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EIGENFUNCTION EXPANSIONS FOR ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA $\boldsymbol{R}^3$

Dedicated to Professor Mutsuhide MATSUMURA on his 60th birthday

By

Senjo SHIMIZU

Abstract. This paper provides eigenfunction expansions associated with the stationary problems for elastic wave propagation in stratified media $\boldsymbol{R}^3$. The eigenfunction expansion is given in terms of generalized eigenfunctions corresponding to incident, reflected, refracted and Stoneley waves.

Contents

§0. Introduction

This paper provides eigenfunction expansions associated with the stationary problems for elastic wave propagation in *stratified* media $\boldsymbol{R}^3$. The eigenfunction expansion is given in terms of a family of generalized eigenfunctions corresponding to incident, reflected, refracted and Stoneley waves.

The eigenfunction expansion theory for wave propagation problems has been studied by several authors (for example, K. Mochizuki [8], J.R. Schulenberger and C.H. Wilcox [11], C.H. Wilcox [18]). S. Wakabayashi [16] provided

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eigenfunction expansions associated with the stationary problems in the half-space $\boldsymbol{R}_+^n$ for symmetric hyperbolic systems with constant coefficients. Such systems were first studied in $\boldsymbol{R}_+^n$ by M. Matsumura [7]. The eigenfunction expansion is given in terms of a family of generalized or improper eigenfunctions corresponding to incident, reflected and surface or boundary waves.

For elastic wave propagation, J. R. Schulenberger [9], [10] gave eigenfunction expansions in the half-space $\boldsymbol{R}_+^n$ ($n=2, 3$), using the method developed by S. Wakabayashi. He transformed the 2×2 second order system of linear elasticity into a 5×5 first order system. But the defect of this approach is to introduce static solutions corresponding to a zero propagation speed which do not appear in the elastic wave propagation. The treatment (for example the definition domain) for the self-adjoint operator associated with non elliptic spatial part is somewhat complicated (see [16, Section 7]). Moreover the relations between the displacement vector solutions of the original system and solutions of the transformed system are complicated.

Y. Dermenjian and J. C. Guillot [3] studied scattering theory for elastic wave propagation starting with the basic elastic operators (symmetric systems of second order). J. C. Guillot [5] proved the existence and uniqueness of a Rayleigh surface wave propagation along the free boundary of a transversely isotropic elastic half space, by reducing the basic operator to a family of operators which is easier to study. Concerning stratified media, there is an interesting work by C. H. Wilcox [17] on eigenfunction expansions for the Pekeris differential operator in terms of free wave eigenfunctions and guided wave eigenfunctions.

In this paper we shall derive eigenfunction expansions associated with the stationary problems for elastic wave propagation in plane-stratified media $\boldsymbol{R}^3$ using the methods due to S. Wakabayashi [16], and also J. C. Guillot [5]. Schulenberger's works [9], [10] are useful references in our study.

We consider the plane stratified medium $\boldsymbol{R}^3=\{x=(x_1, x_2, x_3)\,;\ x_i\in\boldsymbol{R}\}$ with the planar interface $x_3=0$, which is defined by

$$(\lambda(x_3),\ \mu(x_3),\ \rho(x_3))=\begin{cases}(\lambda_1,\ \mu_1,\ \rho_1), & x_3<0,\\ (\lambda_2,\ \mu_2,\ \rho_2), & x_3>0.\end{cases}$$

Here λ_1, λ_2, μ_1, μ_2 are certain quantities called the Lamé constants and ρ_1, $\rho_2>0$ are the densities.

For simplicity, we shall denote the lower half-space $\boldsymbol{R}_-^3=\{x\in\boldsymbol{R}^3\,;\ x_3<0\}$ by *medium I* and the upper half-space $\boldsymbol{R}_+^3=\{x\in\boldsymbol{R}^3\,;\ x_3>0\}$ by *medium II*, as in Figure 1.

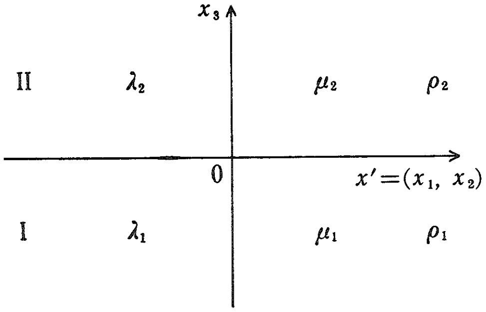

Figure 1. Stratified media I and II.

The equations describing the propagation of elastic waves in the stratified medium are given by

$$\frac{\partial^2 u_i}{\partial t^2}=\sum_{j=1}^{3}\frac{1}{\rho(x_3)}\frac{\partial\sigma_{ij}}{\partial x_j}(u), \qquad i=1, 2, 3, \tag{0.1}$$

where $u(x, t)={}^t(u_1(x, t), u_2(x, t), u_3(x, t))$ is the displacement vector, and the σ_{ij} are the symmetric stress tensors defined by

$$\sigma_{ij}(u)=\lambda(x_3)(\nabla\cdot u)\delta_{ij}+2\mu(x_3)\varepsilon_{ij}(u), \qquad \varepsilon_{ij}(u)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

Here ${}^tM$ denotes the transpose of a matrix M.

The c^I_{kilj}, $c^{II}_{kilj}(i, j, k, l=1, 2, 3)$ are the stress-strain tensors given by

$$\begin{aligned} c^I_{kilj}&=\lambda_1\delta_{ki}\delta_{lj}+\mu_1(\delta_{kl}\delta_{ij}+\delta_{kj}\delta_{il}),\\ c^{II}_{kilj}&=\lambda_2\delta_{ki}\delta_{lj}+\mu_2(\delta_{kl}\delta_{ij}+\delta_{kj}\delta_{il}), \end{aligned} \tag{0.2}$$

with the properties

$$\begin{aligned} c^I_{kilj}&=c^I_{iklj}=c^I_{kijl}=c^I_{ljki},\\ c^{II}_{kilj}&=c^{II}_{iklj}=c^{II}_{kijl}=c^{II}_{ljki}, \end{aligned}$$

and δ_{ki} is the Kronecker delta. We assume that the constants c^I_{kilj}, c^{II}_{kilj} satisfy the following stability conditions

$$\begin{aligned} \lambda_1+\mu_1&>0, \qquad \mu_1>0,\\ \lambda_2+\mu_2&>0, \qquad \mu_2>0, \end{aligned} \tag{0.3}$$

which are equivalent to the conditions

$$\begin{aligned} \sum_{k,i,l,j=1}^{3} c^I_{kilj}s_{lj}\overline{s_{ki}}&\geqq \exists\delta_1\sum_{k,i=1}^{3}|s_{ki}|^2, \qquad \delta_1>0,\\ \sum_{k,i,l,j=1}^{3} c^{II}_{kilj}s_{lj}\overline{s_{ki}}&\geqq \exists\delta_2\sum_{k,i=1}^{3}|s_{ki}|^2, \qquad \delta_2>0, \end{aligned} \tag{0.3'}$$

for all complex symmetric 3×3 matrices (s_{ki}), $s_{ki}=s_{ik}\in \boldsymbol{C}$ (cf. [6]).

The wave equations (0.1) should be supplemented by interface conditions at the interface $x_3=0$ of the medium. We now impose on u the following conditions at the interface $x_3=0$:

$$u^{I}|_{x_3=0}=u^{II}|_{x_3=0}\,, \tag{0.4}$$

$$\sigma_{i3}(u^{I})|_{x_3=0}=\sigma_{i3}(u^{II})|_{x_3=0}\,, \tag{0.5}$$

where $u=u^{I}$ for $x\in \boldsymbol{R}^3_-$, and $u=u^{II}$ for $x\in \boldsymbol{R}^3_+$.

The equations (0.1) may be written in the following form:

$$\frac{\partial^2 u}{\partial t^2}+Mu=0\,, \tag{0.6}$$

(0.7)

$$Mu=-\frac{\lambda+\mu}{\rho}\nabla(\nabla\cdot u)-\frac{\mu}{\rho}\Delta u$$

$$=-\frac{1}{\rho}\begin{pmatrix} (\lambda+2\mu)\dfrac{\partial^2}{\partial x_1^2}+\mu\Big(\dfrac{\partial^2}{\partial x_2^2}+\dfrac{\partial^2}{\partial x_3^2}\Big) & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_2} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_2} & (\lambda+2\mu)\dfrac{\partial^2}{\partial x_2^2}+\mu\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_3^2}\Big) & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_3} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x_1\partial x_3} & (\lambda+\mu)\dfrac{\partial^2}{\partial x_2\partial x_3} & (\lambda+2\mu)\dfrac{\partial^2}{\partial x_3^2}+\mu\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big) \end{pmatrix}\begin{pmatrix} u_1\\ u_2\\ u_3 \end{pmatrix},$$

where $\lambda=\lambda(x_3)$, $\mu=\mu(x_3)$, $\rho=\rho(x_3)$.

We interpret (0.1), (0.4), and (0.5) as an abstract wave equation

$$\frac{d^2u}{dt^2}+Au=0\,.$$

As we shall show later, A is a non-negative self-adjoint operator associated with (0.1), (0.4) and (0.5) in the Hilbert space

$$\mathcal{H}=L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3,\ \rho(x_3)dx)\,,$$

with inner product

$$(u,\ v)=\int_{\boldsymbol{R}^3}u\cdot v\rho(x_3)dx\,,$$

where $u\cdot v$ denotes the usual scalar product in $\boldsymbol{C}^3$: $u\cdot v=\sum_{i=1}^3 u_i\bar{v}_i$.

Let $\eta'=(\eta_1, \eta_2)\in\boldsymbol{R}^2$ be the dual variables of $x'=(x_1, x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x':

$$\hat{u}(\eta', x_3)=(F_{x'}u)(\eta', x_3)=\operatorname*{l.i.m.}_{R\to\infty}\frac{1}{2\pi}\int_{|x'|\leq R}e^{-i(x_1\eta_1+x_2\eta_2)}u(x)dx'$$

for u in $\mathscr{H}$. Let

$$D(\hat{A})=F_{x'}D(A)=\{\hat{u}\,;\ u\in D(A)\},$$

$$\hat{A}\hat{u}=F_{x'}AF_{\eta'}^{-1}\hat{u},\qquad \hat{u}\in D(\hat{A}).$$

For every $\eta'\neq 0$, let

$$\mathrm{U}=\frac{1}{|\eta'|}\begin{pmatrix}\eta_1 & -\eta_2 & 0\\ \eta_2 & \eta_1 & 0\\ 0 & 0 & |\eta'|\end{pmatrix},\qquad \mathrm{C}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

where U and C are unitary matrices and $|\eta'|=(\eta_1^2+\eta_2^2)^{1/2}$. Then we have

$$Au=F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta')\oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u\qquad\text{for}\quad u\in D(A),$$

where $A_1(\eta')$ and $A_2(\eta')$ are non-negative self-adjoint operators (see Proposition 1.7).

