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A SEQUENCE ASSOCIATED WITH THE ZEROS OF THE RIEMANN ZETA FUNCTION

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Let $\zeta(s)$ be the Riemann zeta function, let C_n denote the generalized Euler constants associated with $\zeta(s)$, i. e.,

$$C_n=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\log^n k/k-\log^{n+1}N/(n+1)\right\},$$

and define the numbers $\delta_n(n\geqslant 1)$ by

$$\delta_1=1+\frac{1}{2}C_0-\frac{1}{2}\log\pi-\log 2,$$

and, for $n\geqslant 2$,

$$\delta_n=1-(1-2^{-n})\zeta(n)+n\sum_{h=1}^{n}\frac{1}{h}\sum_{\substack{j_1+\cdots+j_h=n-h\\ j_1\geqslant 0,\cdots,j_h\geqslant 0}}\prod_{b=1}^{h}\frac{C_{j_b}}{j_b!}. \tag{1}$$

Then we have, for $n\geqslant 1$,

$$\delta_n=\sum_{\rho}\rho^{-n}, \tag{2}$$

where the sum Σ_ρ is taken over all complex zeros ρ of $\zeta(s)$(see [2]).

In this paper we shall study the sequence $\{\delta_n\}$ and prove four theorems. In Theorem 1 we shall give an expression of δ_n, and, using it, we shall derive, in Theorem 2, a necessary and sufficient condition for the truth of the Riemann hypothesis. The expression of δ_n will be specified in Theorem 3 under the Riemann hypothesis, and in the final Theorem 4, we shall give an upper bound for the quantity $|\delta_n|$.

We start with the following lemma.

LEMMA. *Let m be a positive integer, let $a_j(j=1, 2, \cdots, m)$ be real numbers, and let $b_j(j=1, 2, \cdots, m)$ be mutually distinct positive numbers $<\pi$. If*

$$\sum_{j=1}^{m}a_j\cos b_jn\longrightarrow 0,$$

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as an integer $n\to\infty$, then $a_j=0$ for all $j=1, 2, \cdots, m$.

PROOF. Suppose that $a_k\neq 0$ for some integer k with $1\leqslant k\leqslant m$, and let ε be a positive number$<\frac{1}{4}|a_k|$. Then, from the condition of the lemma, there exists a positive integer n_0 such that the inequality

$$(3)\qquad \left|\sum_{j=1}^{m} a_j \cos b_j n\right|<\varepsilon$$

holds for all integers $n\geqslant n_0$. Let N be an arbitrary positive integer. If we multiply both sides of (3) by $|2\cos b_k n|$, and add them with respect to n form n_0+1 to n_0+N, then we get

$$\sum_{j=1}^{m} a_j \sum_{n=n_0+1}^{n_0+N} \{\cos(b_k-b_j)n+\cos(b_k+b_j)n\}=r, \text{ say,}$$

with $|r|<2\varepsilon N$. It follows that

$$(4)\qquad a_k\sum_{n=n_0+1}^{n_0+N}\cos 2b_k n+\sum_{\substack{j=1\\ j\neq k}}^{m} a_j \sum_{n=n_0+1}^{n_0+N}\{\cos(b_k-b_j)n+\cos(b_k+b_j)n\}=r-Na_k.$$

Since b_j are mutually distinct positive numbers$<\pi$, $2b_k\not\equiv 0 \pmod{2\pi}$ and $b_k\pm b_j\not\equiv 0 \pmod{2\pi}$ for any pair of k and j with $k\neq j$, so that the left hand side of (4) is $o(N)$, as $N\to\infty$. On the other hand, the absolute value of the right hand side is greater than $\frac{1}{2}|a_k|N$, and hence the equation (4) yields a contradiction. This completes the proof.

THEOREM 1. *For any $n\geqslant 1$, δ_n can be uniquely expressed in the form*

$$(5)\qquad \delta_n=2\sum_{k=1}^{\infty} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}\cos b_{kj} n,$$

where $\{r_k\}$ is an increasing sequence of positive numbers, $m_k (k\geqslant 1)$ and $a_{kj} (j=1, 2, \cdots, m_k)$ are positive integers, and $b_{kj} (j=1, 2, \cdots, m_k)$ are mutually distinct positive numbers$<\frac{1}{2}\pi$ such that $r_k \cos b_{kj}<1$. The constants r_k, m_k, a_{kj} and b_{kj} are independent of n.

