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ON COGENERATOR RINGS AS Φ-TRIVIAL EXTENSIONS

Dedicated to the memory of Professor Akira Hattori

By

Kazunori SAKANO

Let R be a ring with identity and M an (R, R)-bimodule with a pairing $\Phi=[-,-]: M\otimes_R M\rightarrow R$, that is, an (R, R)-bilinear map satisfying $[m, m']m''=m[m', m]$. Then by defining a multiplication on the abelian group $R\oplus M$ as $(r, m)(r', m')=(rr'+[m, m'], mr'+rm')$, $R\oplus M$ becomes a ring, which is called the Φ-trivial extension of R by M and is denoted by $\Lambda=R\underset{\Phi}{\ltimes}M$. Note that $\Phi=0$ corresponds to the trivial extension $R\ltimes M$. In particular, a generalized matrix ring defined by a Morita context can be considered as a special case of a Φ-trivial extension.

The main purpose of this paper is to give a necessary and sufficient condition for Λ to be a right cogenerator ring under the condition that $\operatorname{Im}\Phi$ is nilpotent.

In Section 1, we study the form of the injective hull of a simple right Λ-module and decide the condition for Λ to be a right cogenerator ring under the assumption that $\operatorname{Im}\Phi$ is nilpotent. Furthermore, in case of the trivial extension $R\ltimes M$, we investigate the condition for $M=0$, when $R\ltimes M$ is a right cogenerator ring. In Section 2, we give a sufficient condition for Λ to be right self-injective under the assumption that $\operatorname{Im}\Phi$ is nilpotent. Moreover, in case of the trivial extension $R\ltimes M$, we give a necessary and sufficient condition for $R\ltimes M$ to be a right injective cogenerator ring. Let $\Gamma=\begin{pmatrix} S & 0 \\ U & T \end{pmatrix}$ be a generalized triangular matrix ring, where both S and T are rings with identity and U a (T, S)-bimodule. In the final Section 3, we study an application of results in Sections 1 and 2 to a generalized triangular matrix ring Γ. Especially, we show that Γ is a right injective cogenerator ring if and only if both S and T are right injective cogenerator rings, and $U=0$. This result was mentioned by T. Kato during a conversation and he pointed out whether the similar result as above holds when Γ is a right cogenerator ring (in case of Γ being a QF ring, see [6, Exercise (3)-(2), p. 362]). In case of $S=T$ in Γ, there holds that Γ is a right cogenerator ring if and only if T is a right cogenerator ring and

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$U=0$. But it remains an unsolved problem when $S\neq T$ in Γ.

Throughout this paper, unless otherwise specified, Λ denotes the Φ-trivial extension $R\underset{\Phi}{\ltimes}M$ and $\mathbf{l}_R(K)$ the left annihilator of K in R for a subset K of a right R-module X. For a right R-module Y, $E(Y_R)$ means the injective hull of Y_R.

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1. Cogenerator rings as Φ-trivial extensions.

In this section, we assume that $\operatorname{Im}\Phi$ is nilpotent. By a slight modification of the proof of [14, Lemma 3.1], we have the following.

LEMMA 1.1. *Let X be a right R-module and K a nilpotent ideal of R. Then $\mathbf{l}_X(K)$ is essential in X_R.*

LEMMA 1.2. *$\operatorname{Im}\Phi\oplus M$ is a nilpotent ideal of Λ.*

PROOF. This is found in the proof of [12, LEMMA 1].