We can get an explicit representation of the Green function $G_1(x_3, y_3, \eta'\,;\zeta)$ for the operator $A_1(\eta')-\zeta I(\zeta\notin\boldsymbol{R})$ from the expression of the solution for the following problem:

$$(A_1(\eta', D)-\zeta)v(\eta', x_3)=f(\eta', x_3),\tag{0.8}$$

$$v(\eta', x_3)|_{x_3=-0}=v(\eta', x_3)|_{x_3=+0},\tag{0.9}$$

$$B_1(\eta')v(\eta', x_3)|_{x_3=-0}=B_1(\eta')v(\eta', x_3)|_{x_3=+0}.\tag{0.10}$$

Here (0.9) and (0.10) are the interface conditions for $A_1(\eta', D)$ corresponding to (0.4) and (0.5). $A_1(\eta', D)$ $(D=(1/i)(d/dx_3))$ is the differential operators corresponding to the self-adjoint operator $A_1(\eta')$. Since the solution v of (0.8) should satisfy the interface conditions (0.9) and (0.10), the denominator of v has the Lopatinski determinant $\Delta(\eta', \zeta)$ as follows:

$$\Delta(\eta', \zeta)=|\eta'|^6D(z),$$

$$\begin{aligned}D(z)=&\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}+\frac{\mu_2 z}{c_{s_2}^2}\right)^2+4(\mu_1-\mu_2)^2a_1a_2b_1b_2\\&-a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}\right)^2-a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}\right)^2\\&-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}(a_1b_2+a_2b_1)z^2,\end{aligned}$$

where

$$z=\frac{\zeta}{|\eta'|^2},$$

$$a_1=\sqrt{1-\frac{z}{c_{p_1}^2}},\quad a_2=\sqrt{1-\frac{z}{c_{p_2}^2}},\quad b_1=\sqrt{1-\frac{z}{c_{s_1}^2}},\quad b_2=\sqrt{1-\frac{z}{c_{s_2}^2}}.$$

The squares of propagation speeds of shear (S) and pressure (P) waves are given by

$$c_{s_i}^2=\frac{\mu_i}{\rho_i},\quad c_{p_i}^2=\frac{\lambda_i+2\mu_i}{\rho_i},\quad (i=1,\ 2),$$

respectively. From the conditions (0.3), the minimum speed of $\{c_{s_1}, c_{p_1}, c_{s_2}, c_{p_2}\}$ is either c_{s_1} or c_{s_2}.

We can see that $D(z)$ has the only one real zero when $D(z)$ has zeros. Denote by c_{St}^2 its real zero. Then the zero of $\Delta(\eta', \zeta)$ is $c_{St}^2|\eta'|^2$ and is the origin of the Stoneley wave propagating along the interface $x_3=0$ in the elastic space R^3, and c_{St} is its speed.

By virtue of principle of the argument, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta', \zeta)=|\eta'|^6D(z)$ (the existence of the Stoneley waves) are given as follows:

If $c_{s_1}<c_{s_2}$, then

(i) $D(c_{s_1}^2)>0 \Rightarrow$ The zero $\zeta=c_{St}^2|\eta'|^2$ of $\Delta(\eta', \zeta)$ in ζ exists in $[0, c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of Theorem 6.5 that $c_{St}\neq 0$.

(ii) $D(c_{s_1}^2)=0 \Rightarrow c_{St}=c_{s_1}$ and we shall consider this case under some restricted conditions (cf. Lemma 6.4).

(iii) $D(c_{s_1}^2)<0 \Rightarrow \Delta(\eta', \zeta)$ has no zero.

If $c_{s_2}<c_{s_1}$, then we must replace $D(c_{s_1}^2)$ by $D(c_{s_2}^2)$.

We also obtain an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta')-\zeta I(\zeta\notin \boldsymbol{R})$ by the same method as $G_1(x_3, y_3, \eta'; \zeta)$. The Lopatinski determinant corresponding to the operator $A_2(\eta')-\zeta I(\zeta\notin \boldsymbol{R})$ has no zero. By using the Green functions $G_1(x_3, y_3, \eta'; \zeta)$ and $G_2(x_3, y_3, \eta'; \zeta)$, we define

$$\phi_{1j}(x_3, \eta; \zeta)=F_{y_3}^{-1}[G_1(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta)-\zeta)P_j(\eta)\rho(x_3)^{-1},\qquad j\in M,$$

$$\phi_{1j}^{St}(x_3, \eta; \zeta)=\frac{\zeta-c_{St}^2|\eta'|^2}{\zeta-\lambda_j(\eta)}\phi_{1j}(x_3, \eta; \zeta),\qquad j\in M,$$

$$\phi_{2k}(x_3, \eta; \zeta)=F_{y_3}^{-1}[G_2(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_k(\eta)-\zeta)\rho(x_3)^{-1},\qquad k\in N.$$

Here $\eta=(\eta_1, \eta_2, \xi)=(\eta', \xi)$, $\lambda_j(\eta)=c_j^2|\eta|^2$ are the eigenvalues of $A_1(\eta')$, $P_j(\eta)$ are mutually orthogonal projections for $A_1(\eta')$, $\lambda_k(\eta)=c_k^2|\eta|^2$ are the eigenvalues of $A_2(\eta')$, $M=\{s_1, p_1, s_2, p_2\}$ and $N=\{s_1, s_2\}$. When $\zeta\to\lambda_j(\eta)\pm i0$, $\zeta\to c_{St}^2|\eta'|^2$, and $\zeta\to\lambda_k(\eta)\pm i0$, the limits $\phi_{1j}^{\pm}(x_3, \eta)$, $\phi_{1j}^{St}(x_3, \eta)$, and $\phi_{2k}^{\pm}(x_3, \eta)$ exist and these limit functions are generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, respectively.

Using these generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, we define generalized eigenfunctions for A as follows:

$$\phi_{1j}^{\pm}(x, \eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\phi_{1j}^{\pm}(x_3, \eta)\oplus O_{1\times 1}), \qquad j\in M,$$

$$\phi_{1j}^{St}(x, \eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\phi_{1j}^{St}(x_3, \eta)\oplus O_{1\times 1}), \qquad j\in M,$$

$$\phi_{2k}^{\pm}(x, \eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\phi_{2k}^{\pm}(x_3, \eta)), \qquad k\in N,$$

where $O_{n\times n}$ denotes the $n\times n$ zero matrix.

Now we define the Fourier transform of $f\in\mathcal{H}$ with respect to these generalized eigenfunctions: $f\mapsto(\hat{f}_{1j}^{\pm}, \hat{f}_{1j}^{St}, \hat{f}_{2k}^{\pm})$,

$$\hat{f}_{1j}^{\pm}(\eta)=\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|x|\leq R}\phi_{1j}^{\pm}(x, \eta)^* f(x)\rho(x_3)dx, \qquad j\in M,$$

$$\hat{f}_{1j}^{St}(\eta)=\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|x|\leq R}\phi_{1j}^{St}(x, \eta)^* f(x)\rho(x_3)dx, \qquad j\in M,$$

$$f_{2k}^{\pm}(\eta)=\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|x|\leq R}\phi_{2k}^{\pm}(x, \eta)^* f(x)\rho(x_3)dx, \qquad k\in N.$$

Our main results are the following three theorems. Theorem 0.1 corresponds to the Parseval and Plancherel formulas.

THEOREM 0.1. *We assume that* $D(c_{s_1}^2)>0$ *if* $c_{s_2}<c_{s_2}$ *and that* $D(c_{s_2}^2)>0$ *if* $c_{s_2}<c_{s_1}$. *Let* $f, g\in\mathcal{H}$ *and* $0<a<b<\infty$. *Then we have*

$$(f, g)=\sum_{j\in M}\left(\int_{R^3}\hat{f}_{1j}^{\pm}(\eta)\cdot\hat{g}_{1j}^{\pm}(\eta)d\eta+\int_{R^3}\hat{f}_{1j}^{St}(\eta)\cdot\hat{g}_{1j}^{St}(\eta)d\eta\right)$$
$$+\sum_{k\in N}\int_{R^3}\hat{f}_{2k}^{\pm}(\eta)\cdot\hat{g}_{2k}^{\pm}(\eta)d\eta\,.$$

The first half of Theorem 0.2 expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

THEOREM 0.2. *We assume the same assumption as Theorem* 0.1.

(1) *For* $f\in\mathcal{H}$,

$$f(x)=\sum_{j\in M}\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|\eta|\leq R}(\psi_{1j}^{\pm}(x,\ \eta)\hat{f}_{1j}^{\pm}(\eta)+\psi_{1j}^{St}(x,\ \eta)\hat{f}_{1j}^{St}(\eta))d\eta$$

$$+\sum_{k\in N}\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|\eta|\leq R}\psi_{2k}^{\pm}(x,\ \eta)\hat{f}_{2k}^{\pm}(\eta)d\eta\,.$$

(2) *For* $f\in D(A)$,

$$Af(x)=\sum_{i\in M}\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|\eta|\leq R}(\lambda_j(\eta)\psi_{1j}^{\pm}(x,\ \eta)\hat{f}_{1j}^{\pm}(\eta)+c_{St}^2|\eta'|^2\psi_{1j}^{St}(x,\ \eta)\hat{f}_{1j}^{St}(\eta))d\eta$$

$$+\sum_{k\in N}\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|\eta|\leq R}\lambda_k(\eta)\psi_{2k}^{\pm}(x,\ \eta)\hat{f}_{2k}^{\pm}(\eta)d\eta,$$

and

$$(\widehat{Af})_{1j}^{\pm}(\eta)=\lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta),\qquad j\in M,$$
$$(\widehat{Af})_{1j}^{St}(\eta)=c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta),\qquad j\in M,$$
$$(\widehat{Af})_{2k}^{\pm}(\eta)=\lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta),\qquad k\in N.$$

Theorem 0.3 gives an explicit expression of the ranges $R(\Phi^{\pm})$.

