Hom}(C, N) \longrightarrow \text{Hom}(C, G') \longrightarrow \text{Hom}(C, L') \longrightarrow 0$$

are homotopically equivalent.

Hence if one has zero homology, so does the other, i.e. if one sequence is exact so is the other.

As a consequence we get that $\text{Hom}(C, -)$ leaves any Gorenstein resolution $0 \to N \to G^0 \to G^1 \to \cdots$ exact. A dual argument gives that when G is Gorenstein injective then $\text{Hom}(-, G)$ leaves any Gorenstein projective resolution $\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$ (by finitely generated Gorenstein projective modules C_i) exact.

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3. Dimensions.

A left R-module N is said to have Gorenstein injective dimension $\leqq n$(G-id $N \leqq n$) if there is a Gorenstein injective resolution

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots G^n \longrightarrow 0$$

of N. In a similar manner we define the Gorenstein projective dimension of a finitely generated module to be $\leqq n$(G-pd $M \leqq u$).

THEOREM 3.2. *If R is left and right noetherian, then the following are equivalent:*

1) *R is n-Gorenstein.*
2) *for every R-module N (left or right), N has a Gorenstein injective preenvelope and G-id $N \leqq n$.*
3) *every nth cosyzygy of an R-module (left or right) is Gorenstein injective.*
4) *every finitely generated R-module M (left or right) has a Gorenstein projective precover and G-pd $M \leqq n$.*
5) *every nth syzygy in a projective resolution of a finitely generated R-module (left or right) by finitely generated projective modules is Gorensteiv projective.*

PROOF. 1)$\Rightarrow$2). Let $0 \to N \to G^0 \to G^1 \to \cdots \to G^{n-1}$ be a partial Gorenstein injective resolution of N. We argue that $G = \mathrm{Coker}(G^{n-2} \to G^{n-1})$ (or Coker $(N \to G^0)$ if $n=1$) is Gorenstein injective. By if pd $L < \infty$ then pd $L \leqq n$. Hence $\mathrm{Ext}^1(L, G) = \mathrm{Ext}^{n+1}(L, N) = 0$. So by Prop. 1.11, G is Gorenstein injective.

2)$\Rightarrow$3). Given the partial injective resolution $0 \to N \to E^0 \to \cdots \to E^{n-1}$ let $G =$ Coker $(E^{n-2} \to E^{n-1})$ (or$=$Coker $(N \to E^0)$ if $n=1$). We argue that G is Gorenstein injective.

Let pd $L < \infty$. Then since we have a Gorenstein injective resolution $0 \to N \to G^0 \to \cdots \to G^n \to 0$ and since $\mathrm{Ext}^i(L, G') = 0$ when G' is Gorenstein injective and $i \geqq 1$, we see that $\mathrm{Ext}^{n+1}(L, N) = 0$. Hence pd $L \leqq n$. But then an argument as in 1)$\Rightarrow$2) gives $\mathrm{Ext}^1(L, G) = 0$.

3)$\Rightarrow$1) by Enochs-Jenda ([9], Theorem 2.4).

1)$\Rightarrow$4) and 4)$\Rightarrow$5) are by arguments dual to those for 1)$\Rightarrow$2) and 2)$\Rightarrow$3), 5)$\Rightarrow$ 1) by ([9], Theorem 2.4).

4. Gext.

If R is Gorenstein then the right derived functors of Hom$(-, -)$ computed using either a resolution of M by finitely generated Gorenstein projective modules or one of N by Gorenstein injective modules will be denoted $G\,\mathrm{ext}^i(M, N)$.

Then it is easy to check the following properties of G ext:

a) $G\,\mathrm{ext}^0(-, -) \equiv \mathrm{Hom}(-, -)$

b) $G\,\mathrm{ext}^i(C, -)=0$ if $i \geqq 1$ and C is a finitely generated Gorenstein projective module.

c) $G\,\mathrm{ext}^i(-, G)=0$ for $i \geqq 1$ and G a Gorenstein injective module.

d) For any short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated modules left exact by $\mathrm{Hom}(C, -)$ when C is finitely generated and Gorenstein projective gives rise to a long exact sequence

$$\cdots \to G\,\mathrm{ext}^i(M'', -) \to G\,\mathrm{ext}^i(M, -) \to G\,\mathrm{ext}^i(M'\,'-) \to G\,\mathrm{ext}^{i+1}(M'', -) \to \cdots$$