PROOF. If we put $\rho=\beta+i\gamma$ in (2), we get

$$\delta_n=2\sum_{\gamma>0}(\beta^2+\gamma^2)^{-\frac{1}{2}n}\cos\left(n\arctan\frac{\gamma}{\beta}\right),$$

noting that the complex conjugate of ρ is also the zero of $\zeta(s)$. If we make use of the fact that $0<\beta<1$, we get $0<\arctan(\gamma/\beta)<\frac{1}{2}\pi$, and $(\beta^2+\gamma^2)^{1/2}\cos\{\arctan(\gamma/\beta)\}<1$. Hence, we can obtain the expression (5).

It remains to show the uniqueness. Suppose that δ_n has two expressions

$$(6)\qquad \delta_n=2\sum_{k=1}^{\infty} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}\cos b_{kj}n=2\sum_{k=1}^{\infty} R_k^{-n}\sum_{j=1}^{M_k} A_{kj}\cos B_{kj}n.$$

Then we can get

(7) $$r_1^{-n}\sum_{j=1}^{m_1} a_{1j}\cos b_{1j}n+O(r_2^{-n})=R_1^{-n}\sum_{j=1}^{M_1} A_{1j}\cos B_{1j}n+O(R_2^{-n}).$$

To prove (7), we make use of the estimate ([4], Theorem 9. 4.)

$$N(T)=\frac{T}{2\pi}\log T-\frac{1}{2\pi}(1+\log 2\pi)T+O(\log T),$$

where $N(T)$ is the number of the zeros of $\zeta(\sigma+\mathrm{it})$ in the region $0\leqslant\sigma\leqslant 1$, $0<t\leqslant T$. Then we obtain

$$\sum_{T\leqslant\left(r_k^2-\frac{1}{4}\right)^{1/2}<2T} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}=\sum_{T\leqslant\gamma<2T}|\rho|^{-n}<T^{-n}\{N(2T)-N(T)\}=O(T^{-n+1}\log T),$$

so that

$$\sum_{T\leqslant\left(r_k^2-\frac{1}{4}\right)^{1/2}} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}=O(T^{-n+1}\log T).$$

It follows that, for sufficiently large T,

$$\sum_{k=1}^{\infty} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}\cos b_{kj}n=r_1^{-n}\sum_{j=1}^{m_1} a_{1j}\cos b_{1j}n+O\left(\sum_{\substack{\left(r_k^2-\frac{1}{4}\right)^{1/2}<T\\ k\neq 1}} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}\right)$$
$$+O\left(\sum_{T\leqslant\left(r_k^2-\frac{1}{4}\right)^{1/2}} r_k^{-n}\sum_{j=1}^{m_k} a_{kj}\right)$$
$$=r_1^{-n}\sum_{j=1}^{m_1} a_{1j}\cos b_{1j}n+O(r_2^{-n}T\log T)+O(T^{-n+1}\log T);$$

which leads to (7). Now, if we compare the orders of both sides of (7), we get, using the lemma, $r_1=R_1$. Thus we have

$$\sum_{j=1}^{m_1} a_{1j}\cos b_{1j}n-\sum_{j=1}^{M_1} A_{1j}\cos B_{1j}n=O(r_1^n r_2^{-n}+R_1^n R_2^{-n}),$$

where we may assume that $b_{11}<b_{12}<\cdots<b_{1m_1}$ and $B_{11}<B_{12}<\cdots<B_{1M_1}$. If the equation above holds, then we see that

$$\sum_{j=1}^{m_1} a_{1j}\cos b_{1j}n-\sum_{j=1}^{M_1} A_{1j}\cos B_{1j}n \longrightarrow 0,$$

as the integer $n\to\infty$. We therefore have by the lemma

(8) $$m_1=M_1,\ a_{1j}=A_{1j},\ b_{1j}=B_{1j}.$$

We next consider the case of $k=2$. We have from (6) and (8)

$$r_2^{-n}\sum_{j=1}^{m_2} a_{2j}\cos b_{2j}n+O(r_3^{-n})=R_2^{-n}\sum_{j=1}^{M_2} A_{2j}\cos B_{2j}n+O(R_3^{-n}),$$

which implies that $r_2=R_2$, $m_2=M_2$, $a_{2j}=A_{2j}$ and $b_{2j}=B_{2j}(j=1, 2, \cdots, m_2)$. Repeating this, we get, for all k, $r_k=R_k$, $m_k=M_k$, $a_{kj}=A_{kj}$ and $b_{kj}=B_{kj}(j=1, 2, \cdots, m_k)$. This proves the uniqueness for the expression (5) of δ_n, and the proof is completed.