THEOREM 0.3. *Assume the same assumption as Theorem* 0.1. *We define the mapping by*

$$\Phi_{1j}^{\pm}:\quad \mathcal{H}\ni f \longrightarrow \hat{f}_{1j}^{\pm}(\eta)\in L^2(\boldsymbol{R}_+^3,\ \boldsymbol{C}^3)(\xi>0)\in L^2(\boldsymbol{R}_-^3,\ \boldsymbol{C}^3)(\xi<0),\ j\in M,$$

$$\Phi_{1j}^{St}:\quad \mathcal{H}\ni f \longrightarrow \hat{f}_{1j}^{St}(\eta)\in L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3),\ j\in M,$$

$$\Phi_{2k}^{\pm}:\quad \mathcal{H}\ni f \longrightarrow \hat{f}_{2k}^{\pm}(\eta)\in L^2(\boldsymbol{R}_+^3,\ \boldsymbol{C}^3)(\xi>0)\in L^2(\boldsymbol{R}_-^3,\ \boldsymbol{C}^3)(\xi<0),\ k\in N,$$

and put for $f\in H$

$$\Phi^{\pm}f=\Big(\sum_{j\in M}\Phi_{1j}^{\pm}f,\ \sum_{j\in M}\Phi_{1j}^{St}f,\ \sum_{k\in N}\Phi_{2k}^{\pm}f\Big).$$

Then we have

$$R(\Phi^{\pm})=L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3)\oplus L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)\oplus L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3).$$

This implies that $\Phi^{\pm}$ *are unitary operators in* $\mathcal{H}$, *and that the systems of generalized eigenfunctions* $\{\psi_{1j}^{+},\ \psi_{1j}^{St},\ \psi_{2k}^{+}\}_{j\in M,\,k\in N}$ *and* $\{\psi_{1j}^{-},\ \psi_{1j}^{St},\ \psi_{2k}^{-}\}_{j\in M,\,k\in N}$ *are complete.*

The remainder of this paper consists of seven sections. In Section 1, we prove the selfadjointness of the operator A governing the wave propagation of the elastic waves in plane-stratified media $\boldsymbol{R}^3$. In Section 2, we give a construction and an explicit representation of the Green function $G_1(x_3,\ y_3,\ \eta'\ ;\ \zeta)$ for the operator $A_1(\eta')-\zeta I(\zeta\notin\boldsymbol{R})$. In Section 3, the number and nature of the zeros of the Lopatinski determinant of $A_1(\eta')$ are studied by using Cagniard's method. In Section 4, we define a family of generalized eigenfunctions for $A_1(\eta')$ by using the Green function $G_1(x_3, y_3, \eta'; \zeta)$. In Section 5, we give an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta')-\zeta I(\zeta\notin \boldsymbol{R})$ and a family of generalized eigenfunctions for $A_2(\eta')$. In Section 6, we construct the spectral family of A by means of the generalized eigenfunctions of $A_1(\eta')$ and $A_2(\eta')$. We also prove the Parseval formula (Theorem 0.1). Finally in Section 7, we prove the eigenfunction expansion theorems (Theorem 0.2 and 0.3).

§ 1. The Self-adjoint Operator A

In this section, we shall prove the self-adjointness of the operator A along standard results in the theory of linear operators in Hilbert space.

Let us describe the operator A more carefully. We have

$$Mu=\frac{1}{\rho(x_3)}\sum_{i,j=1}^{3}M_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j},$$

with

$$M_{11}=-\begin{pmatrix}\lambda(x_3)+2\mu(x_3) & 0 & 0\\ 0 & \mu(x_3) & 0\\ 0 & 0 & \mu(x_3)\end{pmatrix},\quad M_{12}=-\begin{pmatrix}0 & \lambda(x_3) & 0\\ \mu(x_3) & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$M_{13}=-\begin{pmatrix}0 & 0 & \lambda(x_3)\\ 0 & 0 & 0\\ \mu(x_3) & 0 & 0\end{pmatrix},\quad M_{22}=-\begin{pmatrix}\mu(x_3) & 0 & 0\\ 0 & \lambda(x_3)+2\mu(x_3) & 0\\ 0 & 0 & \mu(x_3)\end{pmatrix},$$

$$M_{23}=-\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \lambda(x_3)\\ 0 & \mu(x_3) & 0\end{pmatrix},\quad M_{33}=-\begin{pmatrix}\mu(x_3) & 0 & 0\\ 0 & \mu(x_3) & 0\\ 0 & 0 & \lambda(x_3)+2\mu(x_3)\end{pmatrix},$$

$$M_{21}={}^tM_{12},\quad M_{31}={}^tM_{13},\quad M_{32}={}^tM_{23}.$$

We represent M and $M_{ij}(1\leqq i,\ j\leqq 3)$ as follows:

$$M=\begin{cases}M^{I}, & x_3<0,\\ M^{II}, & x_3>0,\end{cases}\qquad M_{ij}=\begin{cases}M^{I}_{ij}, & x_3<0,\\ M^{II}_{ij}, & x_3>0.\end{cases}$$

The interface condition (0.5) can be written as follows:

$$\sum_{j=1}^{3}M^{I}_{3j}\frac{\partial u^{I}}{\partial x_j}\bigg|_{x_3=0}=\sum_{j=1}^{3}M^{II}_{3j}\frac{\partial u^{II}}{\partial x_j}\bigg|_{x_3=0}. \tag{1.1}$$

The Sobolev spaces on an open subset Ω of $\boldsymbol{R}^3$ are defined by

$$H^m(\Omega, \boldsymbol{C}^3)=\{u\in \boldsymbol{C}^3\,;\ D^\alpha u\in L^2(\Omega, \boldsymbol{C}^3),\ \text{for } |\alpha|\leqq m\}.$$

Here m is a non-negative integer and the multi-index notation is used for derivatives. Thus $\alpha=(\alpha_1, \alpha_2, \alpha_3)$ where each α_j is a non-negative integer, $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$, $D_j=(\partial/\partial x_j)(j=1, 2, 3)$ and $|\alpha|=\alpha_1+\alpha_2+\alpha_2$. $H^m(\Omega, \boldsymbol{C}^3)$ is a Hilbert space with inner product

$$(u, v)_m=\int_\Omega \sum_{|\alpha|\leq m} D^\alpha u(x)\cdot D^\alpha v(x)dx\,. \tag{1.2}$$

DEFINITION 1.1. $u\in H^1(\boldsymbol{R}^3, \boldsymbol{C}^3)\cap\{Mu\in\mathcal{H}\}$ is said to satisfy the *generalized free interface condition* on $x_3=0$ if one has

$$\int_{\boldsymbol{R}^3} Mu\cdot v\rho(x_3)dx+\sum_{i,j=1}^{3}\int_{\boldsymbol{R}^3} M_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}dx=0 \tag{1.3}$$

for all $v\in H^1(\boldsymbol{R}^3, \boldsymbol{C}^3)$.

Let $D(A)$ denote the set of functions $u\in H^1(\boldsymbol{R}^3, \boldsymbol{C}^3)\cap\{Mu\in\mathcal{H}\}$ which satisfy the generalized free interface condition (1.3). We then have the following theorem:

THEOREM 1.2. *The following operator A with domain $D(A)$:*

$$Au=Mu, \qquad u\in D(A),$$

is a non-negative self-adjoint operator in the Hilbert space $\mathcal{H}=L^2(\boldsymbol{R}^3, \boldsymbol{C}^3, \rho(x_3)dx)$.

And u belongs to $D(A)$ if and only if u belongs to $H^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3)\oplus H^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)$ and satisfies the interface conditions (0.4) *and* (0.5) *in the sense of trace on $x_3=0$.*

In order to prove Theorem 1.2, we prepare some Lemmas.

LEMMA 1.3. *The operator A is symmetric; that is,*

$$A\subset A^*. \tag{1.4}$$

PROOF. To prove (1.4), note that the set

$$\mathcal{D}_0(\boldsymbol{R}^3, \boldsymbol{C}^3)=\mathcal{D}(\boldsymbol{R}^3, \boldsymbol{C}^3)\cap\{u\,;\ u(x)=0 \text{ in a neighborhood of } x_3=0\}$$

is a subset of $D(A)$. And $\mathcal{D}_0(\boldsymbol{R}^3, \boldsymbol{C}^3)$ is dense in $\mathcal{H}$. Hence $D(A)$ is dense in $\mathcal{H}$, so the adjoint operator A^* is uniquely defined. If u and v are both in $D(A)$, then we have by using interface condition (1.3)

$$\begin{aligned}(1.5)\quad (Au, v)&=(Mu, v)\\ &=\int_{\boldsymbol{R}^3} Mu\cdot v\rho(x_3)dx\\ &=\int_{\boldsymbol{R}^3_-} M^I u^I\cdot v^I\rho_1 dx+\int_{\boldsymbol{R}^3_+} M^{II}u^{II}\cdot v^{II}\rho_2 dx\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i,j=1}^{3}\int_{R_-^3} M_{ij}^{I}\frac{\partial^2}{\partial x_i\partial x_j}u^I\cdot v^I dx+\sum_{i,j=1}^{3}\int_{R_+^3} M_{ij}^{II}\frac{\partial^2}{\partial x_i\partial x_j}u^{II}\cdot v^{II}dx\\
&=-\sum_{i,j=1}^{3}\int_{R_-^3} M_{ij}^{I}\frac{\partial u^I}{\partial x_j}\cdot\frac{\partial v^I}{\partial x_i}dx+\sum_{j=1}^{3}\int_{\partial R_-^3} M_{3j}^{I}\frac{\partial u^I}{\partial x_j}\cdot v^I dx'\\
&\quad-\sum_{i,j=1}^{3}\int_{R_+^3} M_{ij}^{II}\frac{\partial u^{II}}{\partial x_j}\cdot\frac{\partial v^{II}}{\partial x_i}dx-\sum_{j=1}^{3}\int_{\partial R_+^3} M_{3j}^{II}\frac{\partial u^{II}}{\partial x_j}\cdot v^{II} dx'\\
&=\int_{R_-^3} u^I\cdot M^I v^I\rho_1 dx+\int_{R_+^3} u^{II}\cdot M^{II}v^{II}\rho_2 dx\\
&=\int_{R^3} u\cdot Mv\rho(x_3)dx\\
&=(u,\ Av),
\end{aligned}
$$

