

DOMESTIC TRIVIAL EXTENSIONS OF SIMPLY CONNECTED ALGEBRAS

By

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Dedicated to Professor H. Tachikawa on his 60th birthday

Abstract Let A be a finite-dimensional, basic and connected algebra (associative, with 1) over an algebraically closed field. It is called simply connected if it is triangular and, for any presentation of A as a bound quiver algebra, the fundamental group of its bound quiver is trivial. Let $T(A)$ denote the trivial extension of A by its minimal injective cogenerator. We show that, if A is simply connected, then the following conditions are equivalent: (i) $T(A)$ is representation-infinite and domestic, (ii) $T(A)$ is 2-parametric, (iii) there exists a representation-infinite tilted algebra B of Euclidean type \tilde{D}_n or \tilde{E}_p such that $T(A) \simeq T(B)$, (iv) A is an iterated tilted algebra of type \tilde{D}_n or \tilde{E}_p .

Introduction.

Let k denote a fixed algebraically closed field, and A a finite-dimensional k -algebra (associative, with an identity) which we shall moreover assume to be basic and connected. We shall denote by $\text{mod } A$ the category of finite-dimensional right A -modules. The *trivial extension* $T(A)$ of A by its minimal injective cogenerator bimodule $DA = \text{Hom}_k(A, k)$ is the algebra whose additive structure is that of the group $A \oplus DA$, and whose multiplication is defined by:

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in A$ and $f, g \in {}_A(DA)_A$. Then $T(A)$ is a self-injective and, in fact, a symmetric algebra.

Trivial extension algebras have been extensively investigated in representation theory. First, in the representation-finite case, they were studied by Müller [32], Green and Reiten [22] and Iwanaga and Wakamatsu [30] when the radical

square of A equals zero. It was shown by Tachikawa [39] (see also [45]) that, if A is hereditary, then the cardinality of the set of isomorphism classes of indecomposable $T(A)$ -modules is twice that of the set of indecomposable A -modules. He also described the components of the Auslander-Reiten quiver of $T(A)$ if A is tame hereditary. In [46], Yamagata proved that, if $T(A)$ is representation-finite, then A is triangular, that is, its ordinary quiver contains no oriented cycles. Actually, as observed in [47], the proof shows that A is simply connected in the sense of [13], that is, it is representation-finite with a simply connected Auslander-Reiten quiver. Later, Hughes and Waschbüch [29] proved that, if A is a tilted algebra of Dynkin type Δ , then $T(A)$ is representation-finite of Cartan class Δ and conversely, if $T(A)$ is representation-finite of Cartan class Δ , then there exists a tilted algebra B of Dynkin type Δ such that $T(A) \simeq T(B)$ (see also [27] [15]). Finally, it was shown in [2] that $T(A)$ is representation-finite of Cartan class Δ if and only if A is an iterated tilted algebra of Dynkin type Δ .

Our objective in this article is to present a result corresponding to the last two results in the representation-infinite case. First, we shall restrict to the case where the algebra A is *simply connected* in the sense of [6], that is, is triangular and such that, for any presentation of A as a bound quiver algebra, the fundamental group of its bound quiver [31] is trivial (in the representation-finite case, this notion of simple connectedness coincides with the notion introduced in [13]). Next, we recall that an algebra A is called *domestic* [35] if there exists a finite number of (parametrising) functors $F_i: \text{mod } k[X] \rightarrow \text{mod } A$, $1 \leq i \leq n$, where $k[X]$ is the polynomial algebra in one variable, satisfying the following two conditions:

(a) For each i , $F_i = - \otimes_{k[X]} Q_i$, where Q_i is a $k[X]$ - A -bimodule which is finitely generated and free as $k[X]$ -module.

(b) For any dimension d , all but a finite number of isomorphism classes of indecomposable A -modules of k -dimension d are of the form $F_i(M)$, for some i and some indecomposable right $k[X]$ -module M .

Finally, A is called *n-parametric* if the minimal number of such functors is n . Every domestic algebra is tame in the sense of [19]. Equivalent definitions for a domestic algebra can be found in [17]. We may now state our main theorem:

THEOREM. *Let A be a finite-dimensional, basic and connected algebra over*

an algebraically closed field k . If A is simply connected, then the following conditions are equivalent:

- (i) $T(A)$ is representation-infinite and domestic.
- (ii) $T(A)$ is 2-parametric.
- (iii) There exists a representation-infinite tilted algebra B of Euclidean type \tilde{D}_n or \tilde{E}_p such that $T(A) \simeq T(B)$.
- (iv) A is an iterated tilted algebra of Euclidean type \tilde{D}_n or \tilde{E}_p .

The article is organised as follows. After a preliminary section (1), we shall study in section (2) branch enlargements of tame concealed algebras. In section (3), we present the strategy of the proof and some reduction lemmas. Sections (4), (5) and (6) consist of the proof of our main theorem, while section (7) is devoted to some concluding remarks.

1. Preliminaries.

1.1. For a quiver Q , we shall denote by Q_0 its set of vertices and by Q_1 its set of arrows. For a (locally) finite-dimensional k -algebra A (usually assumed to be basic and connected), we shall denote by Q_A its ordinary quiver. For $i \in (Q_A)_0$ we denote by e_i the corresponding primitive idempotent of A , and by $S(i)$ the corresponding simple A -module. We shall denote by $P(i)$ (respectively, $I(i)$) the projective cover (respectively, the injective envelope) of $S(i)$. We recall from [29] that $i \in (Q_A)_0$ is called a *strong sink* if there exists no chain of non-zero non-isomorphisms between indecomposable modules of the form $M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_t \rightarrow I(i)$, where M_0 is injective. We define dually a strong source. Following [13], we shall equivalently consider a bound quiver algebra A as a k -category. We recall that a k -category A is called \tilde{A} -free whenever there exists no full subcategory $A' \simeq kQ'$ of A where the underlying graph of Q' is \tilde{A}_m ($m \geq 1$). It is called *Schurian* if, for any pair of objects x, y of A , $\dim_k A(x, y) \leq 1$. A bound quiver k -category $A \simeq kQ/I$ is called *special biserial* [38] if the number of arrows with a prescribed source or target is at most two, and for any $\alpha \in Q_1$, there is at most one arrow β and one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ do not belong to I . A special biserial k -category $A \simeq kQ/I$ is called *gentle* [4] if it is triangular, I is generated by a set of paths of length two and for any $\alpha \in Q_1$, there is at most one arrow ξ and one arrow ζ such that $\alpha\xi$ and $\zeta\alpha$ belong to I . Finally, we shall denote by τ_A (or τ , if there is no danger of confusion) the Auslander-Reiten translation DTr in $\text{mod } A$, and by Γ_A the Auslander-Reiten quiver of A [9][36].

1.2. Let A be a locally bounded k -category (in the sense of [13]). Then A is called *domestic* if every finite full subcategory of A is domestic (compare [17]). It is *locally support-finite* [16] if, for each object x of A , the full subcategory of A formed by all objects of the support $\text{Supp } M$, where M ranges through all indecomposable finite-dimensional A -modules such that $M(x) \neq 0$, is finite.

Let G be a torsion-free residually finite group acting freely on the objects of A , $F: A \rightarrow A/G$ be the Galois covering [21] which assigns to each object x of A its G -orbit $G \cdot x$, and $F_\lambda: \text{mod } A \rightarrow \text{mod } (A/G)$ the associated push-down functor [13]. We shall need the following results:

PROPOSITION 1. *If A/G is domestic, then A is domestic.*

PROOF. Repeat the second part of the proof of [16], Proposition (2).

PROPOSITION 2. *If A is locally support-finite, then the pushdown functor F_λ induces a bijection between the G -orbits of isomorphism classes of indecomposable A -modules and the isomorphism classes of A/G -modules. In particular, A is domestic if and only if A/G is domestic.*

PROOF. Apply [16], Theorem and Lemma (3).

1.3. For the basic definitions and results of tilting theory, we refer the reader to [25][36]. Two finite-dimensional algebras A and B are called *tilting-cotilting equivalent* if there exists a sequence of algebras $A = A_0, A_1, \dots, A_{m+1} = B$ and a sequence of modules $T_{A_i}^i$ ($0 \leq i \leq m$) such that $A_{i+1} = \text{End } T_{A_i}^i$ and T^i is either a tilting or a cotilting module. It was shown by Tachikawa and Wakamatsu that, if A and B are tilting-cotilting equivalent, then their trivial extensions $T(A)$ and $T(B)$ are stably equivalent [41]. An algebra A is called *iterated tilted of type Δ* [1] if it is tilting-cotilting equivalent to the path algebra of a quiver with underlying graph Δ , and moreover each T^i is a tilting module such that, for any indecomposable A_i -module M , we have either $\text{Hom}_{A_i}(T^i, M) = 0$ or $\text{Ext}_{A_i}^1(T^i, M) = 0$. If $m \leq 1$, we say that A is a *tilted algebra* [25]. It was shown by Happel, that, if Δ is a Dynkin or an Euclidean diagram, then A is iterated tilted of type Δ if and only if A is tilting-cotilting equivalent to the path algebra of a quiver with underlying graph Δ [24]. Moreover, an iterated tilted algebra of Euclidean type is simply connected if and only if it is of type \tilde{D}_n or \tilde{E}_p [6]. Iterated tilted algebras of type \tilde{A}_m were completely classified in [4].

1.4. Let A be a finite-dimensional algebra. Its *repetitive algebra* \hat{A} is the self-injective, locally finite-dimensional algebra [29]:

$$\hat{A} = \begin{bmatrix} & & & & 0 \\ & & & & \\ & & A_{m-1} & & \\ & & Q_m & A_m & \\ & & 0 & Q_{m+1} & A_{m+1} \\ & & 0 & & \end{bmatrix}$$

in which matrices have finitely many non-zero entries, $A_m = A$, $Q_m = {}_A(DA)_A$ for all $m \in \mathbb{Z}$, all the remaining entries are zero, and multiplication is induced from the bimodule structure of DA and the zero maps $DA \otimes_A DA \rightarrow 0$. The identity maps $A_m \rightarrow A_{m+1}$, $Q_m \rightarrow Q_{m+1}$ induce an automorphism ν of \hat{A} (called the *Nakayama automorphism*) and thus \hat{A} is a Galois covering of $T(A)$ with the infinite cyclic group generated by ν . It is shown in [41][44] (see also [23]) that if T_A is a tilting module and $B = \text{End } T_A$ then \hat{A} and \hat{B} are stably equivalent. Also, it is shown in [37] that, if \hat{A} is locally support-finite, then A is triangular. Moreover, if T_A is a tilting module and $B = \text{End } T_A$, then \hat{A} is locally support-finite if and only if \hat{B} is locally support-finite.

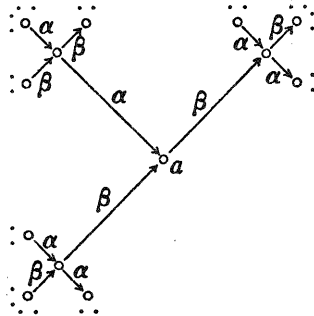
1.5. The one-point extension (respectively, coextension) of an algebra A by an A -module M will be denoted by $A[M]$ (respectively, $[M]A$). In order to handle modules over one-point extensions, we shall use vector-space category methods, for which we refer to [33][35][36]. Let A be a triangular algebra, and i be a sink in Q_A . The *reflection* $S_i^+ A$ of A at i is the quotient of the one-point extension $A[I(i)]$ by the two-sided ideal generated by e_i [29]. Dually, starting with a source j , we define the reflection $S_j^- A$. Clearly, the repetitive algebras of A and $S_i^+ A$ are isomorphic. Also, it is shown in [42] that A and $S_i^+ A$ are tilting-cotilting equivalent. Moreover, by [41], $T(A) \simeq T(S_i^+ A)$. The quiver of $S_i^+ A$ is denoted by $\sigma_i^+ Q_A$ and is called a ν -*reflection* of Q_A . The sink i of Q_A is replaced in $\sigma_i^+ Q_A$ by a source denoted by i' . A ν -*reflection sequence* of sinks i_1, \dots, i_t is a sequence of vertices of Q_A such that i_s is a sink of $\sigma_{i_{s-1}}^+ \dots \sigma_{i_1}^+ Q_A$ for $1 \leq s \leq t$.

1.6. We shall need the following well-known lemma:

LEMMA. *Let e be an idempotent in A , then $T(eAe) \simeq eT(A)e$.*

2. Branch enlargements.

2.1. We first recall from [7] the notion of branch enlargements. An *extension branch* K in a vertex a , called its *root*, is a finite connected full bound subquiver of the following infinite tree, consisting of two types of arrows: the α -arrows and the β -arrows, and bound by all possible relations of the forms $\alpha\beta=0, \beta\alpha=0$:



A coextension branch K in a is defined dually (reversing all arrows in the figure). The number of vertices in a branch K is called its *length* and is denoted by $|K|$. We shall agree to consider the empty quiver as a branch of length zero.

Let $A=kQ/I$ be a bound quiver algebra, and (Q', I') be a full bound subquiver of (Q, I) with a source a . Then A is said to be obtained from kQ'/I' by *rooting an extension branch* (Q'', I'') in a provided that (Q'', I'') is a full bound subquiver of (Q, I) such that:

- (1) $Q'_0 \cap Q''_0 = \{a\}, Q'_0 \cup Q''_0 = Q_0$.
- (2) I is generated by I', I'' and all paths $\beta\gamma$ where $\beta \in Q'_1$ has target a , and $\gamma \in Q''_1$ has source a .

Thus, each extension arrow γ can actually be considered as an α -arrow. For an extension branch K , the full connected subquiver of K consisting of all x in K such that there is a non-zero path from x to the root of K is called the *main line* of K . Thus all arrows on the main line are α -arrows. We define dually the rooting of coextension branches and main lines on coextension branches (on which all arrows are also α -arrows).