LEMMA 1.3. *Let X be a simple right Λ-module. Then the injective hull of X_Λ has the form $\operatorname{Hom}_R(\Lambda_R, E(X_R))_\Lambda$.*

PROOF. Since $E(X_R)$ is injective and ${}_\Lambda\Lambda$ is flat, $\operatorname{Hom}_R(\Lambda_R, E(X_R))_\Lambda$ is injective. Therefore, it suffices to show that X_Λ is essential in $\operatorname{Hom}_R(\Lambda_R, E(X_R))_\Lambda$. Since

$$\begin{aligned}\mathbf{l}_{\operatorname{Hom}_R(\Lambda, E(X))}(\operatorname{Im}\Phi\oplus M)_\Lambda &\cong \operatorname{Hom}_\Lambda(\Lambda/\operatorname{Im}\Phi\oplus M, \operatorname{Hom}_R(\Lambda, E(X)))_\Lambda\\ &\cong \operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M\otimes_\Lambda\Lambda, E(X))_\Lambda\\ &\cong \operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))_\Lambda,\end{aligned}$$

$\operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))_\Lambda$ is essential in $\operatorname{Hom}_R(\Lambda, E(X))_\Lambda$ by Lemmas 1.1 and 1.2. Since $\operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))_R\subsetneq E(X_R)$, we may consider $X_R\subsetneq\operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))_R$ and X_R is essential in $\operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))_R$. Since $\operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))\operatorname{Im}\Phi=0$, X_Λ is also essential in $\operatorname{Hom}_R(\Lambda/\operatorname{Im}\Phi\oplus M, E(X))_\Lambda$. Thus we obtain X_Λ is essential in $\operatorname{Hom}_R(\Lambda_R, E(X_R))_\Lambda$.

In the remainder of this section, let $\alpha: M\to\operatorname{Hom}_R(M_R, R_R)$ be the natural map defined via

$$(\alpha(m))(m')=[m, m'] \quad \text{for} \quad m, m'\in M,$$

and $\sigma: R\to\operatorname{End}(M_R)$ the canonical map. We put $\operatorname{Ker}\alpha=M'$.

Since every simple right Λ-module is isomorphic to $R\oplus M/\mathfrak{m}\oplus M$, where $\mathfrak{m}$ is a maximal right ideal of R, every simple right Λ-module is also simple as a right R-module and vice versa.

THEOREM 1.4. *Λ is a right cogenerator ring if and only if, for each simple right R-module X and $E_R=E(X_R)$, there exists a primitive idempotent e in R satisfying the following condition*

(1) *$E_R\cong eR_R=e\mathrm{l}_R(M')_R$ and $\alpha': eM_R\cong e\,\mathrm{Hom}_R(M_R, R_R)_R$, where α' is the induced map by α,*

or

(2) *$E_R\cong eM_R$ and $\sigma': eR_R\cong e\,\mathrm{End}(M_R)_R$, where σ' is the induced map by σ.*

PROOF. ($\Rightarrow$). Let X be a simple right R-module and $E_R=E(X_R)$. Since every simple right R-module is also simple as a right Λ-module, $\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda$ is the injective hull of X_Λ by Lemma 1.3. Since Λ_Λ is a cogenerator, there exists a primitive idempotent (e, m) in Λ such that $(e, m)\Lambda_\Lambda\cong\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda$. Then it is easily seen that e is a primitive idempotent in R and $[m, m]=0$. Moreover, since $(e, m)^2=(e, m)$, we have $m=em+me$, from which it follows that $eme=0$. Therefore, we have $meR\cap eM=0$ and $eR\cap[me, M]=0$. Hence we get $(e, m)\Lambda_\Lambda\subsetneq(eR\oplus eM)_\Lambda\oplus([me, M]\oplus meR)_\Lambda$. Since $(e, m)\Lambda$ is the injective hull of a simple right Λ-module, there holds $(e, m)\Lambda_\Lambda\subsetneq(eR\oplus eM)_\Lambda$ or $(e, m)\Lambda_\Lambda\subsetneq([me, M]\oplus meR)_\Lambda$. If $(e, m)\Lambda_\Lambda\subsetneq([me, M]\oplus meR)_\Lambda$, then $eR\subsetneq\mathrm{Im}\,\Phi\subsetneq\mathrm{Rad}(R)$. Therefore, we obtain $(e, m)=0$. Hence we must have $(e, m)\Lambda_\Lambda\subsetneq(e, 0)\Lambda_\Lambda$. Since $(e, m)\Lambda_\Lambda$ is injective and $(e, 0)\Lambda_\Lambda$ is indecomposable, we have $(e, m)\Lambda_\Lambda\cong(e, 0)\Lambda_\Lambda$. Therefore, we may take $m=0$. Since $(e, 0)\Lambda_\Lambda\cong\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda$, we have $eR\oplus eM_R\cong E_R\oplus\mathrm{Hom}_R(M, E)_R$. Since E_R is the injective hull of a simple right R-module, there holds $E_R\subsetneq eR_R$ or $E_R\subsetneq eM_R$. Furthermore, since