which is equivalent to (1.4). □

LEMMA 1.4. *The symmetric operator A is non-negative; that is*

$$A\geqq 0. \tag{1.6}$$

PROOF. Putting $u=v\in D(A)$ in the first half of the formula (1.5), we have

$$
\begin{aligned}
(Au,\ u)&=\int_{R_-^3} M^I u^I\cdot u^I\rho_1 dx+\int_{R_+^3} M^{II}u^{II}\cdot u^{II}\rho_2 dx\\
&=-\sum_{i,j=1}^{3}\int_{R_-^3} M_{ij}^{I}\frac{\partial u^I}{\partial x_j}\cdot\frac{\partial u^I}{\partial x_i}dx-\sum_{i,j=1}^{3}\int_{R_+^3} M_{ij}^{II}\frac{\partial u^{II}}{\partial x_j}\cdot\frac{\partial u^{II}}{\partial x_i}dx.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
M_{ij}^{I}\frac{\partial u^I}{\partial x_j}\cdot\frac{\partial u^I}{\partial x_i}&=-\begin{pmatrix} c_{1i1j}^{I} & \cdot & c_{1i3j}^{I}\\ \cdot & \cdot & \cdot\\ c_{3i1j}^{I} & \cdot & c_{3i3j}^{I}\end{pmatrix}\begin{pmatrix}\dfrac{\partial u_1^I}{\partial x_j}\\ \cdot\\ \dfrac{\partial u_3^I}{\partial x_j}\end{pmatrix}\cdot\begin{pmatrix}\dfrac{\partial u_1^I}{\partial x_i}\\ \cdot\\ \dfrac{\partial u_3^I}{\partial x_i}\end{pmatrix}\\
&=-\sum_{k,l=1}^{3} c_{kilj}^{I}\frac{\partial u_l^I}{\partial x_j}\overline{\frac{\partial u_k^I}{\partial x_i}}\\
&=-\sum_{k,l=1}^{3}\frac{1}{2}\left(c_{kilj}^{I}\frac{\partial u_l^I}{\partial x_j}\overline{\frac{\partial u_k^I}{\partial x_i}}+c_{iklj}^{I}\frac{\partial u_l^I}{\partial x_j}\overline{\frac{\partial u_i^I}{\partial x_k}}\right)\\
&=-\sum_{k,l=1}^{3} c_{kilj}^{I}\frac{\partial u_l^I}{\partial x_j}\overline{\varepsilon_{ki}^{I}}\\
&=-\sum_{k,l=1}^{3}\frac{1}{2}\left(c_{kilj}^{I}\frac{\partial u_l^I}{\partial x_j}\overline{\varepsilon_{ki}^{I}}+c_{kijl}^{I}\frac{\partial u_j^I}{\partial x_l}\overline{\varepsilon_{ki}^{I}}\right)\\
&=-\sum_{k,l=1}^{3} c_{kilj}^{I}\varepsilon_{lj}^{I}\overline{\varepsilon_{ki}^{I}},
\end{aligned}
$$

and also

$$M^{II}_{ij}\frac{\partial u^{II}}{\partial x_j}\cdot\frac{\partial v^{II}}{\partial x_i}=-\sum_{k,l=1}^{3}c^{II}_{kilj}\varepsilon^{II}_{lj}\overline{\varepsilon^{II}_{ki}}.$$

From the conditions (0.3) and Korn's inequality (cf. [6], [12]), we obtain

$$\|\nabla u^{I}\|^2_{L^2(\mathbf{R}^3_-)}\leqq c\sum_{k,i=1}^{3}\int_{\mathbf{R}^3_-}|\varepsilon^{I}_{ki}|^2dx,\quad \|\nabla u^{II}\|^2_{L^2(\mathbf{R}^3_+)}\leqq c\sum_{k,i=1}^{3}\int_{\mathbf{R}^3_+}|\varepsilon^{II}_{ki}|^2dx,$$

thus

$$\begin{aligned}(Au,\ u)&=-\sum_{i,j=1}^{3}\int_{\mathbf{R}^3_-}M^{I}_{ij}\frac{\partial u^{I}}{\partial x_j}\cdot\frac{\partial u^{I}}{\partial x_i}dx-\sum_{i,j=1}^{3}\int_{\mathbf{R}^3_+}M^{II}_{ij}\frac{\partial u^{II}}{\partial x_j}\cdot\frac{\partial u^{II}}{\partial x_i}dx\\&=\sum_{i,j=1}^{3}\left(\int_{\mathbf{R}^3_-}c^{I}_{kilj}\varepsilon^{I}_{ll}\overline{\varepsilon^{I}_{ki}}dx+\int_{\mathbf{R}^3_+}c^{II}_{kilj}\varepsilon^{II}_{lj}\overline{\varepsilon^{II}_{ki}}dx\right)\\&\geqq\int_{\mathbf{R}^3_-}\delta_1\sum_{k,i=1}^{3}|\varepsilon^{I}_{ki}|^2dx+\int_{\mathbf{R}^3_+}\delta_2\sum_{k,i=1}^{3}|\varepsilon^{II}_{ki}|^2dx\\&\geqq c\delta\|\nabla u\|^2_{L^2(\mathbf{R}^3)},\end{aligned}$$

which implies (1.6). □

LEMMA 1.5. *The range of $I+A$ is $\mathcal{H}$:*

$$R(I+A)=\mathcal{H}. \tag{1.7}$$

PROOF. If $f\in R(I+A)$, there exists an element $u\in D(A)$ such that $u+Au=f$. Then we have for any $v\in H^1(\mathbf{R}^3,\ \mathbf{C}^3)$

$$\begin{aligned}(f,\ v)&=(Au,\ v)+(u,\ v)\\&=-\int_{\mathbf{R}^3}\sum_{i,j=1}^{3}M_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}dx+\int_{\mathbf{R}^3}u\cdot v\rho(x_3)dx.\end{aligned}\tag{1.8}$$

Now we can define by using the right-hand side of (1.8) an inner product on $H^1(\mathbf{R}^3,\ \mathbf{C}^3)$

$$\{u,\ v\}=-\int_{\mathbf{R}^3}\sum_{i,j=1}^{3}M_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}dx+\int_{\mathbf{R}^3}u\cdot v\rho(x_3)dx$$

$$\text{for}\quad \forall u,\ v\in H^1(\mathbf{R}^3,\ \mathbf{C}^3).$$

It follows from Korn's inequality as in the proof of Lemma 1.4 that

$$\{u,\ u\}\geqq c\delta\|\nabla u\|^2_{L^2(\mathbf{R}^3)}+\|u\|^2_{\mathcal{H}}\qquad\text{for}\quad\forall u\in H^1(\mathbf{R}^3,\ \mathbf{C}^3).$$

This implies that the norm $\{u,\ u\}^{1/2}$ is equivalent to the norm $\|u\|_1$ defined by (1.2), and that $H^1(\mathbf{R}^3,\ \mathbf{C}^3)$ is also an Hilbert space (denoted by $\tilde{H}^1(\mathbf{R}^3,\ \mathbf{C}^3)$) with the inner product $\{u,\ v\}$.

For any $f\in\mathcal{H}$, we consider the linear form on $\tilde{H}^1(\mathbf{R}^3,\ \mathbf{C}^3)$:

$$\tilde{H}^1(\mathbf{R}^3,\ \mathbf{C}^3)\ni v\longmapsto(f,\ v)\in\mathbf{C}.$$

Since

$$|(f, v)| \leqq \|f\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \leqq \|f\|_{\mathcal{H}} \{v, v\}^{1/2},$$

this linear form on $\tilde{H}^1(\boldsymbol{R}^3, \boldsymbol{C}^3)$ is bounded. So by the Riesz representation theorem, there exists a $u \in \tilde{H}^1(\boldsymbol{R}^3, \boldsymbol{C}^3)$ such that for all $v \in \tilde{H}^1(\boldsymbol{R}^3, \boldsymbol{C}^3)$

$$(f, v) = \{u, v\}. \tag{1.9}$$

Next, we shall show $u \in D(A)$. By taking $v \in \mathcal{D}(\boldsymbol{R}^3, \boldsymbol{C}^3)$, the equality (1.9) can be written as follows:

$$\langle \rho(x_3)(f-u), v \rangle = \left\langle -\sum_{i,j=1}^{3} M_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right\rangle = \left\langle \sum_{i,j=1}^{3} M_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, v \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{D}'$ and $\mathcal{D}$. This duality means

$$f - u = Mu = Au \in \mathcal{H}$$

in the distribution sense. Furthermore from (1.9)

$$(Mu, v) + \left(\frac{1}{\rho(x_3)} \sum_{i,j=1}^{3} M_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) = 0 \quad \text{for } \forall v \in H^1(\boldsymbol{R}^3, \boldsymbol{C}^3).$$

This means that u satisfies (1.3). Hence $u \in D(A)$. □

LEMMA 1.6. *A function u belongs to $D(A)$ if and only if it belongs to the space $H^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3) \oplus H^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)$ and satisfies the interface conditions* (0.4) *and* (0.5) *in the sence of trace on $x_3 = 0$.*