2.2. Let C be a tame concealed algebra [36][26] with a tubular family $(\mathcal{F}_\lambda)_{\lambda \in P_1(k)}$, and let E_1, \dots, E_t be pairwise non-isomorphic simple regular C -modules. For each $1 \leq i \leq t$, we let K_i be an extension branch in a_i , and K'_i be a coextension branch in a'_i , where either K_i or K'_i may be empty. We shall

define inductively the *branch enlargement* A of C by the extension branches K_i and the coextension branches K'_i . The algebra $C[E_1, K_1]$ is obtained from the one-point extension $C[E_1]$ with extension vertex a_1 by rooting the branch K_1 in a_1 , and, for $1 < j \leq t$, $C[E_i, K_i]_{i=1}^j$ is obtained from the one point extension $(C[E_i, K_i]_{i=1}^{j-1})[E_j]$ with extension vertex a_j by rooting the branch K_j in a_j . Then $B = C[E_i, K_i]_{i=1}^t$ is called the *branch extension* of C at the modules E_i by the extension branches K_i ($1 \leq i \leq t$). We now let E'_i be the unique indecomposable B -module whose restriction to C is E_i and whose restriction to K_i is the unique indecomposable module with support the main line in K_i . Then $[E'_1, K'_1]B$ is obtained from the one-point coextension $[E'_1]B$ with coextension vertex a'_1 by rooting K'_1 in a'_1 , and, for $1 < j \leq t$, ${}_{i=1}^j[E'_i, K'_i]B$ is obtained from $[E'_j]$ (${}_{i=1}^{j-1}[E'_i, K'_i]B$) with coextension vertex a'_j by rooting K'_j in a'_j . Then $A = {}_{i=1}^t[E'_i, K'_i]B$ is the required branch enlargement of C .

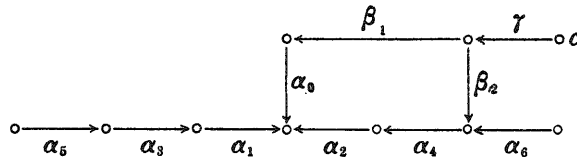
Let r_λ denote the rank of the tube \mathcal{T}_λ ($\lambda \in \mathcal{P}_1(k)$). The *tubular type* $n_A = (n_\lambda)_{\lambda \in \mathcal{P}_1(k)}$ of A is defined by:

$$n_\lambda = r_\lambda + \sum_{E_i \in \mathcal{T}_\lambda} (|K_i| + |K'_i|).$$

We write, instead of $(n_\lambda)_{\lambda \in \mathcal{P}_1(k)}$, the finite sequence consisting of at least two n_λ , keeping those which are larger than 1, and arranged in non-decreasing order. We say that n_A is *domestic*, and that A is a *domestic branch enlargement* of C if n_A is equal to: (p, q) , $p \leq q$, $(2, 2, r)$, $2 \leq r$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$. It is shown in [7] that an algebra A is a domestic branch enlargement of a tame concealed algebra if and only if A is a representation-infinite iterated tilted algebra of Euclidean type Δ . Moreover, in this case n_A equals the tubular type of a hereditary algebra of type Δ . As a direct consequence, we obtain:

LEMMA. *A domestic branch enlargement of a tame concealed algebra is 1-parametric (thus domestic).*

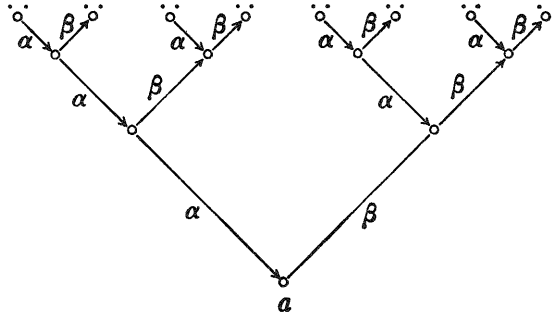
REMARK. The converse of this statement is not true. For instance, let A be given by the quiver:



bound by $\beta_1 \alpha_0 = \beta_2 \alpha_4 \alpha_2$, $\gamma \beta_1 = 0$, $\gamma \beta_2 = 0$. Here, $n_A = (2, 3, 6)$, that is, is not domestic. However, A is a one-point extension (with extension vertex c) of a

tilted algebra of Euclidean type \tilde{E}_8 by a simple injective module, and hence is 1-parametric.

2.3. A *truncated branch* in a (branch in the sense of [36]) is a finite connected full bound subquiver, containing a , of the following infinite tree bound by all possible relations of the form $\alpha\beta=0$:



If K_1, \dots, K_t are truncated branches, then the branch extension $B=C[E_i, K_i]_{i=1}^t$ is a tubular extension in the sense of [36]. It was shown by Ringel that, if A is a domestic truncated branch extension of a tame concealed algebra, then A is a tilted algebra of Euclidean type having a complete slice in its preinjective component, and conversely, every representation-infinite tilted algebra of Euclidean type is either a domestic truncated branch coextension or a domestic truncated branch extension of a tame concealed algebra [36] (4.9).

LEMMA. *Let A be a truncated branch extension of a tame concealed algebra C . Then n_A is domestic if and only if A is a domestic algebra.*

PROOF. The necessity follows from (2.2). In order to prove the sufficiency, assume that n_A is not domestic, and let B be given by a full bound subquiver of A containing C , maximal for the property that n_B is domestic. Then A also contains as full bound subquiver the bound quiver of a one-point extension or coextension B' of B . We shall show that B' is not domestic.

We claim that B' may be assumed to be a one-point extension of B . Indeed, if this is not the case, let a be the root of the branch K of B' containing the coextension vertex, and d denote the maximal distance from a to a vertex in K . If K contains a source i such that the distance from a to i equals d , then we replace B by the algebra B^* given by the full bound subquiver with vertex set given by all the vertices of B' except i . Clearly, B^* contains C and is maximal for the property that its tubular type is domestic. Moreover, B' is a one-point extension of B^* with extension vertex i . If K contains no such source, let j be an arbitrary vertex (thus, a sink) whose distance to a equals d . Since

K is a truncated branch, j is not the terminal point of a zero-relation in K . We replace B by the algebra B^{**} given by the full bound subquiver with vertex set given by all the vertices of B' except j . Again, B^{**} contains C and is maximal with the property that its tubular type is domestic, and B' is a one-point coextension of B^{**} with coextension vertex j . Applying the APR-tilting module at j , we obtain an algebra B'' which is a one-point extension of B^{**} and a truncated branch extension of C . Moreover, by [28], B'' is a domestic algebra if and only if B' is. This proves our claim.

Let thus $B' = B[M]$ with extension vertex i . Then B is a tilted algebra of Euclidean type having a complete slice \mathcal{S} in its preinjective component. Let T_B be the slice module of \mathcal{S} , and $H = \text{End } T_B$. We want to show that the full subcategory \mathcal{U} of the vector space category $\text{Hom}_B(M, \text{mod } B)$ formed by all objects of the form $\text{Hom}_B(M, X)$, where X_B is an indecomposable preinjective which is a proper predecessor of \mathcal{S} , is not domestic. Let $N_H = \text{Ext}_B^1(T, M)$. Since M_B is a regular B -module [36], N_H is an indecomposable regular H -module. Let \mathcal{V} denote the full subcategory of the vector space category $\text{Hom}_H(N, \text{mod } H)$ formed by all objects of the form $\text{Hom}_H(N, Y)$, where Y_H is indecomposable preinjective. It follows directly from the Brenner-Butler theorem [25] that $\mathcal{V} \simeq \mathcal{U}$. Let R_H denote the simple regular socle of N , and \mathcal{W} be the full subcategory of the vector space category $\text{Hom}_H(R, \text{mod } H)$ formed by all objects of the form $\text{Hom}_H(R, Y)$, where Y_H is indecomposable preinjective. Observe that \mathcal{W} is a full subcategory of \mathcal{V} . The one-point extension $H[R]$ is a tubular extension of H of the same tubular type as B' . By [35], (3.5), \mathcal{W} is non-domestic. Hence \mathcal{U} is non-domestic and the proof is complete.

2.4. In order to prove the next lemma, we shall need some notation. Let B be a branch enlargement of a tame concealed algebra, and L be a branch with root b . Let $S(L)$ be the set of all vertices x of L such that the walk w_x in L from b to x is bound by a zero-relation. Thus, $S(L) = \emptyset$ if and only if L is a truncated branch. Suppose $S(L) \neq \emptyset$ and let $x \in S(L)$. We shall denote by $d(x)$ the distance from b to the midpoint of the first zero-relation on w_x and by $d(L)$ the minimum $\min\{d(x) \mid x \in S(L)\}$. Thus $d(L) \geq 1$. Also, let $N(L)$ be the full bound subquiver of L with vertex set $\{x \in S(L) \mid d(x) = d(L)\}$. Let c_1, \dots, c_s denote the midpoints of the first zero-relations on the walks w_x , for $x \in N(L)_0$. Each c_i determines a connected component $N(c_i)$ of $N(L)$. Moreover, the distance from b to each c_i is exactly $d(L)$. For each $1 \leq i \leq s$, let γ_i denote the arrow connecting c_i to $N(c_i)$. If c_i is the source (respectively, the target) of γ_i , we let $n(c_i)$ denote the length of the maximal path starting (respectively,

ending) in c_i , and ending (respectively, starting) in $N(c_i)$. Then let $n(L)$ denote the maximum $\max\{n(c_i) \mid 1 \leq i \leq s\}$. Since $S(L) \neq \emptyset$, $n(L) \geq 1$.

LEMMA. *Let A be a branch enlargement of a tame concealed algebra C , and K be a branch in a . Then there exists a branch enlargement A' of C , obtained by replacing K by a truncated branch K' in a , such that $|K'| = |K|$ and $T(A') \simeq T(A)$.*

PROOF. If K is a truncated branch, there is nothing to show. Thus we can assume that $S(K) \neq \emptyset$. With the above notations, $S(K)$ represents the set of "bad" vertices, $d(K)$ gives a measure of the distance from a to the closer subset $N(K)$ of "bad" vertices, and $n(K)$ measures how large $N(K)$ is. We shall eliminate inductively the "bad" vertices both by reducing the number of those which lie in $N(K)$ and by sending them away from a . More precisely we shall construct a sequence of algebras $(A_i)_{i \geq 1}$ and branches $(K_i)_{i \geq 1}$ such that A_{i+1} exists if $S(K_i) \neq \emptyset$, and is obtained from A_i by replacing K_i by a new branch K_{i+1} such that:

- (i) $|K_{i+1}| = |K_i|$.
- (ii) A_{i+1} is obtained from A_i by a sequence of reflections (1.5).
- (iii) If $S(K_{i+1}) \neq \emptyset$, then either $d(K_{i+1}) = d(K_i)$ and then $n(K_{i+1}) < n(K_i)$, or else $d(K_{i+1}) > d(K_i)$.

Clearly, we reach in this way an algebra A_t and a branch K_t such that $S(K_t) = \emptyset$. But then $A_t = A'$, $K_t = K'$. Moreover, by (ii), $T(A') \simeq T(A_0)$.

Let $A_0 = A$, $K_0 = K$. Inductively, suppose that $S(K_i) \neq \emptyset$ and decompose $N(K_i)$ in disjoint connected components $N(c_j^i)$, $1 \leq j \leq s_i$, as above. We know that $n(K_i) \geq 1$. Let x_l , $1 \leq l \leq m_i$, be the set of all vertices lying in $N(K_i)$ (thus, in some $N(c_j^i)$) such that the distance from x_l to the corresponding c_j^i is exactly $n(K_i)$. In particular, each x_l is either a source or a sink. We let $A_{i+1} = S_{x_1}^{\varepsilon_1} \cdots S_{x_{m_i}}^{\varepsilon_{m_i}} A_i$ (where ε_l is $+$ if x_l is a sink, and $-$ if it is a source). The branch K_i is replaced by a new branch K_{i+1} having the same length, which clearly satisfies (iii). This completes the proof of the lemma.

2.5. LEMMA. *Let A be a branch enlargement of a tame concealed algebra C such that each coextension branch is truncated. Then there exists a branch extension B of C such that $n_A = n_B$ and $T(A) \simeq T(B)$.*

PROOF. As in (2.2), we denote respectively the extension and coextension branches of A at the simple regular modules E_i , $1 \leq i \leq t$, by K_i and K'_i and their roots by a_i and a'_i . For each i , we shall find a ν -reflection sequence (2.5)

of sinks $x(i, 1), \dots, x(i, s_i)$ in K'_i such that $B = S_{x(i, s_1)}^+ \cdots S_{x(i, 1)}^+ S_{x(i, 2)}^+ \cdots S_{x(i, 1)}^+ A$ is a branch extension of C at the modules E_i by extension branches K''_i such that $|K''_i| = |K_i| + |K'_i|$, $1 \leq i \leq t$. This implies the statement.

For each i such that $K'_i \neq \emptyset$, let S_i denote the set of vertices on the main line of K'_i , and s_i denote its cardinality. For each $x \in S_i$, let l_x denote the length of the path from a'_i to x . Now, let $x(i, 1)$ denote the unique sink in S_i , and, for each $1 \leq r \leq s_i$, let $x(i, r)$ be the unique vertex x such that $l_x = s_i - r$. Thus $x(i, s_i) = a'_i$. Clearly, $x(i, 1), \dots, x(i, s_i)$ is a ν -reflection sequence of sinks. Moreover, the branch K'_i is replaced in $S_{x(i, 1)}^+ A$ by a branch having at least one vertex less, while the corresponding extension branch K_i is replaced by a branch having at least one vertex more. This indeed follows from the fact that the restrictions to C of the extension module defining K_i and the coextension module defining K'_i are equal. An obvious induction completes the proof.

2.6. PROPOSITION. *Let A be a branch enlargement of a tame concealed algebra C . Then there exists a truncated branch extension B of C such that $n_A = n_B$ and $T(A) \simeq T(B)$.*

PROOF. By (2.4), there exists a branch enlargement A' of C such that each coextension branch is truncated, $n_{A'} = n_A$, and $T(A') \simeq T(A)$. Next, by (2.5), there exists a branch extension A'' of C such that $n_{A''} = n_{A'}$ and $T(A'') \simeq T(A')$. A further application of (2.4) to A'' yields the result.

2.7. COROLLARY. *Let A be a representation-infinite iterated tilted algebra of Euclidean type Δ . Then there exists a representation-infinite tilted algebra B of type Δ such that $T(A) \simeq T(B)$.*

2.8. COROLLARY. *Let A be a branch enlargement of a tame concealed algebra C . If $T(A)$ is domestic, then n_A is domestic.*

PROOF. By (2.6), we may assume that A is a truncated branch extension. We then apply (2.3).

REMARK. The converse of this corollary is also true, and will follow from our main result.