$$\begin{aligned}\mathrm{Hom}_R(\Lambda/\mathrm{Im}\,\Phi\oplus M, E)_\Lambda &\cong\mathrm{Hom}_R(\Lambda/\mathrm{Im}\,\Phi\oplus M\otimes_\Lambda\Lambda, E)_\Lambda\\ &\cong\mathrm{Hom}_\Lambda(\Lambda/\mathrm{Im}\,\Phi\oplus M, \mathrm{Hom}_R(\Lambda, E))\\ &\cong\mathrm{Hom}_\Lambda(\Lambda/\mathrm{Im}\,\Phi\oplus M, (e, 0)\Lambda)\\ &\cong(e, 0)(\mathrm{l}_R(M)\oplus M')_\Lambda\end{aligned}$$

and $\mathrm{Hom}_R(\Lambda/\mathrm{Im}\,\Phi\oplus M, E)_R\cong\mathrm{Hom}_R(R/\mathrm{Im}\,\Phi, E)_R$ is essential in E_R by Lemma 1.1, there holds the following condition

(i) $eM'=0$ and $e\mathrm{l}_R(M)_R$ is essential in E_R.

or

(ii) $e\mathrm{l}_R(M)=0$ and eM'_R is essential in E_R.

First, we consider the case (i). In this case, $E_R \subsetneq eR_R$. Since eR_R is indecomposable, we have $E_R \cong eR_R$. Since $e \in \mathfrak{l}_R(M')$ by (i), we have $eR \subset e\mathfrak{l}_R(M')$. Therefore, we get $eR = e\mathfrak{l}_R(M')$. It follows that $E_R \cong eR_R = e\mathfrak{l}_R(M')_R$. We define a map $f_1 : (eR \oplus eM)_R \to (eR \oplus e\,\mathrm{Hom}_R(M, R))_R$ via

$$f_1(er, em) = (er, \alpha'(em)) \quad \text{for} \quad r \in R,\ m \in M.$$

Since $\mathrm{Ker}\, f_1 = (0, \mathrm{Ker}\, \alpha') = (0, eM') = 0$, f_1 is a right R-monomorphism. Furthermore, a routine calculation shows that f_1 is also a right Λ-monomorphism. Consider the following composition map:

$$g_1 : (e, 0)\Lambda_\Lambda \overset{f_1}{\subsetneq} (eR \oplus e\,\mathrm{Hom}_R(M, R))_\Lambda \cong \mathrm{Hom}_R(\Lambda_R, eR_R)_\Lambda \cong \mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda.$$

Since $\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda$ is indecomposable and $(e, 0)\Lambda_\Lambda$ is injective, g_1 is an isomorphism. Hence f_1 is an isomorphism. Thus we get $\alpha' : eM_R \cong e\,\mathrm{Hom}_R(M, R)_R$. Hence we conclude that (1) holds. Next, we consider the case (ii). In this case, $E_R \subsetneq eM_R$. We claim that eM_R is indecomposable. Suppose that $eM_R = eM_{1_R} \oplus eM_{2_R}$ with $eM_1 \neq 0$ and $eM_2 \neq 0$. Then $(e, 0)([M_1, M] \oplus M_1)_\Lambda + (e, 0)([M_2, M] \oplus M_2)_\Lambda \subsetneq (e, 0)\Lambda_\Lambda$. We show that the above sum is direct. Let $(e, 0)([m_1, m], m_1') \in (e, 0)([M_1, M] \oplus M_1) \cap (e, 0)([M_2, M] \oplus M_2)$. Then $em_1' \in eM_1 \cap eM_2 = 0$. Moreover, since