PROOF. The implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Since

$$H^1(\boldsymbol{R}^3, \boldsymbol{C}^3) \subset H^1(\boldsymbol{R}^3_-, \boldsymbol{C}^3) \oplus H^1(\boldsymbol{R}^3_+, \boldsymbol{C}^3),$$

every $u \in D(A)$ has a unique decomposition

$$u = u^I + u^{II}, \qquad u^I \in H^1(\boldsymbol{R}^3_-, \boldsymbol{C}^3), \qquad u^{II} \in H^1(\boldsymbol{R}^3_+, \boldsymbol{C}^3),$$

where u^I and u^{II} satisfy (1.3). We have the bilinear forms:

$$\left(\frac{1}{\rho_1} \sum_{i,j=1}^{3} M_{ij}^I \frac{\partial u^I}{\partial x_j}, \frac{\partial v^I}{\partial x_i} \right) = -(M^I u^I, v^I),$$

$$\left(\frac{1}{\rho_2} \sum_{i,j=1}^{3} M_{ij}^{II} \frac{\partial u^{II}}{\partial x_j}, \frac{\partial v^{II}}{\partial x_i} \right) = -(M^{II} u^{II}, v^{II}),$$

where $v^I \in H^1(\boldsymbol{R}^3_-, \boldsymbol{C}^3)$, $v^{II} \in H^1(\boldsymbol{R}^3_+, \boldsymbol{C}^3)$. Since we have by regularity theorem (see for example [1, Theorem 9.6]), if

$$-M^I u^I \in L^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3, \rho_1 dx),$$

$$-M^{II} u^{II} \in L^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3, \rho_2 dx),$$

then it follows that

$$u^I\in H^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3), \qquad u^{II}\in H^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3).$$

Note that for all $\omega\in C_0^\infty(\partial\boldsymbol{R}^3, \boldsymbol{C}^3)$, there exist $v\in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{C}^3)$ such that $v|_{x_3=0}=\omega$. From (1.3) with this $v\in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{C}^3)$, it follows that

$$\left\langle \sum_{i,j=1}^{3} M^I_{3j}\frac{\partial u^I}{\partial x_j} - \sum_{i,j=1}^{3} M^{II}_{3j}\frac{\partial u^{II}}{\partial x_j}, \omega\right\rangle=0.$$

Since ω is arbitrary,

$$\sum_{i,j=1}^{3} M^I_{3j}\frac{\partial u^I}{\partial x_j}\bigg|_{x_3=0} = \sum_{i,j=1}^{3} M^{II}_{3j}\frac{\partial u^{II}}{\partial x_j}\bigg|_{x_3=0}.$$

(1.1) is equivalent to (0.5), so this means that u^I and u^{II} satisfy (0.5). □

PROOF OF THEOREM 1.2. The fact that A is self-adjoint is a direct consequence of Lemmas 1.3 and 1.5 and standard results in the theory of linear operators in Hilbert space as follows:

$$A\subset A^*$$

$$R(I+A)=\mathcal{H}$$

$$\Longrightarrow$$

$$A\text{ ; selfadjoint in } \mathcal{H}$$

(see, for example, [13, Section 187, Theorem 2]). Lemma 1.4 shows that A is non-negative. The latter claim is a direct consequence of Lemma 1.6. □

As shown in Section 0, we transform A into a self-adjoint operator $\hat{A}(\eta')$ depending on a parameter η' and moreover we decompose $\hat{A}(\eta')$ as a direct sum of the simple self-adjoint operators $A_1(\eta')$ and $A_2(\eta')$ which is much easier to study (cf. [3] and [5]).

By a direct computation, we can easily prove the following proposition.

PROPOSITION 1.7. *We have*

$$\hat{A}=\hat{A}(\eta')=\mathrm{UC}(A_1(\eta')\oplus A_2(\eta'))(\mathrm{UC})^{-1} \qquad \textit{for} \quad \eta'\neq 0,$$

and

(1.10) $$Au=F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta')\oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u \qquad \textit{for} \quad u\in D(A),$$

where $A_1(\eta')$ *and* $A_2(\eta')$ *are non-negative self-adjoint operators in* $L^2(\boldsymbol{R}, \boldsymbol{C}^2, \rho(x_3)dx_3)$ *and* $L^2(\boldsymbol{R}, \boldsymbol{C}, \rho(x_3)dx_3)$ *defined respectively as follows:*

$$D(A_1(\eta'))=\left\{\begin{pmatrix}u_1\\u_2\end{pmatrix}\in H^2(\boldsymbol{R}_-,\ \boldsymbol{C}^2)\oplus H^2(\boldsymbol{R}_+,\ \boldsymbol{C}^2);\right.$$

$$\left. u^I|_{x_3=0}=u^{III}|_{x_3=0},\ B_1^I(\eta')u^I|_{x_3=0}=B_1^{II}(\eta')u^{II}|_{x_3=0}\right\},$$

$$A_1\left(\eta',\ \frac{d}{dx_3}\right)\begin{pmatrix}u_1\\u_2\end{pmatrix}=\frac{1}{\rho}\begin{pmatrix}-\mu\dfrac{d^2}{dx_3^2}+(\lambda+2\mu)|\eta'|^2 & -i|\eta'|(\lambda+\mu)\dfrac{d}{dx_3}\\ -i|\eta'|(\lambda+\mu)\dfrac{d}{dx_3} & -(\lambda+2\mu)\dfrac{d^2}{dx_3^2}+\mu|\eta'|^2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix},$$

$$B_1^I(\eta')\begin{pmatrix}u_1^I\\u_2^I\end{pmatrix}=\begin{pmatrix}\mu_1\dfrac{d}{dx_3} & i|\eta'|\mu_1\\ i|\eta'|\lambda_1 & (\lambda_1+2\mu_1)\dfrac{d}{dx_3}\end{pmatrix}\begin{pmatrix}u_1^I\\u_2^I\end{pmatrix},$$

$$B_1^{II}(\eta')\begin{pmatrix}u_1^{II}\\u_2^{II}\end{pmatrix}=\begin{pmatrix}\mu_2\dfrac{d}{dx_3} & i|\eta'|\mu_2\\ i|\eta'|\lambda_2 & (\lambda_2+2\mu_2)\dfrac{d}{dx_3}\end{pmatrix}\begin{pmatrix}u_1^{II}\\u_2^{II}\end{pmatrix},$$

$$D(A_2(\eta'))=\{u\in H^2(\boldsymbol{R}_-)\oplus H^2(\boldsymbol{R}_+);$$

$$u^I|_{x_3=0}=u^{II}|_{x_3=0},\ B_2^I(\eta')u^I|_{x_3=0}=B_2^{II}(\eta')u^{II}|_{x_3=0}\},$$

$$A_2\left(\eta',\ \frac{d}{dx_3}\right)u=-\frac{\mu(x_3)}{\rho(x_3)}\frac{d^2u}{dx_3^2}+\frac{\mu(x_3)}{\rho(x_3)}|\eta'|^2u,$$

$$B_2^I(\eta')u^I=\mu_1\frac{d}{dx_3}u^I,\qquad B_2^{II}(\eta')u^{II}=\mu_2\frac{d}{dx_3}u^{II}.$$

Since $A_2(\eta')$ is an operator corresponding to the usual wave operator, from now on, we shall mainly treat the operator $A_1(\eta')$.

§ 2. The Green Function $G_1(x_3,\ y_3,\ \eta';\ \zeta)$ of $A_1(\eta')-\zeta I$

In this section, we give an explicit representation of the Green function $G_1(x_3,\ y_3,\ \eta';\ \zeta)$ for the operator $A_1(\eta')-\zeta I(\zeta\notin\boldsymbol{R})$ by using a standard technique (cf. [7], [9], [10], [16]) in order to define generalized eigenfunctions for the operator $A_1(\eta')$ in Section 4 below.

Denote by $R(\zeta;\ T)$ the resolvent $(T-\zeta)^{-1}$ of an operator T. The resolvent of the selfadjoint operator $A_1(\eta')$ has the kernel representation; that is, there exists the Green function $G_1(x_3,\ y_3,\ \eta';\ \zeta)$ and for $f(\cdot,\ x_3)\in C_0^\infty(\boldsymbol{R}\setminus\{0\},\ \boldsymbol{C}^2)$ we have

$$R(\zeta;\ A_1(\eta'))f(\eta',\ x_3)=\int_{\boldsymbol{R}}G_1(x_3,\ y_3,\ \eta';\ \zeta)f(\eta',\ y_3)dy_3.$$

From the self-adjointness of $A_1(\eta')$, it follows that the resolvent kernel has the symmetry property

$$G_1(x_3, y_3, \eta' ; \zeta)^* = G_1(y_3, x_3, \eta' ; \bar{\zeta}).$$

In order to find the Green function, we consider the following problem:

$$(2.1)\qquad \begin{aligned} &(A_1^I(\eta', D)-\zeta)v^I(x_3, \eta' ; \zeta)=f(\eta', x_3), \qquad x_3<0, \\ &(A_1^{II}(\eta', D)-\zeta)v^{II}(x_3, \eta' ; \zeta)=f(\eta', x_3), \qquad x_3>0, \end{aligned}$$

$$(2.2)\qquad v^I(x_3, \eta' ; \zeta)|_{x_3=0}=v^{II}(x_3, \eta' ; \zeta)|_{x_3=0},$$

$$(2.3)\qquad B_1^I(\eta')v^I(x_3, \eta' ; \zeta)|_{x_3=0}=B_1^{II}(\eta')v^{II}(x_3, \eta' ; \zeta)|_{x_3=0},$$

where $D=(1/i)(d/dx_3)$.