3. Reduction to the representation-infinite case.

In this section, we shall prove a series of preliminary results, from which we shall deduce the implication (iv) \Rightarrow (iii) of our main theorem. Also, we shall

show that in the proof of the implication (i) \Rightarrow (iv), it may be assumed that the algebra A is representation-infinite. This will allow us to use the characterisation of representation-infinite iterated tilted algebras of Euclidean type as domestic branch enlargements of a tame concealed algebra (2.2).

3.1. LEMMA. *Let A be a representation-finite simply connected algebra, and i be a sink in Q_A . Then $A' = S_{i^+}^{\dagger}A$ is Schurian and simply connected.*

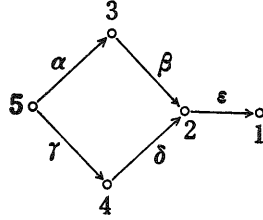
PROOF. Since A is representation-finite, $I(i)_A$ is multiplicity-free (that is, for each $a \in (Q_A)_0$, $\dim_k \text{Hom}_A(P(a), I(i)) \leq 1$). This implies that $A[I(i)]$ is Schurian and therefore that A' is Schurian. Consequently, all relations in the bound quivers of A and A' are zero-relations and commutativity relations.

Let w be a closed walk in $A' = S_{i^+}^{\dagger}A$. We claim that w is contractible. Clearly, we may assume that the walk w is reduced, that is, it contains no pairs of the form $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for $\alpha \in (Q_A)_1$. It w does not pass through i' , it is a walk in $B = A/\langle e_i \rangle$ which is representation-finite and simply connected (because it is a full convex subcategory of the representation-finite simply connected category A [14]) and therefore, w is contractible. If, on the other hand, w passes through i' and is given by two parallel paths from i' to j (say) which are linearly dependent in $A'(i', j)$, there is nothing to show. We may thus assume that w passes through i' but is not of this form. We claim that w is homotopic in A' to another walk w' which does not pass through i' . Since by the previous reasoning w' is contractible, we are done. Let thus $\alpha: i' \rightarrow a$, $\beta: i' \rightarrow b$ be arrows through i' on the walk w . Since w is reduced, $\alpha \neq \beta$. Observe that $S(a)_A$ and $S(b)_A$ belong to the top of $I(i)_A$, therefore of $I(i)/S(i)$. However, since A^{op} is representation-finite and simply connected, it satisfies the (S)-condition [10] and therefore $I(i)/S(i)$ is separated. Since a and b are connected by a subwalk of w lying entirely in B , they belong to the same indecomposable summand of $I(i)/S(i)$. Since $I(i)_A$ is multiplicity-free, there exists in Q_A a sequence of vertices and paths of the form $a = a_0 \xrightarrow{u_1} a_1 \xleftarrow{v_1} a_2 \xrightarrow{u_2} a_3 \xleftarrow{v_2} \dots \xleftarrow{v_m} a_{2m} = b$ such that the compositions αu_1 and βv_m are non-zero (in $A[I(i)]$) and, for each $0 \leq t \leq m$, $S(a_{2t})$ belongs to the top of $I(i)$. But this implies that there exists an arrow $\alpha_{t+1}: i' \rightarrow a_{2t}$ such that $\alpha_1 = \alpha$, $\alpha_m = \beta$ and, for each $0 \leq t \leq m$, there exists a commutativity relation (in $A[I(i)]$, but then in A') between $\alpha_t u_t$ and $\alpha_{t+1} v_t$. Consequently, for each $1 \leq t \leq m$, $v_t u_t^{-1}$ is homotopic to $\alpha_{t+1}^{-1} \alpha_t$ in A' . By symmetry, we can assume that $w = w_1 \beta^{-1} \alpha w_2$. But then w is homotopic to $w_1 v_m u_m^{-1} \dots v_2 u_2^{-1} v_1 u_1^{-1} w_2$ which lies entirely in B . This shows our

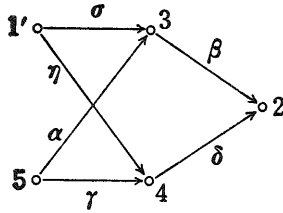
claim and hence the lemma.

REMARKS. 1. Actually, it is possible to show that A' even satisfies the (S)-condition (see [2]).

2. In general, A' is not \tilde{A} -free, even if A is. Indeed, let A be given by the quiver :



bound by $\alpha\beta=\gamma\delta$ and $\alpha\beta\epsilon=0$. Then $A'=S_1^+A$ is given by the quiver :



bound by $\alpha\beta=\gamma\delta$ and $\sigma\beta=\eta\delta$.

3.2. LEMMA. Let A be an algebra, and i a sink in Q_A such that $I(i)_A$ satisfies the following condition :

(*) For every indecomposable A -module $M \not\cong I(i)$ such that $\text{Hom}_A(I(i), M) \neq 0$, we have $\text{Hom}_A(M, I(i))=0$.

Then every indecomposable $A[I(i)]$ -module which is not isomorphic to $P(i')$ is an A -module or a S_i^+A -module.

PROOF. Recall that $\text{mod } A[I(i)]$ is equivalent to the category of triples (V, M, φ) , where V is a finite-dimensional k -vector space, M a finitely generated A -module, and $\varphi: V \rightarrow \text{Hom}_A(I(i), M)$ is a k -linear map, with the obvious morphisms. Let thus $\bar{M}=(V_k, M_A, \varphi)$ be an arbitrary indecomposable $A[I(i)]$ -module. Here the map φ can be assumed to be injective. Suppose $\varphi \neq 0$. It follows from our assumption that we can decompose M as $M \cong I(i)^m \oplus N$, where $\text{Hom}_A(N, I(i))=0$. Suppose now $m > 0$. Then we can write $M \cong I(i) \oplus M'$. Consider the mapping $\phi = \text{Hom}_A(I(i), [1 \ 0]) \varphi: V_k \rightarrow \text{Hom}_A(I(i), I(i) \oplus M') \rightarrow \text{Hom}_A(I(i), I(i))$. We claim that $\phi \neq 0$. Indeed, if $\phi=0$, the commutative diagram :

$$\begin{array}{ccccc}
0 & \xrightarrow{\quad\quad\quad} & V & \xrightarrow{\quad\quad\quad} & 0 \\
\downarrow & & \downarrow \varphi & & \downarrow \\
\text{Hom}_A(I(i), I(i)) & \xrightarrow{\text{Hom}_A\left(I(i), \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)} & \text{Hom}_A(I(i), I(i) \oplus M') & \xrightarrow{\text{Hom}_A(I(i), [10])} & \text{Hom}_A(I(i), I(i))
\end{array}$$

implies that \bar{M} has a direct summand of the form $(0, I(i), 0)$ which contradicts either the indecomposability of \bar{M} or the assumption that $\varphi \neq 0$.

Let thus $v \in V$ be such that $\varphi(v) \neq 0$. Hence $\varphi(v) = \begin{bmatrix} \lambda \\ x \end{bmatrix} \in \text{Hom}_A(I(i), M) \simeq \text{Hom}_A(I(i), I(i)) \oplus \text{Hom}_A(I(i), M')$, with $\lambda \in k^*$. Letting $u = \lambda^{-1}v$, we have $\varphi(u) = \begin{bmatrix} 1 \\ \lambda^{-1}x \end{bmatrix}$ and we can write $V = ku \oplus U$. Thus $\varphi = \begin{bmatrix} 1 & f \\ g & h \end{bmatrix}$. Consider the commutative diagram:

$$\begin{array}{ccc}
ku \oplus U & \xrightarrow{\begin{bmatrix} 1 & f \\ g & h \end{bmatrix}} & \text{Hom}_A(I(i), I(i)) \oplus \text{Hom}_A(I(i), M') \\
\downarrow \begin{bmatrix} 1 & f \\ \lambda^{-1}x & \end{bmatrix} & \quad \quad \quad & \downarrow \begin{bmatrix} 1 & 0 \\ & \end{bmatrix} \\
ku \simeq k & \xrightarrow{\quad 1 \quad} & k \simeq \text{Hom}_A(I(i), I(i)).
\end{array}$$

Clearly, the map $\begin{bmatrix} 1 & f \\ 1 & 0 \end{bmatrix}$ define an epimorphism $\bar{M} \rightarrow P(i') = (k, I(i), 1)$. By the indecomposability of \bar{M} , $\bar{M} \simeq P(i')$. Therefore, if $\bar{M} \simeq P(i')$, then either $\varphi = 0$ or $m = 0$. In the former case, M is an A -module, and in the latter a $S_i^+ A$ -module.

3.3. PROPOSITION. *Let A be a representation-finite simply connected algebra such that, for every strong ν -reflection sequence of sinks $i_1, \dots, i_t, S_{i_t}^+ \dots S_{i_1}^+ A$ is representation-finite. Then \hat{A} is locally representation-finite.*

PROOF. Let B be the full subcategory of \hat{A} with the objects of A_p , for all $p \geq 0$ (see (1.4)). First we shall show that, under the stated hypothesis, B is locally representation-finite. This is done by constructing a component of Γ_B as in [2] (see also [29]). This is possible since by hypothesis and (3.1), all the algebras $S_{i_t}^+ \dots S_{i_1}^+ A$ are representation-finite and simply connected, so we can apply (3.2). We then obtain a bounded length component \mathcal{C} of Γ_B . Indeed, all indecomposables which are not projective-injectives are indecomposables over representation-finite simply connected algebras having the same number of simples. It follows from a theorem of Auslander [8] that $\mathcal{C} = \Gamma_B$ and therefore B is locally representation-finite. This also implies that every indecomposable non-projective-injective B -module is a $S_{i_t}^+ \dots S_{i_1}^+ A$ -module, for some sequence i_1, \dots, i_t . In particular, the support of any indecomposable B -module has at most $n+1$ vertices, where n is the cardinality of $(Q_A)_0$.

We now claim that the last statement holds in fact for any indecomposable \hat{A} -module M . Let $S = \text{Supp } M$. There exists $p \in \mathbb{Z}$ such that $\nu^p S$ is contained

in B . Let N denote the image of M under the automorphism ν^p of $\text{mod } \hat{A}$. Clearly, N is an indecomposable B -module having support equal to $\nu^p S$. Since $\nu^p S$ has at most $n+1$ vertices this is also true for S .

For each object x of \hat{A} , we define inductively a family $\mathcal{C}_m(x)$, $m \in \mathbf{N}$, of finite full subcategories of A as follows: $\mathcal{C}_0(x)$ is the full subcategory of A having x as a single object, and, for $m \geq 0$ $\mathcal{C}_{m+1}(x)$ is the full subcategory of \hat{A} formed by all objects y such that $\hat{A}(y, z) \neq 0$ or $\hat{A}(z, y) \neq 0$ for some $z \in \mathcal{C}_m(x)$. Since \hat{A} is connected, it is the union of the $\mathcal{C}_m(x)$. For each fixed object x , there exists $q \in \mathbf{Z}$ such that $\mathcal{C}_{n+1}(\nu^q x)$ is contained in B . Then ν^q induces a bijection between the isomorphism classes of indecomposable \hat{A} -modules M with $M(x) \neq 0$ and the isomorphism classes of indecomposable B -modules N with $N(\nu^q x) \neq 0$. Indeed, $\nu^q(\text{Supp } M)$ has at most $n+1$ vertices, hence it is contained in $\mathcal{C}_{n+1}(\nu^q x)$. Since B is locally representation-finite, so is \hat{A} .

3.4. COROLLARY. *Let A be a representation-finite simply connected algebra which is not an iterated tilted algebra of Dynkin type. Then there exists a strong ν -reflection sequence of sinks i_1, \dots, i_t such that $S_{i_{t-1}}^+ \cdots S_{i_1}^+ A$ is representation-finite, but $A' = S_{i_t}^+ \cdots S_{i_1}^+ A$ is representation-infinite.*

PROOF. We apply (3.3) and the fact that A is iterated tilted of Dynkin type if and only if \hat{A} is locally representation-finite [2].

3.5. PROPOSITION. *Let A be an iterated tilted algebra of Euclidean type Δ . Then there exists a representation-infinite tilted algebra B of type Δ such that $T(A) \simeq T(B)$.*

PROOF. If A is of type \tilde{A}_m , it follows from the description in [4] that, by applying a sequence of reflections to the sources and sinks of the unique cycle, there exists a representation-infinite iterated tilted algebra A' of the same type such that $T(A) \simeq T(A')$. If A is of type \tilde{D}_n or \tilde{E}_p , it is simply connected (1.3), and by (3.4), there exists a representation-infinite algebra A' obtained from A by a sequence of reflections that is, iterated tilted of the same type (by (1.3)) and such that $T(A') \simeq T(A)$ (by (1.5)). But then a direct application of (2.7) yields the result.

This shows the implication (iv) \Rightarrow (iii) of our main theorem.

4. Proof of the implication (iii) \Rightarrow (ii):

We shall need the following two lemmas:

4.1. LEMMA. *Let B be a domestic truncated branch extension of a tame concealed algebra C . If i is a strong sink in B , then i belongs to C .*

PROOF. Let \mathcal{S} denote a complete slice in the preinjective component of Γ_B containing $I(i)$, T_B the slice module of \mathcal{S} and $H = \text{End } T_B$. We claim that, if j is a sink in a branch, then $I(j)$ does not belong to \mathcal{S} (and thus, in particular, $j \neq i$). Since j lies in a branch, $P(j)_B$ lies in the regular component of Γ_B . Thus, it is not a summand of T and, by the connecting lemma [25], $\tau_H \text{Ext}_B^1(T, P(j)) \simeq \text{Hom}_B(T, I(j))$. Since $\text{Hom}_B(T, I(j))$ belongs to a tube in Γ_H , $I(j)$ does not belong to \mathcal{S} .

4.2. LEMMA. *Let B be a simply connected domestic truncated branch extension of a tame concealed algebra C . If i is a strong sink in B such that $B[I(i)]$ is a finite enlargement of B (in the sense of [35] (2.6)), then $S_+^1 B$ is representation-finite.*

PROOF. Let B' denote the full subcategory of B consisting of all the objects of B except i . By definition of finite enlargement and (3.2), there are only finitely many isomorphism classes of indecomposable $S_+^1 B$ -modules which are not B -modules. Since the remaining indecomposable $S_+^1 B$ -modules are B' -modules, it suffices to show that B' is representation-finite.