$$\begin{aligned}[em_1, m]M &\in [eM_1, M]M \cap [eM_2, M]M \\ &= eM_1[M, M] \cap eM_2[M, M] \\ &\subseteqq eM_1 \cap eM_2 = 0,\end{aligned}$$

we have $[em_1, m] \in e\mathfrak{l}_R(M)$. Since $e\mathfrak{l}_R(M) = 0$, we have $[em_1, m] = 0$. Therefore, we have $(e, 0)([M_1, M] \oplus M_1)_\Lambda \oplus (e, 0)([M_2, M] \oplus M_2)_\Lambda \subsetneq (e, 0)\Lambda_\Lambda$. Since $(e, 0)\Lambda_\Lambda$ is the injective hull of a simple right Λ-module, there holds $(e, 0)([M_1, M] \oplus M_1) = 0$ or $(e, 0)([M_2, M] \oplus M_2) = 0$. Thus eM_R is indecomposable. Hence we get $E_R \cong eM_R$. We define a map $f_2 : (eR \oplus eM)_R \to (e\,\mathrm{End}(M_R) \oplus eM)_R$ via

$$f_2(er, em) = (\sigma'(er), em) \quad \text{for} \quad r \in R,\ m \in M.$$

Since $\mathrm{Ker}\, f_2 = (\mathrm{Ker}\, \sigma', 0) = (e\mathfrak{l}_R(M), 0) = 0$, f_2 is a right R-monomorphism. Furthermore, it is easily verified that f_2 is also a right Λ-monomorphism. Consider the following composition map:

$$g_2 : (e, 0)\Lambda_\Lambda \overset{f_2}{\subsetneq} (e\,\mathrm{End}(M_R) \oplus eM)_\Lambda \cong \mathrm{Hom}_R(\Lambda_R, eM_R)_\Lambda \cong \mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda.$$

Since $\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda$ is indecomposable and $(e, 0)\Lambda_\Lambda$ is injective, g_2 is an isomorphism. Therefore, f_2 is an isomorphism. Hence we get $\sigma' : eR_R \cong e\,\mathrm{End}(M_R)_R$. Thus we conclude that (2) holds.

($\Leftarrow$). Let X be a simple right Λ-module. Then X is simple as a right R-module. Suppose that (1) holds. Then we can take f_1 and g_1 as in the proof of the part ($\Rightarrow$). Since f_1 is a right Λ-isomorphism, g_1 becomes also a right Λ-isomorphism. Thus we obtain $\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda \hookrightarrow \Lambda_\Lambda$. Similarly, in case of (2), we can show that $\mathrm{Hom}_R(\Lambda_R, E_R)_\Lambda \hookrightarrow \Lambda_\Lambda$. Hence we conclude that Λ is a right cogenerator ring.

If $\Phi=0$, that is, Λ is the trivial extension $R \ltimes M$, then Theorem 1.4 is rewrited as follows. In this case, note that $M'=M$.

COROLLARY 1.5. *Assume that* $\mathrm{Im}\,\Phi=0$. *Then Λ is a right cogenerator ring if and only if, for each simple right R-module X and $E_R=E(X_R)$, there exists a primitive idempotent e in R satisfying the following condition*

(1) $E_R \cong e\mathbf{l}_R(M)_R$ *and* $\mathrm{Hom}_R(M_R, E_R)=0$,

or

(2) $E_R \cong eM_R$ *and* $\sigma' : eR_R \cong e\,\mathrm{End}(M_R)$, *where σ' is the induced map by σ.*

EXAMPLE 1.6. Let R be a right cogenerator ring and $\Lambda = R \ltimes R$. Then Λ becomes also a right cogenerator ring in view of Corollary 1.5.