(We note that the condition above is exactly what is needed to construct a commutative diagram

$$\begin{array}{ccccccccc} & & \ddots & & \ddots & & \ddots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_0' & \longrightarrow & C_0 & \longrightarrow & C_0'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

satisfying the obvious conditions).

e) Any short exact sequence $0 \to N' \to N \to N'' \to 0$ left exact by $\mathrm{Hom}(-, G)$ whenever G is Gorenstein injective gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}^i(-, N') \to \mathrm{Ext}^i(-, N) \to \mathrm{Ext}^i(-, N'') \to \mathrm{Ext}^{i+1}(-, N') \to \cdots$$

f) Then are natural transformations

$$G\,\mathrm{ext}^i(-, -) \longrightarrow \mathrm{Ext}^i(-, -)$$

which are also natural in the exact sequences as in d) and e).

PROPOSITION 4.1. *If R is n-Gorenstein then for a left R-module L the following are equivalent*:

1) proj. dim $L < \infty$ (*and* so $\leqq n$)

2) $G\,\mathrm{ext}^i(L, -) \to \mathrm{Ext}^i(L, -)$ *is an isomorphism for all* $i \geqq 0$

3) $\mathrm{Ext}^1(L, G)=0$ *for all Gorenstein injective modules* G.

PROOF. 1)$\Rightarrow$3) is part of Corollary 4.4 of [8].

3)$\Rightarrow$2). 3) implies $\mathrm{Ext}^i(L, G)=0$ for all Gorenstein injective modules G and $i\geqq 1$, again by Corollary 4.4 of [8]. Furthermore $G\,\mathrm{ext}^i(L, G)=0$ for all such G and $i\geqq 1$. So for $i\geqq 0$ we get a commutative diagram

$$\begin{array}{ccccccc} G\,\mathrm{ext}^i(L, G) & \longrightarrow & G\,\mathrm{ext}^i(L, H) & \longrightarrow & G\,\mathrm{ext}^{i+1}(L, N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Ext}^i(L, G) & \longrightarrow & \mathrm{Ext}^i(L, H) & \longrightarrow & \mathrm{Ext}^{i+1}(L, N) & \longrightarrow & 0 \end{array}$$

For $i=0$, the two first vertical maps are isomorphism since $\mathrm{Ext}^0(-, -)= G\,\mathrm{ext}^0(-, -)=\mathrm{Hom}(-, -)$. Hence we get $G\,\mathrm{ext}^1(L, N)\to\mathrm{Ext}^1(L, N)$ is an isomorphism. Then induction on i gives the desired result.

2)$\Rightarrow$1). Since R is n-Gorenstein, $G\text{-id}\,N\leqq n$ for all modules N by Theorem 3.2 above. So $G\,\mathrm{ext}^i(L, N)=0$ for $i\geqq n+1$ and all N. Hence $\mathrm{Ext}^{n+1}(L, N)=0$ for all N and so $\mathrm{pd}\,L\leqq n$.

REMARK 4.2. If R is n-Gorenstein, for all left R-modules M and N $G\,\mathrm{ext}^1(M, N)\to\mathrm{Ext}^1(M, N)$ is an injection.

PROOF. If $N\to G$ is a Gorenstein injective envelope of N let $0\to N\to G\to H\to 0$ be the associated exact sequence. Then we get a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(M, G) & \longrightarrow & \mathrm{Hom}(M, H) & \longrightarrow & G\,\mathrm{ext}^1(M, N) & \longrightarrow & 0 \\ \| & & \| & & \downarrow & & \\ \mathrm{Hom}(M, G) & \longrightarrow & \mathrm{Hom}(M, H) & \longrightarrow & \mathrm{Ext}^1(M, N) & & \end{array}$$

with exact rows. The result then follows.