First, we prove that B' is Schurian, $\tilde{\mathcal{A}}$ -free and simply connected. Indeed, by (4.1), i belongs to C hence the first two assertions. If C is hereditary of type $\tilde{\mathcal{A}}_m$ or a non-Schurian tame concealed algebra, it is clear that B' is simply connected, while if C is $\tilde{\mathcal{A}}$ -free and Schurian then, by [14][11], B' as a full convex subcategory of B is simply connected. Therefore, if B' is representation-infinite, then, by [11][12][26] it contains a tame concealed algebra C' as a full convex subcategory. Observe that $C' \neq C$, since C' does not contain i . Now, since B is a tilted algebra of Euclidean type, its homological quadratic form q_B has corank one. Since C (respectively, C') is tame concealed, the restriction q_C (respectively, $q_{C'}$) of q_B to it has a sincere radical vector x (respectively, x'). Since x and x' are clearly linearly independent in the Grothendieck group $K_0(B)$, we obtain a contradiction and thus B' is representation-finite.

4.3. We now proceed to show the implication (iii) \Rightarrow (ii) of our main theorem, namely we assume that B is a representation-infinite tilted algebra of type $\Delta = \tilde{D}_n$ or \tilde{E}_p , and we claim that its trivial extension $T(B)$ is 2-parametric. We may assume, up to duality, that B is a truncated branch extension of a tame concealed algebra. Let i_1 be a strong sink in B . By (3.2), every indecomposable

$B[I(i_1)]$ -module which is not projective-injective belongs either to $\text{mod } B$ or to $\text{mod}(S_{i_1}^+ B)$. Moreover, $S_{i_1}^+ B$ is also a tilted algebra of type Δ (and thus is simply connected (1.3)). Indeed, there exists a complete slice in the preinjective component of Γ_B having $I(i_1)_B$ as a source. In particular, all indecomposable modules of \mathcal{S} besides $I(i_1)$ do not have $S(i_1)$ as a simple composition factor, thus are also $S_{i_1}^+ B$ -modules. We obtain a complete slice \mathcal{S}' of $\Gamma_{S_{i_1}^+ B}$ by replacing the B -module $I(i_1)$ by the $S_{i_1}^+ B$ -module $\tau_{B[I(i_1)]}^{-1} I(i_1)$, and every arrow of \mathcal{S} of the form $\alpha: I(i_1) \rightarrow X$ by the corresponding arrow $\sigma^{-1}\alpha: X \rightarrow \tau_{B[I(i_1)]}^{-1} I(i_1)$.

Assume that $B[I(i_1)]$ is a finite enlargement of B , then, by (4.2), $S_{i_1}^+ B$ is representation-finite and all indecomposable injective $S_{i_1}^+ B$ -modules lie in the (finite) set of successors of \mathcal{S}' . Let i_2 be a strong sink in $S_{i_1}^+ B$. Then $S_{i_2}^+ S_{i_1}^+ B$ is again a tilted algebra of type Δ and any indecomposable $B[I(i_1)][I(i_2)]$ -module which is not projective-injective belongs to $\text{mod } B$, $\text{mod}(S_{i_1}^+ B)$ or $\text{mod}(S_{i_2}^+ S_{i_1}^+ B)$. By (3.3), there exists a least index $t \in \mathbb{N}$ such that we have a strong ν -reflection sequence of sinks i_1, \dots, i_t such that $B[I(i_1)] \cdots [I(i_t)]$ is an infinite enlargement of B . Then $R = S_{i_t}^+ \cdots S_{i_1}^+ B$ is a representation-infinite tilted algebra of type Δ having a complete slice in its preprojective component and, by (2.3), a truncated branch coextension of a tame concealed algebra C' , say. Moreover, all indecomposable injective R -modules occur either in the regular or the preinjective component of Γ_R .

Suppose that some indecomposable injective R -module lies in the regular component of Γ_R . We shall apply successive reflections in the following way. We start by fixing an (arbitrary) ordering K'_1, \dots, K'_s of the coextension branches of R . For each branch K'_j , we let S_j denote the set of vertices on the main line, and s_j denote its cardinality. For each $x \in S_j$, we let l_x denote the length of the path from the root to x and, for each $1 \leq r \leq s_j$, let $x(j, r)$ denote the unique vertex x such that $l_x = s_j - r$ (see the proof of (2.5)). Then $x(j, 1), \dots, x(j, s_j)$ is, for each j , a ν -reflection sequence of sinks. Let $i_{t+1} = x(1, 1)$ and consider $R[I(i_{t+1})]$. Let \mathcal{U} denote the full subcategory of the vector space category $\text{Hom}_R(I(i_{t+1}), \text{mod } R)$ consisting of the objects of the form $\text{Hom}_R(I(i_{t+1}), X)$, for X_R indecomposable regular in the tube of Γ_R containing $I(i_{t+1})$. It follows from the structure of the tubes in Γ_R that $\mathcal{U} \simeq \text{add}(kS)$ where S is a partially ordered set of the form:

$$p' > \dots > 2' > 1' > 0 < 1 < 2 < \dots < q < \dots$$

($p \geq 0$). Thus the algebra $R[I(i_{t+1})]$ has a preprojective component which coincides with that of R . The regular component of $R[I(i_{t+1})]$ is obtained from that of R by $p+1$ ray insertions in the tube of Γ_R containing $I(i_{t+1})$, and its

preinjective component by a resulting infinite enlargement of that of R [20]. Observe that, since $I(i_{t+1})_{R[I(i_{t+1})]}$ lies in the regular component, the preinjective $R[I(i_{t+1})]$ -modules are $S_{i_{t+1}}^+ R$ -modules. Repeating this procedure on all the sinks of the form $x(1, r)$, $1 \leq r \leq s_i$, we replace the truncated coextension branch K'_i by a corresponding extension branch K_i (2.5). Repeating this procedure on all other coextension branches K'_j , we replace R by a branch extension R' by branches K_1, \dots, K_s which are generally not truncated, but which satisfy the following property: relations of the form $\alpha\beta=0$ can only occur whenever α belongs to the main line.

Our next objective is to replace R' by a truncated branch extension, by applying the method explained in (2.4). Observe that, by the above property, for each branch K_j , the vertices in $N(K_j)$ (with the notations of (2.4)) can be arranged in a ν -reflection sequence of sinks: indeed, if $N(K_j)$ is decomposed in its disjoint connected components $N(c_l^j)$, $1 \leq l \leq m_j$, then the arrow γ_l^j connecting c_l^j to $N(c_l^j)$ has always c_l^j as a source, thus we need only consider the set of all maximal paths starting in c_l^j and ending inside $N(c_l^j)$. Observe also that, for each x in $N(K_i)$, $I(x)$ lies in the tube corresponding to the extension branch: indeed, let R'' be a truncated branch extension of C' , which is maximal for being a full bound subquiver of R' , then, for each (j, l) , $S(c_l^j)_{R''}$ belongs to a tube in $\Gamma_{R''}$, therefore the injective corresponding to the target of γ_l^j (which has $S(c_l^j)$ as a socle factor) lies in the same tube of $\Gamma_{R''}$. We deduce that no indecomposable preinjective R' -module has $S(x)$ as a simple composition factor.

We now apply a sequence of reflections as in (2.4). We have two possibilities. If the walk connecting the sink i under consideration to the root of its branch is bound by at least two zero-relations, the vector space category $\text{Hom}_{R'}(I(i), \text{mod } R')$ is equivalent to the vector space category $\text{add}(kS)$, where S is a partially ordered set of the form

$$p' > \dots > 1' > 0 < 1 < \dots < q$$

($p \geq 0$ and $q > 0$). Then $\text{mod } R'[I(i)]$ is obtained from $\text{mod } R'$ by a finite enlargement in the tube containing $I(i)$ [34]. If this walk contains exactly one zero-relation, $\text{Hom}_{R'}(I(i), \text{mod } R')$ is equivalent to $\text{add}(kS)$, where S is a partially ordered set of the form:

$$p' > \dots > 1' > 0 < 1 < \dots < q < \dots$$

($p \geq 0$). Then $\text{mod } R'[I(i)]$ is obtained from $\text{mod } R'$ by $p+1$ ray insertions in the tubes and an infinite enlargement of the preinjective component [20]. In both cases, the preinjective component of $R'[I(i)]$ is in fact that of $S_i^+ R'$.

Applying the procedure in (2.4), we find a least $s \in \mathcal{N}$ such that $E = S_{i_s}^+ \cdots S_{i_1}^+ R'$ is a tilted algebra having all its injectives in the preinjective component. Thus, E is a truncated branch extension of a tame concealed algebra, and we are in a situation similar to that at the starting point (that is, for B).

As before, there exists a least index r such that we have a ν -reflection sequence of sinks i_1, \dots, i_r with $B' = S_{i_r}^+ \cdots S_{i_1}^+ E$ a tilted algebra having all its injectives in the preinjective component (that is, B' is obtained from E in exactly the same way as E is obtained from B). Since B' is tilting-cotilting equivalent to B , it is also of type Δ (1.3). We shall now show that we have in fact described \hat{B} . We use the notations of (1.4). First, we claim that B' equals the image B_1 of $B = B_0$ under the action of ν .

To prove our claim, we shall consider the tubes of rank 1 in the previous construction, that is, the full subcategories given by the regular homogeneous modules. The first such family of tubes \mathcal{A}_B occurs in Γ_B . Since, clearly, finite enlargements do not create tubes, the next family occurs in Γ_R . Since the modules in these two families have distinct supports in \hat{B} , the two families are distinct. In passing from R to E , we did not affect the tubes of rank 1 in Γ_R neither did we create a new family, that is, the tubes of rank 1 in Γ_R are the same as those in Γ_E (when both are embedded in $\Gamma_{\hat{B}}$). Applying the same process, the next family of tubes of rank 1 $\mathcal{A}_{B'}$ occurs in $\Gamma_{B'}$. Note that \mathcal{A}_B and $\mathcal{A}_{B'}$, considered as subcategories of $\text{mod } \hat{B}$ are isomorphic to their images in $\underline{\text{mod}} \hat{B}$. Now the structures of $\underline{\text{mod}} \hat{B}$ and $\underline{\text{mod}} T(B)$ are known [23][41][44]. In particular, $\underline{\text{mod}} T(B)$ contains two families of tubes of rank 1, each corresponding to a ν -orbit of families of tubes of rank 1 (separated by two transjective components) in $\underline{\text{mod}} \hat{B}$. Hence $\nu(\mathcal{A}_B) = \mathcal{A}_{B'}$. We next observe that in every reflection step in passing from B to B' , we have only used vertices which lie in $B_0 (=B)$: indeed, since B_1 is a truncated branch extension of a tame concealed algebra, all indecomposable injective B_1 -modules lie in its preinjective component and since $\mathcal{A}_{B'} = \mathcal{A}_{B_1}$, it follows that no vertex of B_1 was used in passing from B to B' . Furthermore, since B and B' are both tilted of type Δ , and since the description of $\underline{\text{mod}} \hat{B}$ [23] implies that, for any vertex i of B_0 , the preinjective component of B' does not contain the injective module with socle $S(i)$, all the vertices of B_0 were used in reflection steps. Thus, B' coincides with $B_1 = \nu(B_0)$.

Let now, for $p < q$ in \mathcal{Z} , $B_{p,q}$ denote the full subcategory of \hat{B} consisting of the objects of B_r , $p \leq r \leq q$. We claim that any indecomposable $B_{p,q}$ -module is actually a $B_{r, r+1}$ -module for some $p \leq r \leq q-1$. Indeed, observe that $B_{p, q+1}$ is obtained from $B_{p,q}$ by a sequence of one-point extensions by modules whose

restrictions to $B_{p,q}$ are either 0 or an indecomposable injective B_q -module. From the previous considerations, it follows that any $B_{p,q+1}$ -indecomposable module is either a $B_{p,q}$ -module or a $B_{q,q+1}$ -module. Dually, any indecomposable $B_{p-1,q}$ -module is either a $B_{p-1,p}$ -module or a $B_{p,q}$ -module. This shows our claim. Consequently, any indecomposable \hat{B} -module is actually a $B_{r,r+1}$ -module, for some $r \in \mathbf{Z}$. Since, for each $r \in \mathbf{Z}$, $B_{r,r+1} = \nu^r(B_{0,1})$ it follows that $B_{r,r+1}$ is 3-parametric (thus \hat{B} is locally support-finite and domestic). Therefore, by (1.2), $T(B)$ is 2-parametric and the proof is complete.

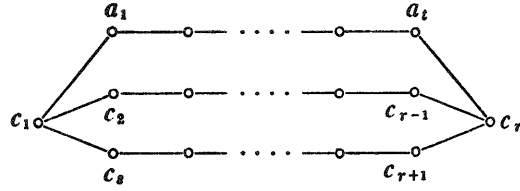
5. Preparatory lemmas.

5.1. LEMMA. *Let $B=C[M]$ be a one-point extension of an algebra C , and let $X_i, i \geq 1$, be an infinite family of C -modules such that $\text{End } X_i \simeq_k$ for all i , with pairwise different dimension-vectors and such that $\dim_k \text{Hom}_C(M, X_i) = 2$ for each i . Then B is not domestic.*

PROOF. We construct, for each i , a family of indecomposable B -modules by setting $Y_i(\lambda) = \left(k, X_i, \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \right)$ ($\lambda \in k^*$). Then $\underline{\dim} Y_i(\lambda) = (1, \underline{\dim} X_i)$ and, by hypothesis, $\underline{\dim} Y_i(\lambda) \neq \underline{\dim} Y_j(\mu)$ for $i \neq j$ and $\lambda, \mu \in k^*$. Suppose that B is domestic, and let $F_l: \text{mod } k[X] \rightarrow \text{mod } B, 1 \leq l \leq n$, be a finite family of parametrising functors. If $\underline{\dim} F_l(k[X]/(X-\lambda)) = (d_l, x_l)$, then $\underline{\dim} F_l(k[X]/(X-\lambda)^m) = (md_l, mx_l)$ for each $m \geq 1$. Assume that infinitely many indecomposable B -modules have $(1, x)$ as a dimension-vector. Since B is domestic, all but finitely many of these modules are of the form $F_l(M)$ where M is an indecomposable $k[X]$ -module and actually, by the previous formula, a simple $k[X]$ -module. This implies that there are only finitely many dimension-vectors of the form $(1, x)$ such that infinitely many non-isomorphic indecomposable B -modules have this dimension-vector. Hence there exists i such that all $Y_i(\lambda), \lambda \in k^*$, are not in the image of one of the F_l , a contradiction.