In the remainder of this section, let Λ denote the trivial extension $R \ltimes M$.

LEMMA 1.7 ([9, Theorem 1]). *If R is a right cogenerator ring, then the following holds.*

(1) *The mapping*

$$Ra \to aR, \qquad a \in R$$

gives a one-to-one, onto, correspondence between isomorphism classes of simple left ideals and isomorphism classes of simple right ideals.

(2) *Each simple left ideal is of the form* $Re/\mathrm{Rad}(R)e$, $e=e^2 \in R$.

LEMMA 1.8 (cf. [13]). $\mathrm{Rad}(\Lambda)=\mathrm{Rad}(R)\oplus M$.

THEOREM 1.9. *If Λ is a right cogenerator ring, then $M=0$ if and only if* $\mathrm{Soc}({}_RM) \hookrightarrow \mathrm{Soc}({}_R\mathbf{r}_R(M))$.

PROOF. ($\Rightarrow$). Obvious.

($\Leftarrow$). We first show that $\mathrm{Soc}(M_R)=0$. Suppose that $\mathrm{Soc}(M_R)\neq 0$ and let mR_R be a simple right R-module contained in $\mathrm{Soc}(M_R)$. Then $(0\oplus mR)_\Lambda=(0, m)\Lambda_\Lambda$ is also simple as a right Λ-module. Since Λ_Λ is a cogenerator, there exists a primitive idempotent e in R such that $(0, m)\Lambda_\Lambda=(e, 0)\mathbf{r}_\Lambda(\mathrm{Rad}(\Lambda))_\Lambda$ (cf. [9, Proof of (2), p. 116]). Since $(e, 0)\mathbf{r}_\Lambda(\mathrm{Rad}(\Lambda))_\Lambda=(e, 0)(\mathbf{r}_\Lambda(\mathrm{Rad}(R)\oplus M)_\Lambda=(e, 0)(\mathbf{r}_R(M)\cap \mathbf{r}_R(\mathrm{Rad}(R))\oplus\mathbf{r}_M(\mathrm{Rad}(R))_\Lambda$ by Lemma 1.8, we have $mR_R=e\mathbf{r}_M(\mathrm{Rad}(R))_R$ and $e(\mathbf{r}_R(M)\cap\mathbf{r}_R(\mathrm{Rad}(R)))=0$. Moreover, by Lemma 1.7, ${}_\Lambda\Lambda(0, m)={}_\Lambda(0\oplus Rm)$ is also a simple left ideal of Λ isomorphic to ${}_\Lambda(\Lambda(e, 0)/\mathrm{Rad}(\Lambda)(e, 0))$. So, we get ${}_R(Re/\mathrm{Rad}(R)e)\cong{}_R(\Lambda(e, 0)/\mathrm{Rad}(\Lambda)(e, 0))\cong{}_RRm\subset\mathrm{Soc}({}_RM)$. Therefore, we see that $\mathrm{Soc}({}_R\mathbf{r}_R(M))\neq 0$ if $\mathrm{Soc}(M_R)\neq 0$. Since $\mathrm{Soc}({}_RM)\subsetneq\mathrm{Soc}({}_R\mathbf{r}_R(M))$, there exists a simple left ideal $Ra\in\mathrm{Soc}({}_R\mathbf{r}_R(M))$ of R which is isomorphic to ${}_R(Re/\mathrm{Rad}(R)e)$. Since ${}_\Lambda(Ra\oplus 0)={}_\Lambda\Lambda(a, 0)$ is a simple left ideal of Λ and Λ is a right cogenerator ring, $(a, 0)\Lambda_\Lambda$ is also a simple right ideal of Λ which is isomorphic to $(e, 0)\mathbf{r}_\Lambda(\mathrm{Rad}(\Lambda))_\Lambda$ by Lemma 1.7. Furthermore, we get $(a, 0)\Lambda_\Lambda=(e', 0)\mathbf{r}_\Lambda(\mathrm{Rad}(\Lambda))_\Lambda$ by Lemma 1.7, where e' is a primitive idempotent in R such that $(e', 0)\Lambda_\Lambda= E((a, 0)\Lambda_\Lambda)$. Therefore, we have $aR_R=e'(\mathbf{r}_R(M)\cap\mathbf{r}_R(\mathrm{Rad}(R)))_R$ and $e'\mathbf{r}_M(\mathrm{Rad}(R)) =0$. Since $(e, 0)\Lambda_\Lambda\cong(e', 0)\Lambda_\Lambda$, we get $eR_R\cong e'R_R$. Therefore, we obtain $e\mathbf{r}_M(\mathrm{Rad}(R))\cong e'\mathbf{r}_M(\mathrm{Rad}(R))=0$. On the other hand, $mR_R=e\mathbf{r}_M(\mathrm{Rad}(R))_R\neq 0$. This is a cotradiction. So, we must have $\mathrm{Soc}(M_R)=0$. Since only (1) of Corollary 1.5 holds, we conclude that $M=0$.