We next prove the following theorem.

THEOREM 2. *Let the notations be as in Theorem 1. Then the Riemann hypothesis holds if and only if* $m_k=1$, *and* $r_k \cos b_{k1}=\frac{1}{2}$ *for all* k.

PROOF. If the Riemann hypothesis is true, then we get $m_k=1$ for all k. Because, if $m_k>1$ for some $k\geqslant 1$, we have at least two different complex zeros of $\zeta(s)$ on a circle with center 0 and radius r_k in the upper plane $\mathrm{Im}\, s>0$; a contradiction. We now consider the modulus of the complex zero $\rho=\beta+i\gamma$ of $\zeta(s)$. Then we get $|\rho|=r_k$ for some k, and therefore we have $r_k \cos b_{k1}=\frac{1}{2}$ under the hypothesis, since $\beta=|\rho| \cos \{\arctan (\gamma/\beta)\}=\frac{1}{2}$. The uniqueness of the expression (5) completes the proof.

At present, we do not have a formula for δ_n which enables us to determine r_k, m_k, a_{kj} and b_{kj}. However we have the following theorem. We define inductively the sequence $\delta(n, k)$ with positive integers n and k. We put

$$\delta(n, 1)=\delta_n,$$

and define $\delta(n, k)$ for $k\geqslant 2$ by

$$\delta(n, k)=\delta(n, k-1)-2R_{k-1}^{-n} \sum_{\substack{v=0 \\ v \text{ even}}}^{n} (-1)^{v/2} \binom{n}{v} \left(\frac{1}{2} R_{k-1}^{-1}\right)^{n-v} \left(1-\frac{1}{4} R_{k-1}^{-2}\right)^{v/2}$$

with

$$R_{k-1}=\lim_{n\to\infty} \left\{\frac{1}{2} \delta^2(n, k-1)-\frac{1}{2} \delta(2n, k-1)\right\}^{-1/2n}.$$

THEOREM 3. *Let the notations be as in Theorem* 1. *Suppose that the Riemann hypothesis is true, and all of the zeros of* $\zeta(s)$ *are simple. Then, for all* k, $m_k=1$, $r_k=R_k$, $a_{k1}=1$, *and* $b_{k1}=\arctan (R_k^2-\frac{1}{4})^{1/2}$.

PROOF. By Theorem 2, we get $m_k=1$, $b_{k1}=\arctan (r_k^2-\frac{1}{4})^{1/2}$, and easily obtain $a_{k1}=1$ from the fact that all of the zeros are simple. Hence, it is enough to prove that, for every k,

$$\delta(n, k)=2 \sum_{h=k}^{\infty} r_h^{-n} \cos b_{h1} n, \quad r_k=R_k. \tag{9}$$

We prove this by induction on k. In case $k=1$, we have $\delta(n, 1)=\delta_n$, and hence we have only to show $r_1=R_1$. We get from the distribution of the zeros of $\zeta(s)$

$$\delta(n, 1)=2r_1^{-n} \cos b_{11} n+O(r_2^{-n}).$$

It follows that

$$\frac{1}{2}\delta^2(n, 1)-\frac{1}{2}\delta(2n, 1)=r_1^{-2n}+O(r_2^{-n}r_1^{-n}),$$

since, for any θ, $2\cos^2\theta-\cos 2\theta=1$. Therefore

$$\left\{\frac{1}{2}\delta^2(n, 1)-\frac{1}{2}\delta(2n, 1)\right\}^{1/2n}=r_1^{-1}\left[1+O\left\{\frac{1}{n}\left(\frac{r_2}{r_1}\right)^{-n}\right\}\right],$$

and hence we get $r_1=R_1$. We next assume that the condition (9) holds in case $k-1$. We then have from the definition of $\delta(n, k)$

$$(10)\quad \delta(n, k)=2\sum_{h=k-1}^{\infty} r_h^{-n}\cos b_{h1}n-2r_{k-1}^{-n}\sum_{\substack{v=0\\ v\text{ even}}}^{n}(-1)^{v/2}\binom{n}{v}\left(\frac{1}{2}r_{k-1}^{-1}\right)^{n-v}\left(1-\frac{1}{4}r_{k-1}^{-2}\right)^{v/2},$$

since $r_{k-1}=R_{k-1}$. By Theorem 2, we get $r_{k-1}\cos b_{k-1,1}=\frac{1}{2}$, and hence