Let us seek solutions $v^I(x_3, \eta' ; \zeta)$ and $v^{II}(x_3, \eta' ; \zeta)$ in the form

$$v^I(x_3, \eta' ; \zeta)=E^I(x_3, \eta' ; \zeta)-K^I(x_3, \eta' ; \zeta),$$

$$v^{II}(x_3, \eta' ; \zeta)=E^{II}(x_3, \eta' ; \zeta)-K^{II}(x_3, \eta' ; \zeta).$$

Let

$$(2.4)\qquad \begin{aligned} &\mathcal{E}^I(x_3-y_3, \eta', \zeta)=\frac{1}{\sqrt{2\pi}}F_\xi^{-1}[(A^I(\eta', \xi)-\zeta)^{-1}e^{-iy_3\xi}], \\ &\mathcal{E}^{II}(x_3-y_3, \eta', \zeta)=\frac{1}{\sqrt{2\pi}}F_\xi^{-1}[(A^{II}(\eta', \xi)-\zeta)^{-1}e^{-iy_3\xi}], \end{aligned}$$

where ξ is the dual variable of x_3 and F_ξ^{-1} denotes the inverse or conjugate Fourier transformation with respect to ξ:

$$(F_\xi^{-1}f)(\eta', x_3)=\underset{R\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{|\xi|\le R}e^{ix_3\xi}f(\xi)d\xi \qquad \text{for } f\in\mathcal{H}.$$

$\mathcal{E}^I$ and $\mathcal{E}^{II}$ are fundamental solutions of $A_1^I(\eta')-\zeta$ and $A_1^{II}(\eta')-\zeta$, respectively, that is, $\mathcal{E}^I$ and $\mathcal{E}^{II}$ are distribution solutions of the equations

$$(A_1^I(\eta', D)-\zeta)\mathcal{E}^I(x_3-y_3, \eta', \zeta)=\delta(x_3-y_3)I, \qquad x_3<0,$$

$$(A_1^{II}(\eta', D)-\zeta)\mathcal{E}^{II}(x_3-y_3, \eta', \zeta)=\delta(x_3-y_3)I, \qquad x_3>0.$$

Then we have in the sense of distributions

$$E^I(x_3, \eta' ; \zeta)=\int_{\boldsymbol{R}}\mathcal{E}^I(x_3-y_3, \eta' ; \zeta)f(\eta', y_3)dy_3,$$

$$E^{II}(x_3, \eta' ; \zeta)=\int_{\boldsymbol{R}}\mathcal{E}^{II}(x_3-y_3, \eta' ; \zeta)f(\eta', y_3)dy_3.$$

$K^I(x_3, \eta' ; \zeta)\in H^2(\boldsymbol{R}_-, \boldsymbol{C}^2)$ and $K^{II}(x_3, \eta' ; \zeta)\in H^2(\boldsymbol{R}_+, \boldsymbol{C}^2)$ are solutions of the equations

$$
\begin{aligned}
&(A_1^I(\eta', D)-\zeta)K^I(x_3, \eta', \zeta)=0, \qquad x_3<0,\\
&(A_1^{II}(\eta', D)-\zeta)K^{II}(x_3, \eta', \zeta)=0, \qquad x_3>0,
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
E^I(x_3, \eta' ; \zeta)|_{x_3=0}&-E^{II}(x_3, \eta' ; \zeta)|_{x_3=0}\\
&=K^I(x_3, \eta' ; \zeta)|_{x_3=0}-K^{II}(x_3, \eta' ; \zeta)|_{x_3=0},
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
B_1^I(\eta')E^I(x_3, \eta' ; \zeta)|_{x_3=0}&-B_1^{II}(\eta')E^{II}(x_3, \eta' ; \zeta)|_{x_3=0}\\
&=B_1^I(\eta')K^I(x_3, \eta' ; \zeta)|_{x_3=0}-B_1^{II}(\eta')K^{II}(x_3, \eta' ; \zeta)|_{x_3=0}.
\end{aligned} \tag{2.7}
$$

First, we find an explicit representation of the fundamental solution $\mathcal{E}^I$. The characteristic matrix $A_1^I(\eta', \xi)$ of $A_1^I(\eta', D)$ is a 2×2 Hermitian matrix with characteristic polynomial

$$
\det(A_1^I(\eta', \xi)-\zeta I)=\left(\zeta-\frac{\mu_1}{\rho_1}|\eta|^2\right)\left(\zeta-\frac{\lambda_1+2\mu_1}{\rho_1}|\eta|^2\right),
$$

where $\eta=(\eta', \xi)=(\eta_1, \eta_2, \xi)$ and $|\eta|^2=|\eta'|^2+\xi^2$. $(\mu_1/\rho_1)^{1/2}$ and $((\lambda_1+2\mu_1)/\rho_1)^{1/2}$ are the propagation speeds of shear and pressure waves, usually called S wave and P wave respectively by physicists and engineers. Thus, from now on, we use the following notation

$$
c_{s_1}^2=\frac{\mu_1}{\rho_1}, \qquad c_{p_1}^2=\frac{\lambda_1+2\mu_1}{\rho_1}. \tag{2.8}
$$

The distinct eigenvalues of $A_1^I(\eta', \xi)$ are

$$
\lambda_{s_1}(\eta)=c_{s_1}^2|\eta|^2, \qquad \lambda_{p_1}(\eta)=c_{p_1}^2|\eta|^2. \tag{2.9}
$$

Introducing the set of indices

$$
M_1=\{s_1, p_1\}, \tag{2.10}
$$

the resolution of the identity for $A_1^I(\eta', \xi)$ is given by

$$
I=\sum_{j\in M_1} P_j(\eta).
$$

Here the $P_j(\eta)$ $(j\in M_1)$ are mutually orthogonal projections defined by

$$
P_j(\eta)=\frac{1}{2\pi i}\int_{|\lambda_j(\eta)-\zeta|=\delta}(\zeta I-A_1^I(\eta))^{-1}d\zeta, \qquad j\in M_1,
$$

where the integration goes over a small circle in the complex plane enclosing only the eigenvalue $\lambda_j(\eta)$ $(j\in M_1)$ in the positive direction.

Since $P_j(\eta)$ $(j\in M_1)$ satisfy the following properties:

$$
\begin{aligned}
&P_j^*=P_j, \qquad \delta_{jk}P_j=P_jP_k,\\
&A_1^I(\eta', \xi)P_j(\eta)=\lambda_j(\eta)P_j(\eta),
\end{aligned}
$$

we have

$$(2.11)\qquad (A_1^I(\eta',\xi)-\zeta I)^{-1}=\sum_{j\in M_1}\frac{1}{\lambda_j(\eta)-\zeta}P_j(\eta).$$

$P_{s_1}(\eta)$ and $P_{p_1}(\eta)$ have more explicit representations. In fact, the $\lambda_j(\eta)\,(j\in M_1)$ are simple poles of $(\zeta I-A_1^I(\eta))^{-1}$

$$(2.12)\qquad P_{s_1}(\eta)=\lim_{\zeta\to\lambda_{s_1}(\eta)}(\zeta-\lambda_{s_1}(\eta))(\zeta I-A_1^I(\eta))^{-1}$$

$$=\lim_{\zeta\to c_{s_1}^2|\eta|^2}\frac{1}{\zeta-c_{p_1}^2|\eta|^2}\begin{pmatrix}\zeta-(c_{p_1}^2\xi^2+c_{s_1}^2|\eta'|^2) & |\eta'|(c_{p_1}^2-c_{s_1}^2)\xi\\ |\eta'|(c_{p_1}^2-c_{s_1}^2)\xi & \zeta-(c_{s_1}^2\xi^2+c_{p_1}^2|\eta'|^2)\end{pmatrix}$$

$$=\frac{1}{|\eta|^2}\begin{pmatrix}\xi^2 & -\xi|\eta'|\\ -\xi|\eta'| & |\eta'|^2\end{pmatrix},$$

and similarly,

$$(2.13)\qquad P_{p_1}(\eta)=\frac{1}{|\eta|^2}\begin{pmatrix}|\eta'|^2 & \xi|\eta'|\\ \xi|\eta'| & \xi^2\end{pmatrix}.$$

From (2.4) and (2.11), we have

$$(2.14)\qquad \mathcal{E}^I(x_3-y_3,\eta';\zeta)=\frac{1}{2\pi}\int_R e^{i(x_3-y_3)\xi}\frac{P_{s_1}(\eta',\xi)}{\lambda_{s_1}(\xi)-\zeta}d\xi$$

$$+\frac{1}{2\pi}\int_R e^{i(x_3-y_3)\xi}\frac{P_{p_1}(\eta',\xi)}{\lambda_{p_1}(\xi)-\zeta}d\xi,\qquad x_3>0.$$

Now, we calculate the two integrals on the right-hand side of (2.14). To do so, we change the real variable ξ to the complex variable $\tau=\xi+i\kappa$ and define

$$(2.15)\qquad \tau_{s_1}=\sqrt{\frac{\zeta}{c_{s_1}^2}-|\eta'|^2},\quad \operatorname{Im}\tau_{s_1}\geqq 0,\quad \tau_{p_1}=\sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2},\quad \operatorname{Im}\tau_{p_1}\geqq 0.$$

Then the determinant $\det(A_1^I(\eta',\tau)-\zeta I)$ is equal to:

$$\det(A_1^I(\eta',\tau)-\zeta I)=c_{s_1}^2c_{p_1}^2(\tau+\tau_{s_1})(\tau-\tau_{s_1})(\tau+\tau_{p_1})(\tau-\tau_{p_1}).$$

Let us consider the first term in the right-hand side of (2.14). In the case where $x_3-y_3>0$, we may deform the path of integration into the τ upper half plane as indicated in Figure 2.

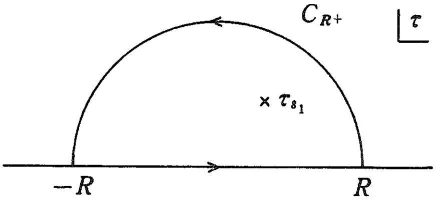

Figure 2. A path of integration

We obtain

$$2\pi i \operatorname*{Res}_{\tau=\tau_{s_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}\right)=\left(\int_{-R}^{R}+\int_{C_R^+}\right)e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}d\tau .$$

On the half-circle C_R^+, $\tau=Re^{i\theta}(0\leqq\theta\leqq\pi)$, we have as $R\to\infty$

$$\left|\int_{C_R^+}e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}d\tau\right|\leqq\text{const.}\int_0^\pi\frac{e^{-(x_3-y_3)R\sin\theta}}{R^2}d\theta\to 0,$$

and so

$$\int_{-\infty}^{\infty}e^{i(x_3-y_3)\xi}\frac{P_{s_1}(\eta',\xi)}{\lambda_{s_1}(\eta',\xi)-\zeta}d\xi=2\pi i\operatorname*{Res}_{\tau=\tau_{s_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}\right),$$
$$x_3-y_3>0.$$

In the case where $x_3-y_3<0$, we may deform the path of integration into the τ lower half plane, and so we have in a similar way

$$\int_{-\infty}^{\infty}e^{i(x_3-y_3)\xi}\frac{P_{s_1}(\eta',\xi)}{\lambda_{s_1}(\eta',\xi)-\zeta}d\xi=-2\pi i\operatorname*{Res}_{\tau=-\tau_{s_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}\right),$$
$$x_3-y_3<0.$$