If we consider the elements of $\mathrm{Ext}^1(M, N)$ as classes of short exact sequences $0\to N\to L\to M\to 0$ we get that

COROLLARY 4.3. *The short exact sequence* $0\to N\to L\to M\to 0$ *corresponds to an element of* $G\,\mathrm{ext}^1(M, N)\subset\mathrm{Ext}^1(M, N)$ *is and only if for every Gorenstein injective module* G, $\mathrm{Hom}(L, G)\to\mathrm{Hom}(N, G)\to 0$ *is exact.*

PROOF. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \bar{K} & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows and $N\to G$ a Gorenstein injective preenvelope of N and E an injective module. Given a map $M\to K$, using a pull-back diagram we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

and so get the element corresponding to $0 \to N \to L \to M \to 0$ of $G\,\mathrm{ext}^1(M, N)$. Then if G' is Gorenstein injective, any map $N \to G'$ gives rise to a map $G \to G'$ (since $N \to G$ is a preenvelope) and so we get the desired map $L \to G'$.

Now given a map $M \to \bar{K}$ form the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \bar{K} & \longrightarrow & 0 \end{array}$$

with exact rows. If $0 \to N \to L \to M \to 0$ has the desired property, then the map $N \to G$ gives rise to a map $L \to G$ and so to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

This shows that $0 \to N \to L \to M \to 0$ corresponds to an element of $G\,\mathrm{ext}^1(M, N)$ and completes the proof.

If M is taken finitely generated in the previous result, a dual argument gives that the elements of $G\,\mathrm{ext}^1(M, N)$ correspond to the sequences $0 \to N \to L \to M \to 0$ such that $\mathrm{Hom}(C, L) \to \mathrm{Hom}(C, M) \to 0$ is exact when C is a finitely generated Gorenstein projective module. vpp

5. G tor.

LEMMA 5.1. *If R is n-Gorenstein then for a left R-module C the following are equivalent:*

1) *C is Gorenstein projective*
2) $\mathrm{Ext}^i(C, L)=0$ *for all L with* $\mathrm{id}\, L<\infty$ *and all* $i \geqq 1$
3) $\mathrm{Ext}^1(C, L)=0$ *for all L with* $\mathrm{id}\, L<\infty$
4) $\mathrm{Tor}_i(L, C)=0$ *for all right R-modules L with* $\mathrm{id}\, L<\infty$ *and* $i \geqq 1$
5) $\mathrm{Tor}_1(L, C)=0$ *for all right R-modules L with* $\mathrm{pd}\, L<\infty$
6) *$\mathrm{Hom}_z(C, Q/Z)$ is a Gorenstein injective right R-module.*

PROOF. 1)⇒2) is by 2)⇒3) is trivial.

3)⇒1). By [2], there is an exact sequence $0 \to L \to \bar{C} \to C \to 0$ with $\bar{C}$ a finitely generated Gorenstein projective module and $\mathrm{id}\, L < \infty$. By 3), this splits and so C is Gorenstein projective.

2)⇒4). Letting $M^+ = \mathrm{Hom}_Z(M, Q/Z)$ for any left or right R-module, we have

$$\mathrm{Tor}_i(L, C)^+ = \mathrm{Ext}^i(C, L^+)$$

If $\mathrm{pd}\, L < \infty$ then $\mathrm{id}\, L^+ < \infty$, so $\mathrm{Ext}^i(C, L^+) = 0$ for $i \geqq 1$ and $\mathrm{pd}\, L < \infty$. Hence $\mathrm{Tor}_i(L, C) = 0$ for such i and L.

4)⇒5) is trivial.

5)⇒6) and 6)⇒3). $\mathrm{Tor}_1(L, C) = 0$ if and only if $\mathrm{Ext}^1(L, C^+) = \mathrm{Tor}_1(L, C)^+ = 0$ so using Proposition 1.11 we get our claims.