5.2. LEMMA. *Let B be an algebra whose bound quiver consists of a full subcategory C which is hereditary of type \tilde{A}_m and objects of a walk w connecting two different objects of C , and assume that B is bound only by zero-relations. Then $T(B)$ is not domestic.*

PROOF. The quiver of B has the form:



where c_i are the vertices of C , and the walk w is equal to $c_1 - a_1 - \dots - a_t - c_r$. If w is a path, then, since C is a full subcategory of the zero-relations algebra B , w must be bound by zero-relations. In particular, $t \geq 1$. Taking a suitable full subcategory of B , we can assume that the walks w , $c_1 - c_2 - \dots - c_r$, $c_1 - c_s - \dots - c_r$ have radical square zero. Let α denote the arrow joining c_1 and c_2 .

Suppose that $T(B)$ is domestic, and let $F_l: k[X] \rightarrow \text{mod } T(B)$, $1 \leq l \leq n$, be a finite family of parametrising functors. We first observe that B is gentle. Indeed, if one of the subcategories formed by the objects a_1, c_1, c_2, c_s or $a_t, c_r, c_{r-1}, c_{r+1}$ is not gentle, then $\text{mod } T(B)$ contains $\text{mod } H$, for H a wild hereditary algebra which is a one-point extension or coextension of C , contrary to the assumption that $T(B)$ is domestic. Thus B is gentle. It is easily seen that this implies that B , and consequently $T(B)$, are special biserial. The full subcategory D of B formed by the objects $a_1, \dots, a_t, c_1, \dots, c_r$ is a gentle cycle. Let m denote the absolute value of the difference between the numbers of clockwise and counterclockwise oriented zero-relations in D . By [37], Lemma (2), $T(B)$ con-

tains a free closed walk $v: c_1 \xrightarrow{\alpha} c_2 - \dots - a_1 - c_1$ containing α and passing through each object of D once if m is even and twice if m is odd. Let u be the (free) closed walk around C in $B: u: c_1 \xrightarrow{\alpha} c_2 - \dots - c_{r+1} - c_r - \dots - c_s$, and consider the (non-periodic) closed walks in $T(B)$ defined by vu^j , $j \geq 1$. By [18], each of them defines a functor $G_j: \text{mod } k[X, X^{-1}] \rightarrow \text{mod } T(B)$ such that:

(i) For each j , $G_j = - \bigoplus_{k[X, X^{-1}]} Q'_j$ where Q'_j is a $k[X, X^{-1}] - T(B)$ bimodule, finitely generated and free as a left $k[X, X^{-1}]$ -module. Moreover, for any $1 \leq i \leq t$, $Q'_j(a_i) = k[X, X^{-1}]$ if m is even, and $Q'_j(a_i) = k[X, X^{-1}]^2$ if m is odd.

(ii) For any fixed j , the $T(B)$ -modules $X_j(\lambda) = G_j(k[X, X^{-1}]/(X - \lambda))$, $\lambda \in k^*$, are indecomposable non-isomorphic, and have the same dimension-vector.

(iii) $\underline{\dim} X_j(\lambda) \neq \underline{\dim} X_l(\mu)$ for $j \neq l$ and $\lambda, \mu \in k^*$.

As in (5.1), it follows from the hypothesis that $T(B)$ is domestic that there are only finitely many dimension-vectors having 1 or 2 at the vertices a_1, \dots, a_t such that infinitely many non-isomorphic indecomposable $T(B)$ -modules have this

dimension-vector. Hence there exists an index j such that all $X_j(\lambda)$, $\lambda \in k^*$, are not in the image of one of the functors F_i , a contradiction.

5.3. LEMMA. *Let $B=C[M]$ be a one-point extension of a tame concealed algebra C such that $T(B)$ is domestic. Then M is a regular C -module.*

PROOF. If M_C has a preprojective direct summand, then, as in [35], Lemma (3), p. 211, B is not tame and thus $T(B)$ is not domestic. Suppose M has a preinjective direct summand N , and let a be the extension vertex defining $B=C[M]$. Let $B'=S_{\bar{a}}(B)=[M]C$, then $B'^{op}=C^{op}[DM]$ and DM has the preprojective direct summand DN . Hence $T(B) \simeq T(B') \simeq T(B'^{op})^{op}$ is not domestic.

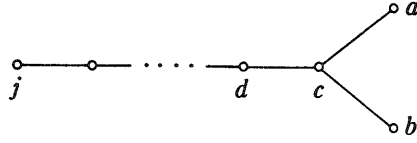
5.4. LEMMA. *Let $B=C[M]$ be a domestic one-point extension of a tame concealed algebra C of type \tilde{D}_n or \tilde{E}_p . Then M_C is a simple regular non-homogeneous C -module.*

PROOF. Let T be the slice module of a complete slice in Γ_C , and $H=\text{End } T$. Then $\text{Hom}_C(T, M)$ is a regular H -module, and is simple non-homogeneous if and only if M is. The result then follows from (5.1) and [35] (3.5).

5.5. LEMMA. *Let $B=C[M]$ be a one-point extension of a hereditary algebra of type \tilde{A}_m such that $T(B)$ is domestic. Then M is either simple regular or regular indecomposable of regular length two lying in a tube of rank at least two.*

PROOF. Since $T(B)$ is domestic, so is B and by [35] (3.5), M is regular of regular length at most two with non-isomorphic regular composition factors. Hence if M is indecomposable and not simple regular, it lies in a tube of rank at least two. Suppose $M \simeq N_1 \oplus N_2$ where N_1 and N_2 are simple regular. If N_1 and N_2 are in different tubes then, by [35] (3.5), the vector space category $\text{Hom}_C(M, \text{mod } C)$ is of type (\tilde{A}_{pq}, p, q) and by (5.1), B is non-domestic. If N_1 and N_2 are in the same tube, then, since $N_1 \not\cong N_2$, this tube has rank at least two. Then B satisfies the hypothesis of (5.2), and we obtain a contradiction.

5.6. LEMMA. *Let C be a tame concealed algebra of type \tilde{D}_n or \tilde{E}_p , X_C a simple regular non-homogeneous module and $B=C[X]$ (respectively, $B=[X]C$). If i denotes the extension (respectively, coextension) vertex corresponding to X , and A is obtained from B by identifying i to the vertex j in a quiver with underlying graph as follows:*



then A is not domestic.

PROOF. By duality, we can assume that $B=C[X]$. Let A' be given by the full bound subquiver of A consisting of all vertices except a . Then A' is a truncated branch extension of C . If $n_{A'}$ is not domestic, we are done by (2.3). Assume thus that $n_{A'}$ is domestic. It follows from the hypothesis that $n_{A'} \neq (p, q)$ ($1 \leq p \leq q$) and therefore A' is tilted of type \tilde{D}_n or \tilde{E}_p . By definition, A is a one-point extension or coextension of A' . In the latter case, applying an APR-tilting module corresponding to the sink a , we replace A by a new algebra A^* which is a one-point extension of A' and which is domestic if and only if A is [28]. We may thus assume that A is a one-point extension of A' , and also that c is either a source or a sink in A' . Since $\text{rad } P(a) = P(c)$, we have $A = A'[P(c)]$. Let now \mathcal{S} be a complete slice in the preinjective component of A' . We shall prove that the full subcategory \mathcal{U} of the vector space category $\text{Hom}_{A'}(P(c), \text{mod } A')$ formed by all objects of the form $\text{Hom}_{A'}(P(c), Z)$, where Z is an indecomposable preinjective A' -module which is a proper predecessor of \mathcal{S} , is not domestic.

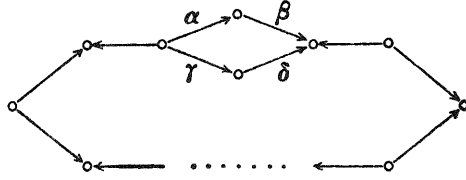
Let $T_{A'}$ be the slice module of \mathcal{S} and $H = \text{End } T$. We claim that $R_H = \text{Ext}_{A'}^1(T, P(c))$ is a regular H -module of regular length at least two. Clearly, it is regular, since $P(c)_{A'}$ is. We shall now apply the connecting lemma [25], assuming that $d \leftarrow c \rightarrow b$: then $\text{rad } P(c) = P(b) \oplus P(d)$ and $I(c) = S(c)$, and the connecting sequence for $P(c)$ (which is not a summand of T) is:

$$0 \longrightarrow \text{Hom}(T, I(c)) \longrightarrow \text{Ext}^1(T, P(b)) \oplus \text{Ext}^1(T, P(d)) \longrightarrow \text{Ext}^1(T, P(c)) = R \rightarrow 0.$$

Since both middle terms are non-zero, R_H is not simple regular.

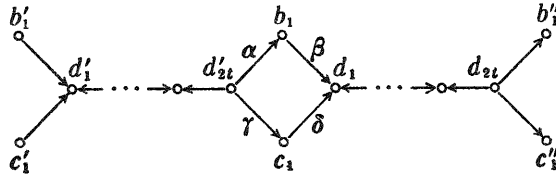
Repeating this reasoning for the opposite orientation of the subgraph $d-c-b$, we prove our claim. It then follows, by (5.1) and [35] (3.5), that the full subcategory \mathcal{CV} of the vector space category $\text{Hom}_H(R, Y)$, where Y_H is indecomposable preinjective, is not domestic. However, as in (2.3), $\mathcal{CV} \simeq \mathcal{U}$ and consequently A is not domestic.

5.7. COROLLARY. *Let B be given by the quiver:*



bound by $\alpha\beta=\lambda$. $\gamma\delta$ ($\lambda \in k^*$) and these are the only paths of length at least two. Then B is not domestic.

PROOF. The universal cover of B contains a full subcategory B' given by the quiver:



bound by $\alpha\beta=\lambda\cdot\gamma\delta$. The full subcategory K of B' formed by the objects $d'_{2t}, b_1, c_1, d_1, \dots, d_{2t}, b''_1, c''_1$ is a one-point extension of the hereditary algebra C , formed by all its objects except d'_{2t} , by a simple regular C -module. By (5.6), B' is non-domestic and hence, by (1.2), B also is non-domestic.

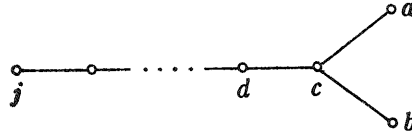
5.8. Let now C be a hereditary algebra of type \tilde{A}_m , of tubular type (p, q) , $1 \leq p \leq q$. We shall denote by \bar{C} an algebra of one of the following types:

- (i) If $p, q \geq 2$, and M_1 denotes a simple regular C -module lying in a tube of rank 1, we let $\bar{C} = C[M_1]$ or $[M_1]C$.
- (ii) If $p=1, q \geq 2$ and M_1, M_2 denote non-isomorphic simple regular C -modules lying in tubes of rank 1, we let $\bar{C} = C[M_1][M_2], [M_1]C[M_2]$ or $[M_1][M_2]C$.
- (iii) If $p=q=1$ and M_1, M_2, M_3 denote pairwise non-isomorphic simple regular C -modules lying in tubes of rank 1, we let $\bar{C} = C[M_1][M_2][M_3], [M_1]C[M_2][M_3], [M_1][M_2]C[M_3]$ or $[M_1][M_2][M_3]C$.

Observe that \bar{C} contains exactly three tubes of rank at least two. Furthermore, it is simply connected (and is actually a smallest simply connected algebra containing C).

LEMMA. Let B be an algebra of one of the following types: $B = \bar{C}$ or $B = \bar{C}[X]$, where \bar{C} is as defined above but with the restriction that it is obtained from C using only extensions. In the first case, we let i be an extension vertex of C inside \bar{C} and, in the second, we assume that X is a simple regular module non-isomorphic to M_1, M_2 or M_3 and let i denote the corresponding extension

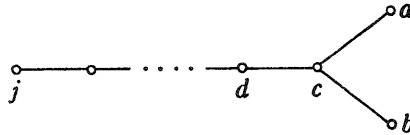
vertex. Further, let A be obtained from B by identifying i to the vertex j in a quiver with underlying graph:



Then A is not domestic.

PROOF. By construction, the full subcategory A' of A consisting of all vertices except a is a truncated branch extension of C of tubular type different from (p, q) , $1 \leq p \leq q$. Thus either A' is not domestic or it is a tilted algebra of type \tilde{D}_n or \tilde{E}_p . Continuing the proof as in (5.6), we prove the lemma.

5.9. COROLLARY. Let B be an algebra of one of the following types: $B = \bar{C}$, $\bar{C}[X]$ or $[X]\bar{C}$. In the first case, we let i be an extension or coextension vertex of C inside \bar{C} . In the remaining cases, X is a simple regular module non-isomorphic to M_1, M_2, M_3 and i is the corresponding extension or coextension vertex. Further, let A be obtained from B by identifying i to the vertex j in a quiver with underlying graph:



Then $T(A)$ is not domestic.

PROOF. It follows from the definition of B that, by applying suitable reflections to A we obtain an algebra A^* such that the full subcategory A' of A^* consisting of all vertices except a is either a truncated branch extension (if i is an extension vertex) or coextension (if i is a coextension vertex) of C . Since $T(A^*) \simeq T(A)$, we may replace A by A^* . Passing, if necessary, to the opposite algebra, we may assume that A is such that A' is a truncated branch extension of C . We then apply (5.8).

5.10. LEMMA. Let B be as defined in (5.6) or (5.9) and Y be an indecomposable C -module such that the trivial extension of $B[Y]$ is domestic. Then Y is not isomorphic to X, M_1, M_2 or M_3 .

PROOF. Suppose that Y is isomorphic to one of these modules. Since

$B[Y]$ is domestic, so is B . As in (5.9), we may assume that B is a truncated branch extension of C , thus is a tilted algebra of type \tilde{D}_n or \tilde{E}_p having a complete slice \mathcal{S} in its preinjective component. In order to obtain a contradiction, it suffices to show that the full subcategory \mathcal{U} of the vector space category $\text{Hom}_B(Y, \text{mod } B)$ formed by all objects of the form $\text{Hom}_B(Y, Z)$, where Z_B is an indecomposable preinjective which is a proper predecessor of \mathcal{S} , is not domestic. Let T_B be the slice module of \mathcal{S} , and $H = \text{End } T_B$. As in (5.6), it suffices to prove that $\text{Ext}_B^1(T, Y)$ is an indecomposable regular H -module of regular length two. Let i denote the extension vertex corresponding to Y inside $B(!)$. Then $Y_B = \text{rad } P(i)_B$ and $I(i) = S(i)$. Since $P(i)_B$ is regular, it is not a direct summand of T and the corresponding connecting sequence is:

$$0 \longrightarrow \text{Hom}_B(T, I(i)) \longrightarrow \text{Ext}_B^1(T, Y) \longrightarrow \text{Ext}_B^1(T, P(i)) \longrightarrow 0$$

with indecomposable middle term. This completes the proof.