2. Injective cogenerator rings.

Let $\alpha: M\to\mathrm{Hom}_R(M_R, R_R)$ and $\sigma: R\to\mathrm{End}(M_R)$ be the natural maps as in Section 1. We set $\mathrm{Ker}\,\alpha=M'$.

LEMMA 2.1 ([15, Theorem 2.4]). *Assume that* $\mathrm{Im}\,\Phi$ *is nilpotent. Then the injective hull of* Λ *has the form*

$$\mathrm{Hom}_R(\Lambda_R, E(\mathbf{l}_R(M)\oplus M')_R).$$

THEOREM 2.2. *Assume that* $\mathrm{Im}\,\Phi$ *is nilpotent. Then* Λ *is right self-injective if the following conditions are satisfied*:

(1) $\mathbf{l}_R(M)_R$ *and* M'_R *are injective.*

(2) (i) *For each* $f\in\mathrm{Hom}_R(M_R, \mathbf{l}_R(M)_R)$, *there exists* $m_0\in M$ *such that* $f=[m_0, -]$.

(ii) *For each* $g\in\mathrm{Hom}_R(M_R, M'_R)$, *there exists* $r_0\in R$ *such that* $g=\bar{r}_0$, *where* $\bar{r}_0$ *denotes left multiplication by* r_0.

PROOF. Suppose that (1) and (2) are satisfied. Since $\mathbf{l}_R(M)_R$ and M'_R are injective, the inclusion maps $M'_R \subsetneq M_R$ and $\mathbf{l}_R(M)_R \subsetneq R_R$ split. So, let $p: M \to M'$ and $q: R \to \mathbf{l}_R(M)$ be the natural projection maps. We define a map $\Psi: \Lambda \to \mathrm{Hom}_R(\Lambda_R, \mathbf{l}_R(M)_R \oplus M'_R)$ via

$$(\Psi(r, m))(a, x) = (q(r)a + q([m, x]), p(m)a + (pr)(x)) \quad \text{for } (r, m), (a, x) \in \Lambda.$$

It is easily verified that Ψ is a right Λ-homomorphism. We claim that Ψ is an isomorphism. Let $f = ({}^t f_1, f_2) \in \mathrm{Hom}_R(\Lambda_R, \mathbf{l}_R(M)_R \oplus M'_R)$, where $f_1 \in \mathrm{Hom}_R(R_R, \mathbf{l}_R(M)_R \oplus M'_R)$ and $f_2 \in \mathrm{Hom}_R(M_R, \mathbf{l}_R(M)_R \oplus M'_R)$. Then by (2), there exist $m_0 \in M$ and $r_0 \in R$ such that $f_2(m) = ([m_0, m], r_0 m)$ for every $m \in M$. We put $f_1(1) = (a_1, x_1)$. Since