The second term in the right-hand side of (2.14) is also calculated similarly. Summing up, we have

$$\mathcal{E}^I(x_3-y_3,\eta';\zeta)$$
$$=i\begin{cases}\operatorname*{Res}_{\tau=\tau_{s_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}\right)+\operatorname*{Res}_{\tau=\tau_{p_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{p_1}(\eta',\tau)}{\lambda_{p_1}(\eta',\tau)-\zeta}\right)\\ -\operatorname*{Res}_{\tau=-\tau_{s_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta}\right)-\operatorname*{Res}_{\tau=-\tau_{p_1}}\left(e^{i(x_3-y_3)\tau}\frac{P_{p_1}(\eta',\tau)}{\lambda_{p_1}(\eta',\tau)-\zeta}\right)\end{cases}$$
$$=\frac{i}{2}\begin{cases}\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta',\tau_{s_1})+\frac{e^{i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta',\tau_{p_1}), & x_3-y_3>0,\\ \frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta',-\tau_{s_1})+\frac{e^{-i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta',-\tau_{p_1}), & x_3-y_3<0.\end{cases}$$

So we have for $x_3<0$

(2.16) $E^I(x_3,\eta';\zeta)$

$$=\frac{i}{2}\times\left[\int_{-\infty}^{x_3}\left(\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta',\tau_{s_1})+\frac{e^{i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta',\tau_{p_1})\right)f(\eta',y_3)dy_3\right.$$
$$\left.+\int_{x_3}^{\infty}\left(\frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta',-\tau_{s_1})+\frac{e^{-i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta',-\tau_{p_1})\right)f(\eta',y_3)dy_3\right].$$

As to the $\mathcal{E}^{II}$, let

$$(2.17)\quad c_{s_2}^2=\frac{\mu_2}{\rho_2},\qquad c_{p_2}^2=\frac{\lambda_2+2\mu_2}{\rho_2},$$

$$\lambda_{s_2}(\eta)=c_{s_2}^2|\eta|^2,\qquad \lambda_{p_2}(\eta)=c_{p_2}^2|\eta|^2,$$

$$M_2=\{s_2,\ p_2\},$$

$$\tau_{s_2}=\sqrt{\frac{\zeta}{c_{s_2}^2}-|\eta'|^2},\quad \operatorname{Im}\tau_{s_2}\geqq 0,\quad \tau_{p_2}=\sqrt{\frac{\zeta}{c_{p_2}^2}-|\eta'|^2},\quad \operatorname{Im}\tau_{p_2}\geqq 0.$$

$$P_{s_2}(\eta)=\frac{1}{|\eta|^2}\begin{pmatrix}\xi^2 & -\xi|\eta'| \\ -\xi|\eta'| & |\eta'|^2\end{pmatrix},\qquad P_{p_2}(\eta)=\frac{1}{|\eta|^2}\begin{pmatrix}|\eta'|^2 & \xi|\eta'| \\ \xi|\eta'| & \xi^2\end{pmatrix}.$$

Using these notations and taking the same procedure as $\mathcal{E}^I$, we have for $x_3>0$

$$(2.18)\quad E^{II}(x_3,\ \eta'\,;\ \zeta)$$

$$=\frac{i}{2}\times\Bigg[\int_{-\infty}^{x_3}\Bigg(\frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{s_2}(\eta',\ \tau_{s_2})+\frac{e^{i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2\tau_{p_2}}P_{p_2}(\eta',\ \tau_{p_2})\Bigg)f(\eta',\ y_3)dy_3$$

$$+\int_{x_3}^{\infty}\Bigg(\frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{s_2}(\eta',\ -\tau_{s_2})+\frac{e^{-i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2\tau_{p_2}}P_{p_2}(\eta',\ -\tau_{p_2})\Bigg)f(\eta',\ y_3)dy_3\Bigg].$$

From the equation (2.5), we may suppose that

$$K^I(x_3,\ \eta'\,;\ \zeta)=C_1e^{-i\tau_{p_1}x_3}+C_2e^{i\tau_{p_1}x_3}+C_3e^{-i\tau_{s_1}x_3}+C_4e^{i\tau_{s_1}x_3},\qquad x_3<0,$$

where $C_1,\ \cdots,\ C_4$ are 2×1 matrices. Since $x_3<0$ and (2.15), the hypothesis that $K^I(x_3,\ \eta'\,;\ \zeta)\in H^2(\boldsymbol{R}_-,\ \boldsymbol{C}^2)$ implies $C_2=C_4=0$. Moreover, since K^I is a solution of (2.5), K^I can be represented as follows:

$$(2.19)\quad K^I(x_3,\ \eta'\,;\ \zeta)=\alpha_1\begin{pmatrix}|\eta'| \\ -\tau_{p_1}\end{pmatrix}e^{-i\tau_{p_1}x_3}+\alpha_2\begin{pmatrix}\tau_{s_1} \\ |\eta'|\end{pmatrix}e^{-i\tau_{s_1}x_3},\qquad x_3<0,$$

where $\alpha_1,\ \alpha_2\in\boldsymbol{C}$. As for $K^{II}(x_3,\ \eta'\,;\zeta)$, we have the following representation:

$$(2.20)\quad K^{II}(x_3,\ \eta'\,;\ \zeta)=\beta_1\begin{pmatrix}|\eta'| \\ \tau_{p_2}\end{pmatrix}e^{i\tau_{p_2}x_3}+\beta_2\begin{pmatrix}-\tau_{s_2} \\ |\eta'|\end{pmatrix}e^{i\tau_{s_2}x_3},\qquad x_3>0,$$

where $\beta_1,\ \beta_2\in\boldsymbol{C}$.

Let us determine the constants α_1, α_2, β_1 and β_2 so that K^I and K^{II} satisfy (2.6) and (2.7). Note that

$$P_{p_1}(\eta',\ \tau_{p_1})=\frac{c_{p_1}^2}{\zeta}\begin{pmatrix}|\eta'|^2 & \tau_{p_1}|\eta'| \\ \tau_{p_1}|\eta'| & \tau_{p_1}^2\end{pmatrix},\quad P_{s_1}(\eta',\ \tau_{s_1})=\frac{c_{s_1}^2}{\zeta}\begin{pmatrix}\tau_{s_1}^2 & -\tau_{s_1}|\eta'| \\ -\tau_{s_1}|\eta'| & |\eta'|^2\end{pmatrix},$$

$$P_{p_2}(\eta',\ \tau_{p_2})=\frac{c_{p_2}^2}{\zeta}\begin{pmatrix}|\eta'|^2 & \tau_{p_2}|\eta'| \\ \tau_{p_2}|\eta'| & \tau_{p_2}^2\end{pmatrix},\quad P_{s_2}(\eta',\ \tau_{s_2})=\frac{c_{s_2}^2}{\zeta}\begin{pmatrix}\tau_{s_2}^2 & -\tau_{s_2}|\eta'| \\ -\tau_{s_2}|\eta'| & |\eta'|^2\end{pmatrix}.$$

Then, if we multiply the both sides of (2.7) by $1/i$, then the equations on α_1, α_2, β_1 and β_2 can be written in the matrix form as follows:

(2.21)

$$\begin{pmatrix} |\eta'| & \tau_{s_1} & -|\eta'| & \tau_{s_2} \\ -\tau_{p_1} & |\eta'| & -\tau_{p_2} & -|\eta'| \\ -2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| & -\rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) & -2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| & \rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) & -2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| & -\rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) & -2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \begin{pmatrix} |\eta'| \\ \tau_{p_1} \\ 2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) \end{pmatrix} g_1(\eta', \zeta) + \begin{pmatrix} -\tau_{s_1} \\ |\eta'| \\ -\rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) \\ 2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| \end{pmatrix} g_2(\eta', \zeta)$$

$$+ \begin{pmatrix} -|\eta'| \\ -\tau_{p_2} \\ -2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| \\ -\rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) \end{pmatrix} g_3(\eta', \zeta) + \begin{pmatrix} \tau_{s_2} \\ -|\eta'| \\ \rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) \\ -2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{pmatrix} g_4(\eta', \zeta)$$

$$+ \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \\ -2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) \end{pmatrix} g_1'(\eta', \zeta) + \begin{pmatrix} \tau_{s_1} \\ |\eta'| \\ -\rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) \\ -2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| \end{pmatrix} g_2'(\eta', \zeta)$$

$$+ \begin{pmatrix} -|\eta'| \\ \tau_{p_2} \\ 2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| \\ -\rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) \end{pmatrix} g_3'(\eta', \zeta) + \begin{pmatrix} -\tau_{s_2} \\ -|\eta'| \\ \rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) \\ 2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{pmatrix} g_4'(\eta', \zeta),$$

where

$$g_1(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{p_1}\zeta}\int_{-\infty}^{0} e^{-i\tau_{p_1}y_3}(|\eta'|, \tau_{p_1})f(\eta', y_3)dy_3,$$

$$g_2(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{s_1}\zeta}\int_{-\infty}^{0} e^{-i\tau_{s_1}y_3}(-\tau_{s_1}, |\eta'|)f(\eta', y_3)dy_3,$$

$$g_3(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{p_2}\zeta}\int_{-\infty}^{0} e^{-i\tau_{p_2}y_3}(|\eta'|, \tau_{p_2})f(\eta', y_3)dy_3,$$

$$g_4(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{s_2}\zeta}\int_{-\infty}^{0} e^{-i\tau_{s_2}y_3}(-\tau_{s_2}, |\eta'|)f(\eta', y_3)dy_3,$$

$$g_1'(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{p_1}\zeta}\int_{0}^{\infty} e^{i\tau_{p_1}y_3}(|\eta'|, -\tau_{p_1})f(\eta', y_3)dy_3,$$

$$g_2'(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{s_1}\zeta}\int_{0}^{\infty} e^{i\tau_{s_1}y_3}(\tau_{s_1}, |\eta'|)f(\eta', y_3)dy_3,$$

$$g_3'(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{p_2}\zeta}\int_{0}^{\infty} e^{i\tau_{p_2}y_3}(|\eta'|, -\tau_{p_2})f(\eta', y_3)dy_3,$$

$$g_4'(\eta', \zeta) = \frac{i}{2}\frac{1}{\tau_{s_2}\zeta}\int_{0}^{\infty} e^{i\tau_{s_2}y_3}(\tau_{s_2}, |\eta'|)f(\eta', y_3)dy_3.$$

Put

$$(2.22)\quad \Delta(\eta', \zeta)=$$

$$\begin{vmatrix} |\eta'| & \tau_{s_1} & -|\eta'| & \tau_{s_2} \\ -\tau_{p_1} & |\eta'| & -\tau_{p_2} & -|\eta'| \\ -2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| & -\rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) & -2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| & \rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2) & -2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| & -\rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2) & -2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{vmatrix}.$$