5.11. LEMMA. *Let \bar{C} be as in (5.9), X_C be an indecomposable regular C -module which is not isomorphic to M_1, M_2 or M_3 and let $B = \bar{C}[X]$ or $[X]\bar{C}$. If $T(B)$ is domestic, then X_C is simple regular non-homogeneous.*

PROOF. It is easy to see that X_C is not homogeneous: for, if it were, either $T(B)$ is wild or, since n_B is not domestic, we obtain a contradiction by (2.8).

It now follows from the definition of B that we may assume (applying, if necessary, suitable reflections) that $B = \bar{C}[X]$ and that moreover \bar{C} is obtained from C by successive extensions. Thus \bar{C} is a tilted algebra of type \tilde{D}_n or \tilde{E}_p having a complete slice \mathcal{S} in its preinjective component. Let T be the slice module of \mathcal{S} and $H = \text{End } T$. Then $X' = \text{Ext}^1(T, X)$ is an indecomposable regular H -module and the vector space category $\text{Hom}_H(X', \text{mod } H)$ is a full subcategory of $\text{Hom}_{\bar{C}}(X, \text{mod } \bar{C})$. Since $B = \bar{C}[X]$ is domestic, then $\text{Hom}_H(X', \text{mod } H)$ is also domestic. By (5.1) and [35] (3.5), this implies that X' is a simple regular non-homogeneous H -module. Therefore X is a simple regular non-homogeneous C -module.

6. Proof of the implication (i) \Rightarrow (iv):

Let A be a simply connected algebra such that $T(A)$ is representation-infinite and domestic. We claim that A is iterated tilted of type \tilde{D}_n or \tilde{E}_p . If A is representation-finite, then, by (3.4), there exists a representation-infinite simply connected algebra A' obtained from A by a sequence of reflections, that is, A' is tilting-cotilting equivalent to A and $T(A') \simeq T(A)$. We may thus assume

that A is representation-infinite. It will then suffice, by (2.2), to show that A is a domestic branch enlargement of a tame concealed algebra C with tubular type not of the form (p, q) , $1 \leq p \leq q$.

6.1. LEMMA. *A contains a full convex subcategory C which is a tame concealed algebra.*

PROOF. a) Assume first that A is not Schurian, and let B be a non-Schurian full convex subcategory of A having the least number of vertices. Thus B contains a source x and a sink y such that $\dim_k A(x, y) = m \geq 2$ and $\dim_k A(t, z) \leq 1$ for all pairs $(t, z) \neq (x, y)$ of objects of B . We claim that $m = 2$. Indeed, if $m \geq 3$, $S_y^+ B$ contains a wild hereditary full subcategory H consisting of m arrows from y' to x , and then $T(B) \simeq T(S_y^+ B)$ is wild, a contradiction to the fact that $T(A)$ is domestic. Let $u_1, u_2 \cdots u_n$ denote a maximal set of linearly independent paths in $A(x, y)$. It follows from the minimality of B that the starting arrows of the u_i have distinct targets.

We now claim that all the objects of B lie on one of the u_i . Observe that, if $n \geq 3$, the u_i have length at least two: for, if one of them is an arrow, it does not belong to the subspace of $A(x, y)$ generated by the remaining ones and consequently $\dim_k A(x, y) \geq 3$, a contradiction. Suppose now that w is an additional path in B , say from a to b . If $(a, b) = (x, y)$, then w is of length at least two and non-zero, since otherwise $\text{mod } B$ contains a subcategory $\text{mod } H$, where H is wild hereditary given by the quiver $\circ \leftarrow \circ \rightrightarrows \circ$. Thus, by definition of the u_i , w depends linearly on one of the paths u_i . If $(a, b) \neq (x, y)$, we claim that a and b lie on the same path u_i and that w depends linearly (in $A(a, b)$) on the subpath of u_i from a to b . Indeed, suppose that a lies on u_1 and b on u_2 , say. Let u'_1 (respectively, u''_2) denote the subpath of u_1 (respectively, u_2) from x to a (respectively, b to y). Since u_1 and u_2 are linearly independent and $\dim_k B(x, b) \leq 1$, $\dim_k B(a, y) \leq 1$, both paths $u'_1 w$ and $w u''_2$ are bound by zero-relations. Thus B contains a full subcategory satisfying the conditions of (5.2), a contradiction to the fact that $T(B)$ is domestic. Thus a and b lie on the same path u_i and, as above, we conclude from (5.2) that w depends on the subpath of u_i from a to b . But then, in both cases, B contains a full convex subcategory given by the quiver of (5.7), a contradiction. We have thus shown that all objects of B lie on the paths u_i . Hence, since $T(A)$ is tame, then $n \leq 4$. We have three cases to consider:

(i) If $n = 4$, since $T(A)$ is tame, u_1, u_2, u_3 and u_4 are of length exactly two. Therefore x is the source of four arrows forming a hereditary convex full sub-

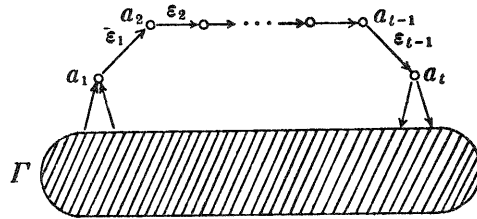
category C of type \tilde{D}_4 .

(ii) If $n=3$, let l_1, l_2, l_3 denote respectively the lengths of u_1, u_2, u_3 . Since B contains no wild hereditary full subcategory, $1/l_1+1/l_2+1/l_3 \geq 1$. If equality is strict, take C equal to B ; if not, take C equal to the full subcategory of all the objects except y .

(iii) If $n=2$, $C=B$ is a full convex subcategory of A of type \tilde{A}_m .

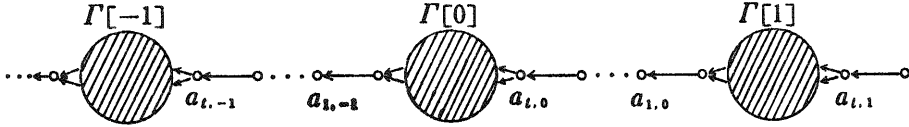
b) If A is Schurian and \tilde{A} -free then, since it is simply connected, its first homology vanishes by [14][11]. But then, by [11][12][26], A contains a full convex subcategory C which is a tame concealed algebra of type \tilde{D}_n or \tilde{E}_p .

c) If A is Schurian, but contains a full subcategory $K \simeq kQ$, where Q is some quiver with underlying graph \tilde{A}_m , we let C denote the convex hull of K . Since K is full, and $\text{mod } \hat{C}$ does not contain a subcategory of the form $\text{mod } H$, for H wild hereditary given by the quiver $\circ \leftarrow \circ \rightrightarrows \circ$ or its opposite, then A either contains a full subcategory given by the bound quiver of (5.7), a contradiction, or a full subcategory L of the form:

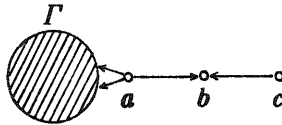


Here, Γ is a hereditary algebra of type \tilde{A}_m , $t \geq 1$, the full subcategory of L formed by Γ and a_1 (respectively, a_t) is a non-point coextension (respectively, extension) of Γ by an indecomposable regular Γ -module M (respectively, N) of regular length at most two, and the full subcategory of L consisting of the objects a_1, \dots, a_t has radical square zero. Observe that, if $a_1 = a_t$, then there is no non-zero path from Γ to Γ through a_1 (because Γ is full). Since A is triangular, M (respectively, N) is not an indecomposable homogeneous Γ -module. Moreover, if a_1 and a_t are connected to Γ by just one arrow, then L is only bound by zero-relations and, by (5.2), $T(L)$ is not domestic, a contradiction.

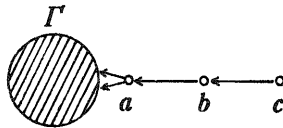
Suppose that N is of regular length two and a_t is connected to Γ by two arrows. Since N does not have two isomorphic simple regular factors, it must belong to a tube of rank at least two. Consider the following Galois covering $R \rightarrow L$ with infinite cyclic group:



where $\Gamma[i]$, $i \in \mathbb{Z}$, denotes a copy of Γ . Observe that this induces a Galois covering $\hat{R} \rightarrow \hat{L}$ (with infinite cyclic group). Then R contains a full subcategory D of the form:



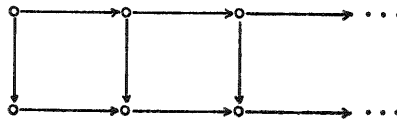
where the full subcategory of D formed by Γ and a is isomorphic to $\Gamma[N]$, the full subcategory of D formed by a , b and c has radical square zero and $\text{rad } P(b)_D \not\cong P(a)_D$. We claim that $\text{rad } P(b)_D \cong S(a)$. Indeed, if this is not the case, then the largest Γ -submodule X of $P(b)_D$ is non-zero, there is a non-zero map from N to X , consequently X has an indecomposable direct summand which is either preinjective or regular of regular length at least three. Let D' be the full subcategory of D formed by Γ and b . It follows from (5.3) and (5.5) that \hat{D}' , and thus \hat{L} , are not domestic, and this contradicts the fact that $T(A)$ is domestic. Therefore, $\text{rad } P(b)_D \cong S(a)$ and \hat{R} contains a full subcategory E of the form:



where the full subcategory of E formed by Γ and a is again isomorphic to $\Gamma[N]$. Let H be the full subcategory of E formed by b and c , and $F = \Gamma \times H$. Clearly, E is a one-point extension of F by the F -module $V = N \oplus S(b)$ and [35] the vector space category $\text{Hom}_F(Y, \text{mod } F)$ contains a full subcategory $U \cong \text{add}(kS)$ where S is the disjoint union of the two partially ordered sets $\text{Hom}_H(S(b), \text{mod } H)$:



and $\text{Hom}_\Gamma(N, \text{mod } \Gamma)$:



Consequently E , and so \hat{R} , are wild. Since \hat{R} is a Galois covering of the full

subcategory \hat{L} of \hat{A} , we obtain a contradiction to the fact that $T(A)$ is domestic.

We obtain similarly a contradiction if we assume that M is of regular length two and a_1 is connected to I by two arrows. Therefore C is hereditary of type \tilde{A}_m and the proof is complete.

6.2. LEMMA. *With the notations of (5.8), if A contains a hereditary algebra C of type \tilde{A}_m as a full convex subcategory, then it contains an algebra of the form \bar{C} .*

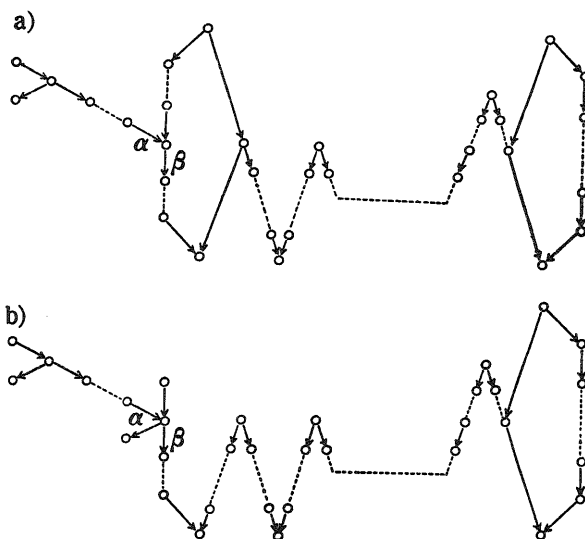
PROOF. (i) We first claim that, for any presentation (Q, I) of A , any arrow of C is involved in a minimal relation in the sense of [31]. Indeed, let (Q, I) be a presentation of A having an arrow α which is not involved in a minimal relation. Let $w = \alpha_1^{\epsilon_1} \cdots \alpha_m^{\epsilon_m}$, $\epsilon_j = \pm 1$, $1 \leq j \leq m$, denote a reduced closed walk around the cycle C . There exists an index $1 \leq j \leq m$ such that $\alpha = \alpha_j$. Since A is simply connected, there exists a sequence of closed walks

$$w = w_0 \sim w_1 \sim \cdots \sim w_t = \{x\}.$$

On the other hand, our assumption implies that each walk w_i contains the term $\alpha_j^{\epsilon_j}$, a contradiction.

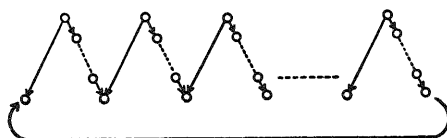
(ii) We shall now prove that, for any presentation (Q, I) of A , there is (inside A) a one-point extension or coextension of C by a simple homogeneous C -module. Let (Q, I) be a presentation of A such that this statement is not true. Let B be the full subcategory of A consisting of C and all its neighbours. It follows from (i) that, up to duality, B contains a one-point extension of C by an indecomposable regular C -module of regular length two (lying in a tube \mathcal{T} of rank at least two) such that the extension vertex is connected to C by two arrows. We claim that any vertex of B which is not in C and is connected to C by two arrows is the extension vertex of a one-point extension of C by an indecomposable regular C -module of regular length two lying in \mathcal{T} .

Since B contains neither one-point extensions nor one-point coextensions of C by a simple homogeneous C -module, it admits a universal Galois covering (in the sense of [31]) $\tilde{B} \rightarrow B$ with the infinite cyclic group (induced by the cycle C). Now suppose that B contains a one-point coextension of C by an indecomposable regular C -module of regular length two (respectively, a one-point extension of C by an indecomposable regular C -module of regular length two lying in the tube of rank at least two distinct from \mathcal{T}), such that the coextension (respectively, extension) vertex is connected to C by at least two arrows. Then R contains a full subcategory D of one of the forms:

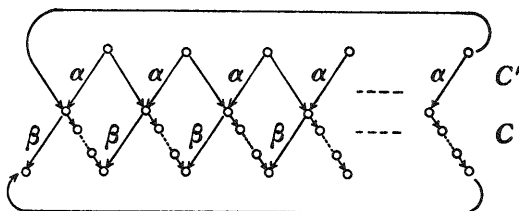


bound in each case by $\alpha\beta=0$, and all possible commutativity relations. It follows from (5.6) that R is not domestic, and consequently, by (1.2), that B is not domestic, a contradiction.