$$\begin{aligned}
&(\Psi((1-q)(r_0) + a_1, (1-p)(m_0) + x_1))(a, x) \\
&= (q((1-q)(r_0) + a_1)a + q([(1-p)(m_0) + x_1, x]), p((1-p)(m_0) + x_1)a \\
&\quad + (p \cdot ((1-q)(r_0) + a_1))(x)) = (a_1 a + [(1-p)(m_0), x], x_1 a + (1-q)(r_0)x) \\
&= (a_1 a + [m_0, x], x_1 a + r_0 x) = f_1(a) + f_2(x) = f(a, x) \qquad \text{for } (a, x) \in \Lambda,
\end{aligned}$$

Ψ is an epimorphism. Let $\iota: (\mathbf{l}_R(M) \oplus M')_\Lambda \subsetneq \Lambda_\Lambda$ be the inclusion map. Then ι is an essential monomorphism by Lemmas 1.1 and 1.2. Since $\Psi\iota$ is a monomorphism, Ψ is also a monomorphism. Thus Ψ is an isomorphism. Since $(\mathbf{l}_R(M) \oplus M')_R$ is injective, $\mathrm{Hom}_R(\Lambda_R, \mathbf{l}_R(M)_R \oplus M'_R)_\Lambda$ is injective, from which it follows that Λ is right self-injective.

Following [2], a right R-module X is called lower distinguished if it contains a copy of each simple right R-module.

THEOREM 2.3. *Assume that* $\mathrm{Im}\,\Phi$ *is nilpotent. Then* Λ_Λ *is lower distinguished if and only if* $(\mathbf{l}_R(M) \oplus M')_R$ *is lower distinguished.*

PROOF. Since every maximal right ideal X of Λ has the form $\mathfrak{m} \oplus M$, where $\mathfrak{m}$ is a maximal right ideal of R, and

$$\begin{aligned}
\mathrm{Hom}_\Lambda(\Lambda/X, \Lambda) &\cong \mathrm{Hom}_\Lambda(\Lambda/\mathfrak{m} \oplus M, \Lambda) \\
&\cong \mathbf{l}_\Lambda(\mathfrak{m} \oplus M) \\
&= \mathbf{l}_{(\mathbf{l}_R(M) \oplus M')}(\mathfrak{m}),
\end{aligned}$$

we conclude that Λ_Λ is lower distinguished if and only if $(\mathbf{l}_R(M) \oplus M')_R$ is lower distinguished.

From now on, let Λ be the trivial extension $R \ltimes M$.

LEMMA 2.4 ([13, Theorem 1.4.1]). *Λ is right self-injective if and only if the* *following conditions are satisfied:*

(1) $\mathbf{l}_R(M)_R$ *and* M_R *are injective.*

(2) $\sigma: R\to\mathrm{End}(M_R)$ *is an epimorphism.*

(3) $\mathrm{Hom}_R(M_R, \mathbf{l}_R(M)_R)=0$.

The following is derived from Theorem 2.3 and Lemma 2.4, directly.

THEOREM 2.5. *Λ is a right injective cogenerator ring if and only if the following coditions are satisfied:*

(1) $(\mathbf{l}_R(M)\oplus M)_R$ *is an injective cogenerator.*

(2) $\sigma: R\to\mathrm{End}(M_R)$ *is an epimorphism.*

(3) $\mathrm{Hom}_R(M_R, \mathbf{l}_R(M)_R)=0$.

REMARK. Y. Kitamura also obtained the above Theorem 2.5 independently (cf. [10, Theorem 3]).

3. Generalized triangular matrix rings.

In this section, let

$$\Gamma=\begin{pmatrix} S & 0 \\ U & T \end{pmatrix}$$

be a generalized triangular matrix ring, where S and T are rings with identity, and U a (T, S)-bimodule. Since U is regarded as an $(S\oplus T, S\oplus T)$-bimodule in the natural way, Γ is isomorphic to $(S\oplus T)\ltimes U$.