$$\begin{aligned}\cos b_{k-1,1}n&=\sum_{\substack{v=0\\ v\text{ even}}}^{n}(-1)^{v/2}\binom{n}{v}\cos^{n-v}b_{k-1,1}\sin^{v}b_{k-1,1}\\ &=\sum_{\substack{v=0\\ v\text{ even}}}^{n}(-1)^{v/2}\binom{n}{v}\left(\frac{1}{2}r_{k-1}^{-1}\right)^{n-v}\left(1-\frac{1}{4}r_{k-1}^{-2}\right)^{v/2},\end{aligned}$$

which implies

$$2r_{k-1}^{-n}\cos b_{k-1,1}n=2r_{k-1}^{-n}\sum_{\substack{v=0\\ v\text{ even}}}^{n}(-1)^{v/2}\binom{n}{v}\left(\frac{1}{2}r_{k-1}^{-1}\right)^{n-v}\left(1-\frac{1}{4}r_{k-1}^{-2}\right)^{v/2}.$$

It follows from (10) that

$$\delta(n, k)=2\sum_{h=k}^{\infty} r_h^{-n}\cos b_{h1}n.$$

We can easily prove that $r_k=R_k$ by means of the formula $\frac{1}{2}\{\delta^2(n, k)-\delta(2n, k)\}=r_k^{-2n}+O(r_k^{-n}r_{k+1}^{-n})$, and the proof is completed.

Recently, we obtained an expression of γ_1([3], Theorem 8), where $\frac{1}{2}+i\gamma_1\,(\gamma_1>0)$ is the first complex zero of $\zeta(s)$, which states that

$$\gamma_1=\lim_{n\to\infty}\left\{\left(\frac{1}{2}\delta_n^2-\frac{1}{2}\delta_{2n}\right)^{-1/n}-\frac{1}{4}\right\}^{1/2}.$$

Let γ_k be the kth ordinate of the zeros of $\zeta(s)$. Noticing that $r_k=|\frac{1}{2}+i\gamma_k|$ for all k under the assumptions of Theorem 3, we have, in general, the following corollary.

COROLLARY 3. *With the same assumptions as in Theorem* 3, *we have, for every k,*

$$\gamma_k=\lim_{n\to\infty}\left[\left\{\frac{1}{2}\delta^2(n, k)-\frac{1}{2}\delta(2n, k)\right\}^{-1/n}-\frac{1}{4}\right]^{1/2}.$$

It is desirable to have a precise estimate for δ_n and $\delta(n, k)$. In this direction, we can show the following upper bound for the quantity $|\delta_n|$.

THEOREM 4. $\delta_n=O\{(1+C_0)^n\}$.

PROOF, Let us recall an estimate for C_n([3], Theorem 4) which states that $|C_n|<0.0001\exp(n\log\log n)$ for all $n\geqslant 10$, and the numerical values for $0\leqslant n\leqslant 9$ (see [1]), that is, $C_0=0.57721566$, $C_1=-0.07281584$, $C_2=-0.00969036$, $C_3=0.00205387$, $C_4=0.00232537$, $C_5=0.00079332$, $C_6=-0.00023876$, $C_7=-0.00052728$, $C_8=-0.00035212$ and $C_9=-0.00003439$. Then we can prove that $|C_n|/n!$ has the maximum $|C_0|/0!=C_0$ at $n=0$. Hence, for $n\geqslant 2$, we have from (1)

$$\delta_n=1-(1-2^{-n})\zeta(n)+n\sum_{h=1}^{n}\frac{1}{h}\sum_{\substack{j_1+\cdots+j_n=n-h\\ j_1\geqslant 0,\cdots,j_n\geqslant 0}}\prod_{b=1}^{h}\frac{C_{j_b}}{j_b!}$$

$$=O(1)+O\left\{\sum_{h=1}^{n}\binom{n}{h}C_0^h\right\}=O\{(1+C_0)^n\},$$

since $1-(1-2^{-n})\zeta(n)=O(1)$, and $\Sigma_j C_0^h=\binom{n-1}{h-1}C_0^h$. This proves the theorem.

If we use Theorem 3 and Theorem 4, we obtain $r_1\geqslant(1+C_0)^{-1}$, so that the first complex zero ρ_1 satisfies $|\rho_1|\geqslant(1+C_0)^{-1}$, which leads to the following corollary.

COROLLARY 4. $\gamma_1>\left\{(1+C_0)^{-2}-\frac{1}{4}\right\}^{1/2}$.

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