Therefore it follows from (i) that C is of the form:



and that B contains a full subcategory B' of the form:

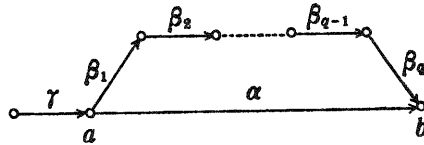


bound by all possible relations of the form $\alpha\beta=0$, and all possible commutativity relations. Let us denote by C' the full subcategory of B formed by all sources of B' and all sources of C . Thus C' is a radical square zero hereditary algebra of type \tilde{A}_m . Observe that B does not contain a one-point extension or coextension of C' by a simple homogeneous C' -module (for, if this were the case, then B would contain a one-point extension of C by the direct sum of all non-

isomorphic simple injective C -modules, a contradiction to (5.3)). As above, we can show that A does not contain a one-point extension of C' by an indecomposable regular C' -module of regular length two whose extension vertex is connected to C' by two arrows. Thus, any neighbour of B' in A is connected to B' by one arrow.

Let now w be a reduced closed walk around C . There exists a sequence of closed walks $w = w_0 \sim w_1 \sim \dots \sim w_i = \{x\}$. However, each closed walk w_i contains all terms of some closed walk w'_i around a cycle in B' . This contradicts the simple connectedness of $A \simeq kQ/I$. Consequently, for each presentation (Q, I) of A , A contains a one-point extension or coextension of C by a simple homogeneous C -module.

(iii) Let (p, q) denote the tubular type of C . If $p \geq 2, q \geq 2$, the existence of \bar{C} follows directly from (ii). Assume $p=1, q \geq 2$ and let (Q, I) be an arbitrary presentation of A . It follows from (ii) that (Q, I) contains (up to duality) a full bound subquiver of the form:



where $\gamma\alpha - \lambda \cdot \gamma\beta_1 \dots \beta_q \in I, \lambda \in k^*$. Replacing the representative of α by $\alpha' = \alpha - \lambda \cdot \beta_1 \dots \beta_q$, we obtain a new presentation (Q, I') of A such that $\gamma\alpha' \in I'$. Applying (ii) to (Q, I') , we deduce that there exists an arrow $\delta: c \rightarrow a$ with $\delta\alpha' - \lambda' \cdot \delta\beta_1 \dots \beta_q \in I', \lambda' \in k^*$, or an arrow $\varepsilon: b \rightarrow d$ with $\alpha'\varepsilon - \lambda'' \cdot \beta_1 \dots \beta_q\varepsilon \in I', \lambda'' \in k^*$. Consequently A contains an algebra \bar{C} of the form (5.8) (ii). Similarly, if $p=q=1$, then A contains an algebra \bar{C} of the form (5.8) (iii). This completes the proof.

6.3. LEMMA. Any one-point extension or coextension of C which is a full subcategory of C is by a simple regular C -module.

PROOF. If C is not hereditary of type \tilde{A}_m , this follows from (5.4) while if it is, this follows from (6.2) and (5.11).

6.4. LEMMA. Let $B=C[M]$ be a one-point extension of C by a simple regular C -module M , with extension vertex a , and $E=B[X]$ be a one-point extension of B , with extension vertex b . Suppose further that E is a full subcategory of A , and let N be an indecomposable direct summand of X containing $S(a)$ in its top. Then either $N \simeq P(a)_B$ or $N \simeq S(a)_B$.

PROOF. Suppose that N is neither isomorphic to $S(a)$, nor to $P(a)$, and let R (respectively, Y) be the largest C -submodule of N (respectively, X). Since $N \not\cong S(a)$, R is non-zero and is obviously a direct summand of Y . Since $N \not\cong P(a)$, $M \not\cong R$ but there exists a non-zero map from M to R and consequently R is not simple regular. Then the full subcategory of A with objects b and the objects of C is a one-point extension of C by a module which is not simple regular, a contradiction to (6.3).

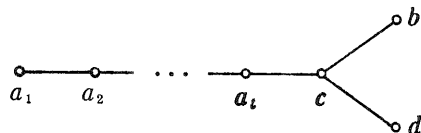
6.5. LEMMA. *Let D be a full subcategory of A of one of the following types:*

(i) $D=C[X]$ if C is not hereditary of type \tilde{A}_m , where X is a simple regular C -module. Let i denote the corresponding extension vertex.

(ii) $D=\bar{C}$ if C is hereditary of type \tilde{A}_m . Let i denote an extension or a coextension vertex of C inside \bar{C} .

(iii) $D=\bar{C}[X]$ if C is hereditary of type \tilde{A}_m , where X is a simple regular C -module not isomorphic to M_1, M_2 and M_3 . Let i denote the corresponding extension vertex.

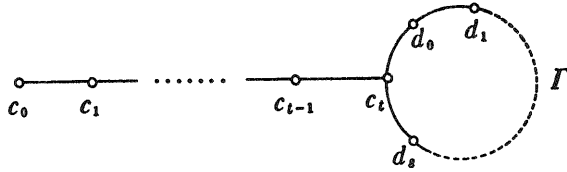
Further, let B be obtained from D by identifying i to the vertex a_1 in a bound quiver with underlying graph:



where the full subcategory formed by the objects a_t, b, c, d is free. Then B is not a full subcategory of A .

PROOF. Suppose A contains a full subcategory of the form B . We may obviously assume that the walk $a_1-a_2-\dots-a_t-c$ has radical square zero. In the case (ii), we can also assume that i is an extension vertex of C (by passing, if necessary, to the opposite algebra). By (6.3), the largest C -submodule of $P(a_1)_B$ is simple regular. Thus, if $a_2 \rightarrow a_1$, we can assume, by (6.4), that the largest C -submodule of $\text{rad } P(a_2)$ is zero. Hence \hat{B} contains a full subcategory B' of the same form as B but in which the radical square zero walk $a_1-a_2-\dots-c$ is not bound, has its first edge oriented as follows: $a_1 \rightarrow a_2$, and moreover, in the cases (ii) and (iii), if \bar{C} contains coextension vertices of C , then these are replaced by corresponding extension vertices of C . We then apply (5.6) and (5.8) to deduce that B' (and hence \hat{B}) are not domestic. This contradicts the fact that $T(A)$, and hence (by (1.2)) \hat{A} , are domestic.

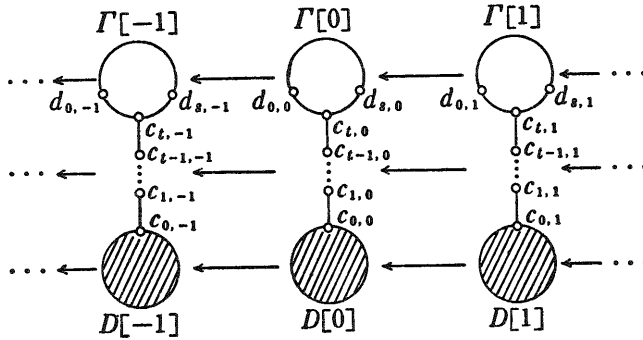
6.6. LEMMA. Let D be a full subcategory of A defined as in (6.5), and let B be obtained from D by identifying i to the vertex c_0 in a bound quiver with underlying graph:



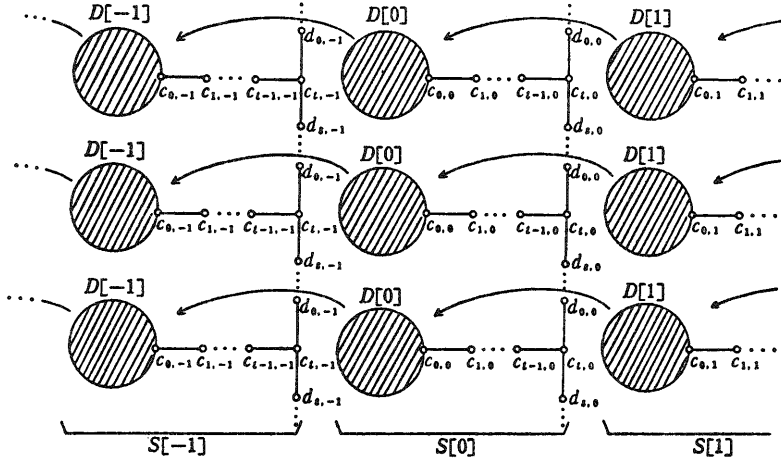
where Γ is a non-commutative cycle. Then B is not a full subcategory of A .

PROOF. Assume that A contains such a full subcategory B . It follows from (6.5) that the full subcategory of B formed by all objects outside C is bound only by zero-relations. We can assume that the walks $c_0 - c_1 - \dots - c_t$ and $c_t - d_0 - \dots - d_s - c_t$ have radical square zero. By duality, we may also assume in the case (ii) that c_0 is an extension vertex of C . Also, by (6.4), we may assume that the restriction of $P(c_1)$ to C is zero.

Let now E denote the full subcategory of B consisting of Γ and c_0, \dots, c_{t-1} . Then the repetitive algebra $\hat{B} = kQ_{\hat{B}}/I_{\hat{B}}$ of B has the following form:



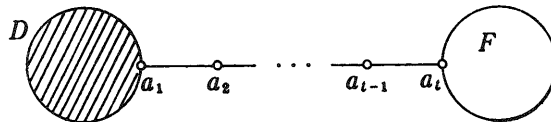
where $D[i]$ (respectively, $\Gamma[i]$) denotes the copy of D (respectively, Γ) indexed by $i \in \mathbb{Z}$. The arrows of $Q_{\hat{B}}$ are all arrows of $Q_{\hat{D}}$ and $Q_{\hat{E}}$. The ideal $I_{\hat{B}}$ is generated by $I_{\hat{D}}$, $I_{\hat{E}}$, all paths $x \rightarrow c_{0,i} \rightarrow y$, $i \in \mathbb{Z}$, of length two with one endpoint in \hat{D} and the second in \hat{E} , and the differences $u - v$, where u (respectively, v) is a non-zero path in \hat{D} (respectively, \hat{E}) from $c_{0,i+1}$ to $c_{0,i}$. In particular, \hat{D} (respectively, \hat{E}) is a full subcategory of B formed by all objects of $D[i]$ (respectively, $E[i]$), $i \in \mathbb{Z}$. Let us consider the following Galois covering $A: \Gamma \rightarrow \hat{B} = \Gamma/G$ with the infinite cyclic group G generated by the vertical shift:



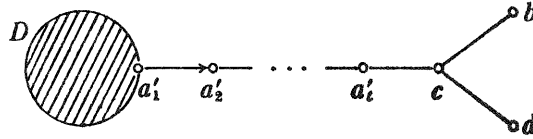
Let S be the full subcategory of \mathcal{A} formed by the objects of the column $S[0]$. Then, clearly, $\mathcal{A} = \hat{S}$. On the other hand, D contains a D_t -frame F [5], that is, a bound quiver of one of the following forms:

- (F1) $\alpha\beta = \gamma\beta$
- (F2) $\alpha\beta\gamma = 0$
- (F3) $\alpha\beta = 0, \alpha\gamma = 0$
- (F4) $\alpha\beta = 0, \alpha\gamma = 0$
- (F5) $\alpha\gamma = 0, \beta\gamma = 0$

Therefore, S contains a full subcategory of the form:



where a_i is identified to the vertex i of D . Then $A = \hat{S}$ contains a full subcategory K of the form:

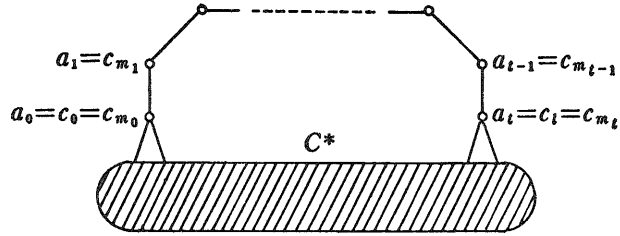


where the full subquiver consisting of all vertices outside of D is free. Further, in the cases (ii) and (iii), we can assume that \bar{C} is given by extensions of C (see the proof of (6.5)). By (5.6) and (5.8), K , and hence A , are not domestic. This, by (1.2), contradicts the fact that $T(A)$ is domestic.

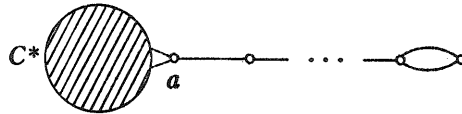
6.7. LEMMA. *Let a and b be two objects of A outside C , each of them connected to C by an edge. Then any walk in A connecting a and b must intersect C .*

PROOF. Suppose that there is a walk $a = c_0 - c_1 - \dots - c_s = b$ in A which does not intersect C . We shall deduce a contradiction to the fact that $T(A)$ is domestic. Observe that, if C is hereditary of type \tilde{A}_m , then there exists such a walk in A which does not intersect \bar{C} . We shall thus, in this case, replace C by \bar{C} . We shall use the letter C^* to denote C in all cases except if it is hereditary of type \tilde{A}_m in which case it denotes \bar{C} .