As a result we get

THEOREM 5.2. *If R is n-Gorenstein then $-\otimes-$ is left balanced on $\mathcal{FG} \times \mathcal{FG}$ (finitely generated right R-modules for the first $\mathcal{FG}$ and left for the second) by* $\mathcal{FG}$ Gor Proj $\times$ $\mathcal{FG}$ Gor Proj *(again left and right).*

PROOF. Let $\cdots \to C_1 \to C_0 \to M \to 0$ be a Gorenstein projective resolution of a finitely generated right R-module and let D be a finitely generated Gorenstein projective left R-module. Then

$$\cdots \longrightarrow C_1 \otimes D \longrightarrow C_0 \otimes D \longrightarrow M \otimes D \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow (M \otimes D)^+ \longrightarrow (C_0 \otimes D)^+ \longrightarrow (C_1 \otimes D)^+ \longrightarrow \cdots$$

i.e. if and only if

$$0 \longrightarrow \mathrm{Hom}(M, D^+) \longrightarrow \mathrm{Hom}(C_0, D^+) \longrightarrow \mathrm{Hom}(C_1, D^+)$$

is exact. But D^+ is Gorenstein injective by the Lemma 5.1. Hence this sequence is exact by Theorem 2.1.

So we can now compute left derived functors of $-\otimes-$ computed using Gorenstein projective resolutions of finitely generated modules (see [6]).

These derived functors will defined denoted $G\,\mathrm{tor}_i(M, N)$.

Then it is easy to check the following:

a) $G\,\mathrm{tor}_0(-, -) \cong - \otimes -$

b) $G\,\mathrm{tor}_i(C, -) = 0$ if $i \geqq 1$ and C is a finitely generated Gorenstein projective right R-module.

c) $G\,\mathrm{tor}_i(-, D) = 0$ if $i \geqq 1$ and D is a finitely generated Gorenstein projective left R-module

d) for any exact sequence $0\to M'\to M\to M''\to 0$ of finitely generated right R-modules which remains exact when $-\otimes D$ is applied to it with finitely generated Gorenstein projective, there is a long exact sequence

$$\cdots\to G\,\mathrm{tor}_{i+1}(M'', -)\to G\,\mathrm{tor}_i(M', -)\to G'\,\mathrm{tor}_i(M, -)\to G\,\mathrm{tor}_i(M'', -)\to\cdots$$

e) same as d) but for an exact sequence of left R-modules $0\to N'\to N\to N''\to 0$

f) there are natural transformations

$$\mathrm{Tor}_i(-, -)\longrightarrow G\,\mathrm{tor}_i(-, -)$$

for each $i\geqq 0$ and these natural transformations commute with the connecting homomorphisms associated with short exact sequences as in d) and e). If $i=0$ the natural transformations is an isomorphism.

PROPOSITION 5.3. *If R is n-Gorenstein then for a finitely generated right R-module L the following are equivalent*:

1) $\mathrm{pd}\,L<\infty$ (*so* $\mathrm{pd}\,L\leqq n$)
2) $\mathrm{Tor}_i(L, -)\to G\,\mathrm{tor}_i(L, -)$ *is an isomorphism for all* $i\geqq 0$
3) $\mathrm{Tor}_1(L, D)=0$ *fcr all finitely generated Gorenstein projective modules* D
4) $\mathrm{Tor}_i(L, D)=0$ *for all finitely generated Gorenstein projective modules* D *and all* $i\geqq 1$.

PROOF. 1)$\Rightarrow$4). $\mathrm{Tor}_i(L, D)^+=\mathrm{Ext}^i(L, D^+)$. By Lemma 5.1 6), D^+ is Gorenstein injective. Then Lemma 1.3 says $\mathrm{Ext}^i(L, D^+)=0$ for $i\geqq 1$ and so $\mathrm{Tor}_i(L, D)=0$ for $i\geqq 1$.

4)$\Rightarrow$3) is trivial.

2)$\Rightarrow$1). By Theorem 3.2 $G-\mathrm{pd}\,L\leqq n$ so $G\,\mathrm{tor}_i(L, N)=0$ for $i\geqq n+1$. Therefore $\mathrm{Tor}_i(L, N)=0$ for all $i\geqq n+1$ and all n. Hence ph $L\leqq n$.

4)$\Rightarrow$2). This is standard. For example, it is straightforward modification of ([4], Proposition 4.4).

3)$\Rightarrow$4). By the definition of a Gorenstein projective module, there is an exact sequence $0\to K\to P\to D\to 0$ with P finitely generated and projective and K Gorenstein projective. So $\mathrm{Tor}_2(L, D)=\mathrm{Tor}_1(K, D)=0$. Similarly we get $\mathrm{Tor}_i(L, D)=0$ for all $i\geqq 1$.

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