LEMMA 3.1 ([13, Theorem 1.5.1]). *Γ is semiperfect if and only if both S and T are semiperfect.*

LEMMA 3.2 ([11, Theorem 1]). *If R is a right injective cogenerator ring, then R is semiperfect.*

LEMMA 3.3 ([8, Theorem 1]). *The following conditions on a ring R are equivalent:*

(1) *R is a right injective cogenerator ring.*

(2) *$E(R_R)$ is torsionless, and both R_R and ${}_RR$ are lower distinguished.*

(3) *R_R is a cogenerator and there are only finitely many non-isomorphic simple right (or left) ideals.*

If we apply Theorem 1.9 to Γ, then we have the following.

COROLLARY 3.4. *If Γ is a right cogenerator ring, then $U=0$ if and only if* $\mathrm{Soc}({}_TU)\subsetneq\mathrm{Soc}({}_TT)$.

The following indicates that Γ can not be a right injective cogenerator ring except the trivial case.

THEOREM 3.5. *Γ is a right injective cogenerator ring if and only if both S and T are right injective cogenerator rings, and $U=0$.*

PROOF. ($\Leftarrow$). Obvious.

($\Rightarrow$). Since Γ_Γ is an injective cogenerator, Γ is semiperfect by Lemma 3.2. Therefore, T is semiperfect by Lemma 3.1. On the other hand, since $\mathbf{l}_T(U)_T$ is an injective cogenerator in view of Theorem 2.5, T_T is an injective cogenerator by Lemma 3.3, from which it follows that ${}_TT$ is lower distinguished together with Lemma 3.3. Thus we get $\mathrm{Soc}({}_TU)\subsetneq\mathrm{Soc}({}_TT)$. Hence $U=0$ by Corollary 3.4, from which it follows that S_S and T_T are injective cogenerators in view of Theorem 2.5.

THEOREM 3.6. *If $S=T$ in Γ, then Γ is a right cogenerator ring if and only if T is a right cogenerator ring, and $U=0$.*

PROOF. ($\Leftarrow$). Obvious.

($\Rightarrow$). If $\mathrm{Soc}({}_TU)=0$, then $0=\mathrm{Soc}({}_TU)\subsetneq\mathrm{Soc}({}_TT)$. Therefore, $U=0$ by Corollary 3.4. Next, we suppose that $\mathrm{Soc}({}_TU)\neq 0$ and let ${}_TTu$ be a simple left T-module contained in $\mathrm{Soc}({}_TU)$. Then ${}_\Gamma\Gamma\begin{pmatrix}0&0\\u&0\end{pmatrix}$ is also a simple left ideal of Γ. Since Γ_Γ is a cogenerator, $\mathbf{l}_T(U)_T$ is a cogenerator in view of Corollary 1.5 and $\begin{pmatrix}0&0\\u&0\end{pmatrix}\Gamma_\Gamma=\begin{pmatrix}0&0\\uT&0\end{pmatrix}_\Gamma$ is a simple right ideal of Γ by Lemma 1.7, from which it follows that uT_T is isomorphic to a simple right ideal aT_T of T together with the fact that $\mathbf{l}_T(U)_T$ is a cogenerator. Since $\begin{pmatrix}a&0\\0&0\end{pmatrix}\Gamma_\Gamma$ is a simple right ideal of Γ and Γ_Γ is a cogenerator, ${}_\Gamma\Gamma\begin{pmatrix}a&0\\0&0\end{pmatrix}$ is also a simple left ideal of Γ which is isomorphic to ${}_\Gamma\Gamma\begin{pmatrix}0&0\\u&0\end{pmatrix}$ by Lemma 1.7. Hence ${}_TTa$ is a simple left ideal of T which is isomorphic to ${}_TTu$. Therefore, we have $\mathrm{Soc}({}_TU)\subsetneq\mathrm{Soc}({}_TT)$. Hence we have $U=0$ by Corollary 3.4, and T is a right cogenerator ring.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305, Japan