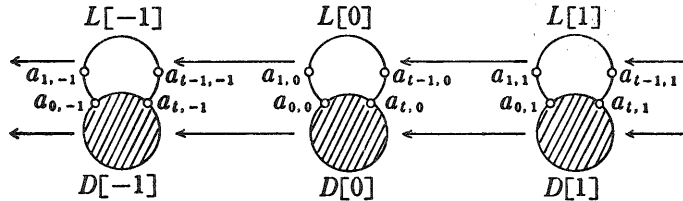
Assume that there exists an index $1 \leq i < s$ and a non-zero path from c_i to C^* or from C^* to c_i which does not pass through a or b . Let l be the least such index, and denote by B the full subcategory of A consisting of the objects of C^* and c_0, \dots, c_l . Observe that c_l is connected (in B) to C^* by at least one edge, and that any non-zero path between an object c_i ($1 \leq i < l$) and an object x of C^* passes through c_0 or c_l . Let K be the full subcategory of B consisting of the vertices c_0, \dots, c_l . We shall define inductively a radical square zero connected full subcategory L of K containing c_0 and c_l : we start with $c_{m_0} = c_0$, and, for each i , let m_i be the largest index $m_{i-1} \leq m_i \leq l$ such that there is a non-zero path in K from $c_{m_{i-1}}$ to c_{m_i} , or from c_{m_i} to $c_{m_{i-1}}$. We then let B' be the full subcategory of A consisting of C^* and L . We claim that B' is of the form:



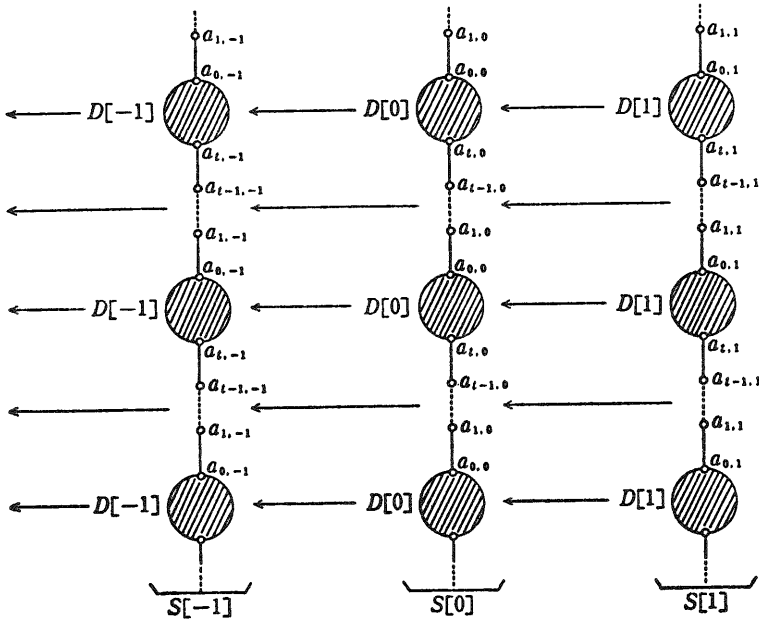
Indeed, suppose first that L has only two objects a_0 and a_t such that we have a double arrow $a_0 \rightrightarrows a_t$ in L . Since C is convex, $\text{mod } B'$ contains $\text{mod } H$, for H wild hereditary given by the quiver $\circ \leftarrow \circ \rightrightarrows \circ$ or its opposite. Next, consider the case where L has more than two objects and assume there exists a double arrow in L . Then B' contains a full subcategory of the form:



where a is either a_0 or a_t . We then obtain a contradiction by (6.6). This shows our claim. By duality, we may assume that a_0 is an extension vertex of C and, by (6.4), that there does not exist a non-zero path between C^* and a_1 or a_{t-1} . Let D be the full subcategory of B' formed by all objects of C^* , a_0 and a_t . Then B' has the form:



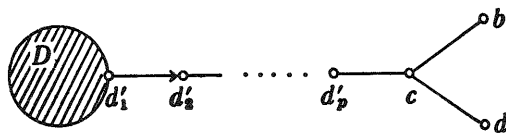
where $D[i]$ (respectively, $L[i]$) denotes the copy of D (respectively, L) indexed by $i \in \mathbb{Z}$. The arrows of $Q_{\hat{B}'}$ are all the arrows of $Q_{\hat{D}}$ and $Q_{\hat{L}}$. The ideal $I_{\hat{B}'}$ is generated by $I_{\hat{D}}$, $I_{\hat{L}}$, all paths $x \rightarrow a_{0,i} \rightarrow x'$, $y \rightarrow a_{t,i} \rightarrow y'$ ($i \in \mathbb{Z}$) of length two with one endpoint in \hat{D} and the second in \hat{L} , and the differences $u - v$, where u (respectively, v) is a non-zero path in \hat{D} (respectively, \hat{L}) from $a_{0,i+1}$ to $a_{0,i}$ or from $a_{t,i+1}$ to $a_{t,i}$, $i \in \mathbb{Z}$. In particular, \hat{D} (respectively, \hat{L}) is a full subcategory of \hat{B}' formed by the objects of $D[i]$ (respectively, $L[i]$), $i \in \mathbb{Z}$. Let us consider the following Galois covering $F: A \rightarrow \hat{B}' = A/G$ with the infinite cyclic group G generated by the vertical shift:



Let S be the full subcategory of A formed by all the objects of the column $S[0]$. Then, clearly, $A = \hat{S}$. On the other hand, since D contains a D_i -frame, then S contains a full subcategory of the form:



where d_1 is identified to the vertex a_0 of D . Then $A = \hat{S}$ contains a full subcategory K of the form:



where the full subquiver consisting of all the objects outside of D is free. Furthermore, if C is hereditary of type \tilde{A}_m , we can assume that \bar{C} is given by extensions of C . By (5.6) and (5.8), K and hence A , are not domestic. This implies, by (1.2), that $T(A)$ is not domestic, a contradiction.

REMARK. The proofs of (6.6) and (6.7) use the same ideas as, respectively, the proofs of (4.8) and (4.9) of [7]. For the convenience of the reader, they are nevertheless written in detail.

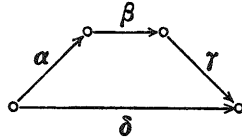
6.8. LEMMA. *A is an iterated tilted algebra of Euclidean type \tilde{D}_n or \tilde{E}_p .*

PROOF. It suffices to show that A is a domestic branch enlargement of C of tubular type $n_A \neq (p, q)$, $1 \leq p \leq q$. If A is not a branch enlargement of C , it follows from (6.6) and (6.7) that A contains a full subcategory consisting of C connected by a walk to a D_4 -frame F . Then \hat{A} contains a full subcategory of the same form, but in which F is a free D_4 -frame. This contradicts (6.5), and therefore A is a branch enlargement of C . In order to show that n_A is domestic, we observe that, by (2.6), there exists a truncated branch enlargement B of C such that $n_A = n_B$ and $T(A) \simeq T(B)$. The result then follows at once from (2.3) since $n_A \neq (p, q)$ because of (6.2).

7. Remarks.

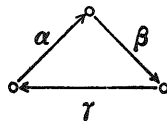
7.1. It follows directly from our theorem and [41] that a representation-infinite domestic trivial extension of a simply connected algebra is stably equivalent to the trivial extension of a radical square zero hereditary algebra of Euclidean type \tilde{D}_n or \tilde{E}_p . This generalises results of [40] and [3].

7.2. Iterated tilted algebras of type \tilde{A}_m were described in [4]. In particular, they are not simply connected. Moreover, it follows from their description that their trivial extensions are special biserial and, by [37][18], they are 2-parametric. On the other hand, if A is given by the quiver:



bound by $\alpha\beta = \beta\gamma = 0$, then $T(A)$ is 2-parametric but A is not iterated tilted (because $\text{mod } A$ is not directed).

7.3. Domestic trivial extension algebras may arise from non-triangular algebras. For instance, if k has characteristic two, the group algebra kA_4 on the alternating group A_4 is isomorphic to the trivial extension of the algebra given by the oriented cycle:



bound by $\alpha\beta=\beta\gamma=\gamma\alpha=0$. Then kA_4 is a 1-parametric algebra. The second author has obtained a complete classification of the Nakayama algebras for which the trivial extension is representation-infinite and domestic.

Acknowledgements

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References

- [1] Assem, I. and Happel, D., Generalized tilted algebras of type A_n , *Comm. Algebra* 9 (1981), No. 20, 2101-2125.
- [2] Assem, I., Happel, D. and Roldán, O., Representation-finite trivial extension algebras, *J. Pure Appl. Algebra* 33 (1984), 235-242.
- [3] Assem, I. and Iwanaga, Y., Stable equivalence of representation-finite trivial extension algebras, *J. Algebra* 102 (1986), No. 1, 33-38.
- [4] Assem, I. and Skowroński, A., Iterated tilted algebras of type \tilde{A}_n , *Math. Z.*, Band 195, Heft 2 (1987), 269-290.
- [5] Assem, I. and Skowroński, A., Algèbres pré-inclinées et catégories dérivées, *Séminaire M.-P. Malliavin*, Springer Lecture Notes, to appear.
- [6] Assem, I. and Skowroński, A., On some classes of simply connected algebras, *Proc. London Math. Soc.*, (3) 56 (1988), 417-450.
- [7] Assem, I. and Skowroński, A., Algebras with cycle-finite derived categories, *Math. Annalen*, 280 (1988), 441-463.
- [8] Auslander, M., Applications of morphisms determined by objects. *Proc. Conf. on Representation Theory (Philadelphia, 1976)*, Marcel Dekker (1978), 245-327.
- [9] Auslander, M. and Reiten, I., Representation theory of artin algebras III and IV, *Comm. Algebra* 3 (1975), 239-294 and 5 (1977), 443-518.
- [10] Bautista, R., Larrión, F. and Salmerón, L., On simply connected algebras, *J. London Math. Soc.* (2) 27 (1983), No. 2, 212-220.
- [11] Bongartz, K., A criterion for finite representation type, *Math. Annalen*, Band 269, Heft 1 (1984), 1-12.
- [12] Bongartz, K., Critical simply connected algebras, *Manuscripta Math.* 46 (1985), 117-136.
- [13] Bongartz, K. and Gabriel, P., Covering spaces in representation-theory, *Invent. Math.* 65 (1981/82), No. 3, 331-378.
- [14] Bretscher, O. and Gabriel, P., The standard form of a representation-finite algebra, *Bull. Soc. Math. France* 111 (1983), 21-40.
- [15] Bretscher, O., Läser, C. and Riedtmann, C., Self-injective and simply connected algebras, *Manuscripta Math.* 36 (1981/82), No. 3, 253-307.
- [16] Dowbor, P. and Skowroński, A., On Galois coverings of tame algebras, *Arch. Math.*, Vol. 44 (1985), 522-529.
- [17] Dowbor, P. and Skowroński, A., On the representation type of locally bounded categories, *Tsukuba J. Math.*, 10 (1986), no 1, 63-72.
- [18] Dowbor, P. and Skowroński, A., Galois coverings of representation infinite algebras, *Comment. Math. Helv.*, Vol. 62, No. 2 (1987), 311-337.

- [19] Drozd, Ju. A., Tame and wild matrix problems, Proc. ICRA II (Ottawa, 1979), Springer Lecture Notes No. 832 (1980), 242-258.
- [20] D'Este, G. and Ringel, C.M., Coherent tubes, J. Algebra 87 (1984), 150-201.
- [21] Gabriel, P., The universal cover of a representation-finite algebra, Proc. ICRA III (Puebla, 1980), Springer Lecture Notes No. 903, (1981), 68-105.
- [22] Green, E.L. and Reiten, I., On the construction of ring extensions, Glasgow Math. J. 17 (1979), 1-11.
- [23] Happel, D., On the derived category of a finite-dimensional algebra, Comment. Math. Helv., Vol. 62, No. 3 (1987), 339-389.
- [24] Happel, D., Iterated tilted algebras of affine type, Comm. Algebra 15 (1 & 2), 29-45 (1987).
- [25] Happel, D. and Ringel, C.M., Tilted algebras, Trans. Amer. Math. Soc. 274 (1982), No. 2, 399-443.
- [26] Happel, D. and Vossieck, D., Minimal algebras of infinite representation type with preprojective component, Manuscripta Math. 42 (1983), 221-243.
- [27] Hoshino, M., Trivial extensions of tilted algebras, Comm. Algebra 10 (1982), No. 18, 1965-1999.
- [28] Hoshino, M., Splitting torsion theories induced by tilting modules, Comm. Algebra 11 (1983), No. 4, 427-441.
- [29] Hughes, D. and Waschbüsch, J., Trivial extensions of tilted algebras, Proc. London Math. Soc. 46 (3) (1983), 347-364.
- [30] Iwanaga, Y. and Wakamatsu, T., Trivial extensions of artin algebras, Proc. ICRA II (Ottawa, 1979), Springer Lecture Notes No. 832 (1980), 295-301.
- [31] Martinez-Villa, R. and de la Peña, J.A., The universal cover of a quiver with relations, J. Pure Appl. Algebra 30 (1983), 277-292.
- [32] Müller, W., Unzerlegbare Moduln über artinschen Ringen, Math. Z. 137 (1974), 197-226.
- [33] Nazarova, L.A. and Rojter, V.A., Categorical matrix problems and the Brauer-Thrall conjecture, Inst. Math. Acad. Sci. Kiev (1973), German translation Mitt. Math. Sem. Giessen 115 (1975).
- [34] Pogorzały, Y. and Skowroński, A., On algebras whose indecomposable modules are multiplicity-free, Proc. London Math. Soc. 47 (1983), 463-479.
- [35] Ringel, C.M., Tame algebras, Proc. Workshop ICRA II (Ottawa, 1979), Springer Lecture Notes No. 831 (1980), 137-287.
- [36] Ringel, C.M., Tame algebras and integral quadratic forms, Springer Lecture Notes No. 1099 (1984).
- [37] Skowroński, A., Generalisation of Yamagata's theorem on trivial extensions, Arch. Math. Vol. 48 (1987), 68-76.
- [38] Skowroński, A. and Waschbüsch, J., Representation-finite biserial algebras, J. Reine Angew. Math. 345 (1983), 172-181.
- [39] Tachikawa, H., Representation of trivial extensions of hereditary algebras, Proc. ICRA II (Ottawa, 1979), Springer Lecture Notes No. 832 (1980), 579-599.
- [40] Tachikawa, H., Reflection functors and Auslander Reiten translations for trivial extensions of hereditary algebras, J. Algebra 90 (1984), No. 1, 98-118.
- [41] Tachikawa, H. and Wakamatsu, T., Tilting functors and stable equivalences for self-injective algebras, J. Algebra 109, No. 1 (1987), 138-165.
- [42] Tachikawa, H. and Wakamatsu, T., Applications of reflection functors for self-injective algebras, Proc. ICRA IV (Ottawa, 1984), Springer Lecture Notes No. 1177 (1986), 308-327.
- [43] Wakamatsu, T., Note on trivial extensions of artin algebras, Comm. Algebra 12

- (1984), No. 1, 33-41.
- [44] Wakamatsu, T., Stable equivalence between universal covers of trivial extension self-injective algebras, *Tsukuba J. Math.*, Vol. 9 (1985), No. 2, 299-316.
- [45] Yamagata, K., Extensions over hereditary Artinian rings with self-dualities I, *J. Algebra* 73 (1981), No. 2, 386-433.
- [46] Yamagata, K., On algebras whose trivial extensions are of finite representation type, *Proc. ICRA III (Puebla, 1980)*, Springer Lecture Notes No. 903 (1981), 364-371.
- [47] Yamagata, K., On algebras whose trivial extensions are of finite representation type II, Preprint (1983).

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