$\Delta(\eta', \zeta)$ is called the Lopatinski determinant. $\Delta_{p_1}^l(\eta', \zeta)$, $\Delta_{s_1}^l(\eta', \zeta)$, $\Delta_{p_2}^l(\eta', \zeta)$, and $\Delta_{s_2}^l(\eta', \zeta)$ ($l=1, 2, 3, 4$) denote the determinants respectively obtained from $\Delta(\eta', \zeta)$ by replacing the lth column by

$$^t(|\eta'|,\quad \tau_{p_1},\quad 2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'|,\quad \rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2)),$$

$$^t(-\tau_{s_1},\quad |\eta'|,\quad -\rho_1 c_{s_1}^2(\tau_{s_1}^2-|\eta'|^2),\quad 2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'|),$$

$$^t(-|\eta'|,\quad \tau_{p_2},\quad 2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'|,\quad -\rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2)),$$

and by

$$^t(-\tau_{s_2},\quad -|\eta'|,\quad \rho_2 c_{s_2}^2(\tau_{s_2}^2-|\eta'|^2),\quad 2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'|).$$

If $\Delta(\eta', \zeta)\neq 0$, then (2.21) has a unique solutions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ which are given in the following form:

$$\alpha_1=\frac{\Delta_{p_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_1'(\eta', \zeta)$$
$$+\frac{\Delta_{p_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta),$$

$$\alpha_2=\frac{\Delta_{p_1}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_2'(\eta', \zeta)$$
$$+\frac{\Delta_{p_2}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta),$$

$$\beta_1=\frac{\Delta_{p_1}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_3(\eta', \zeta)$$
$$+\frac{\Delta_{p_2}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta),$$

$$\beta_2=\frac{\Delta_{p_1}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_4(\eta', \zeta)$$
$$+\frac{\Delta_{p_2}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta).$$

Thus we have for $x_3<0$

(2.23)

$$\begin{aligned}
&v^{I}(x_3, \eta'; \zeta)\\
&=E^{I}(x_3, \eta'; \zeta)-K^{I}(x_3, \eta'; \zeta)\\
&=\frac{i}{2}\Bigl(\int_{-\infty}^{x_3}\Bigl(\frac{e^{i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta', \tau_{p_1})+\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta', \tau_{s_1})\Bigr)f(\eta', y_3)dy_3\\
&\quad+\int_{x_3}^{\infty}\Bigl(\frac{e^{-i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta', -\tau_{p_1})+\frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta', -\tau_{s_1})\Bigr)f(\eta', y_3)dy_3\Bigr)\\
&\quad-\Bigl(\frac{\Delta_{p_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_1'(\eta', \zeta)\\
&\qquad+\frac{\Delta_{p_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta)\Bigr)\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}e^{-i\tau_{p_1}x_3}\\
&\quad-\Bigl(\frac{\Delta_{p_1}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_2'(\eta', \zeta)\\
&\qquad+\frac{\Delta_{p_2}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta)\Bigr)\begin{pmatrix}\tau_{s_1}\\ |\eta'|\end{pmatrix}e^{-i\tau_{s_1}x_3},
\end{aligned}$$

and for $x_3>0$

(2.24)

$$\begin{aligned}
&v^{II}(x_3, \eta'; \zeta)\\
&=E^{II}(x_3, \eta'; \zeta)-K^{II}(x_3, \eta'; \zeta)\\
&=\frac{i}{2}\Bigl(\int_{-\infty}^{x_3}\Bigl(\frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{p_2}(\eta', \tau_{p_2})+\frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{s_2}(\eta', \tau_{s_2})\Bigr)f(\eta', y_3)dy_3\\
&\quad+\int_{x_3}^{\infty}\Bigl(\frac{e^{-i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2\tau_{p_2}}P_{p_2}(\eta', -\tau_{p_2})+\frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{s_2}(\eta', -\tau_{s_2})\Bigr)f(\eta', y_3)dy_3\Bigr)\\
&\quad-\Bigl(\frac{\Delta_{p_1}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_3(\eta', \zeta)\\
&\qquad+\frac{\Delta_{p_2}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta)\Bigr)\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}e^{i\tau_{p_2}x_3}\\
&\quad-\Bigl(\frac{\Delta_{p_1}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta)+\frac{\Delta_{s_1}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta)+g_4(\eta', \zeta)\\
&\qquad+\frac{\Delta_{p_2}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta)+\frac{\Delta_{s_2}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta)\Bigr)\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}e^{i\tau_{s_2}x_3}.
\end{aligned}$$

In summary, the Green functions $G_1^{I}(x_3, y_3, \eta'; \zeta)$ and $G_1^{II}(x_3, y_3, \eta'; \zeta)$ for $x_3<0$ and $x_3>0$ are given respectively in the following form:

(2.25)

$$\begin{aligned}
&G_1^{I}(x_3, y_3, \eta'; \zeta)\\
&=\frac{i}{2}\Bigl[-H(-y_3)\Bigl\{\frac{\Delta_{p_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{p_1}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}(|\eta'|\tau_{p_1})
\end{aligned}$$

$$+\frac{\Delta_{s_1}^1(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{p_1}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(-\tau_{s_1}|\eta'|)\Big\}$$

$$-H(y_3)\Big\{e^{-i\tau_{p_1}x_3}e^{i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(|\eta'|-\tau_{p_1})$$

$$+\frac{\Delta_{p_2}^1(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{p_1}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(|\eta'|-\tau_{p_2})$$

$$+\frac{\Delta_{s_2}^1(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{p_1}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}$$

$$-H(-y_3)\Big\{\frac{\Delta_{p_1}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(|\eta'|\tau_{p_1})$$

$$+\frac{\Delta_{s_1}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(-\tau_{s_1}|\eta'|)\Big\}$$

$$-H(y_3)\Big\{e^{-i\tau_{s_1}x_3}e^{i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_1}|\eta'|)$$

$$+\frac{\Delta_{p_2}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(|\eta'|-\tau_{p_2})$$

$$+\frac{\Delta_{s_2}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}e^{i\tau_{2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}$$

$$+H(x_3-y_3)\Big\{e^{i\tau_{p_1}(x_3-y_3)}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_1}\end{pmatrix}(|\eta'|\tau_{p_1})$$

$$+e^{i\tau_{s_1}(x_3-y_3)}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_1}\\|\eta'|\end{pmatrix}(-\tau_{s_1}|\eta'|)\Big\}$$

$$+H(y_3-x_3)\Big\{e^{-i\tau_{p_1}(x_3-y_3)}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(|\eta'|-\tau_{p_1})$$

$$+e^{-i\tau_{s_1}(x_3-y_3)}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_1}|\eta'|)\Big\}\Big],\quad x_3<0,$$

(2.26)

$$G_1^{II}(x_3,\ y_3,\ \eta';\ \zeta)$$

$$=\frac{i}{2}\Big[-H(-y_3)\Big\{\frac{\Delta_{p_1}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\tau_{p_1})$$

$$+\frac{\Delta_{s_1}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(-\tau_{s_1}|\eta'|)$$

$$+e^{i\tau_{p_2}x_3}e^{-i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\tau_{p_2})\Big\}$$

$$-H(y_3)\Big\{\frac{\Delta_{p_2}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|-\tau_{p_2})$$

$$
\begin{aligned}
&\qquad+\frac{\Delta_{s_2}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Bigg\}\\
&-H(-y_3)\Bigg\{\frac{\Delta_{p_1}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}-\tau_{s_1}\\ |\eta'|\end{pmatrix}(|\eta'|\ \tau_{p_1})\\
&\qquad+\frac{\Delta_{s_1}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}(-\tau_{s_1}\ |\eta'|)\\
&\qquad+e^{i\tau_{s_2}x_3}e^{-i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}(-\tau_{s_2}\ |\eta'|)\Bigg\}\\
&-H(y_3)\Bigg\{\frac{\Delta_{p_2}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}(|\eta'|\ -\tau_{p_2})\\
&\qquad+\frac{\Delta_{s_2}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Bigg\}\\
&+H(x_3-y_3)\Bigg\{e^{i\tau_{p_2}(x_3-y_3)}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}(|\eta'|\ \tau_{p_2})\\
&\qquad+e^{i\tau_{s_2}(x_3-y_3)}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}(-\tau_{s_2}\ |\eta'|)\Bigg\}\\
&+H(y_3-x_3)\Bigg\{e^{-i\tau_{p_2}(x_3-y_3)}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_2}\end{pmatrix}(|\eta'|\ -\tau_{p_2})\\
&\qquad+e^{-i\tau_{s_2}(x_3-y_3)}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}\tau_{s_2}\\ |\eta'|\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Bigg\}\Bigg],\quad x_3>0.
\end{aligned}
$$

Here $H(y_3)$ denotes the Heaviside function.

§ 3. Zeros of the Lopatinski Determinant of $A_1(\eta')$

In this section, we investigate the number and nature of the zeros of the Lopatinski determinant defined in Section 2 in order to obtain the speed of the Stoneley wave. Our Lopatinski determinant seems to be equivalent to Cagniard's one. But Cagniard expressed the solutions of the elastic equation in cylindrical coordinates by using the Bessel transformation, and investigated the existence of the Stoneley wave. So our parameters are different from Cagniard's one. For the sake of completeness, we present the proof by a method due to L. Cagniard [2, Section 4].

Put

$$
\begin{aligned}
&z=\frac{\zeta}{|\eta'|^2},\\
(3.1)\qquad &a_1=\sqrt{1-\frac{z}{c_{p_1}^2}},\quad a_2=\sqrt{1-\frac{z}{c_{p_2}^2}},\quad b_1=\sqrt{1-\frac{z}{c_{s_1}^2}},\quad b_2=\sqrt{1-\frac{z}{c_{s_2}^2}},
\end{aligned}
$$

then

$$\tau_{p_1}=i|\eta'|a_1, \quad \tau_{p_2}=i|\eta'|a_2, \quad \tau_{s_1}=i|\eta'|b_1, \quad \tau_{s_2}=i|\eta'|b_2.$$

With this notation, the Lopatinski determinant (2.22) is rewritten as follows:

$$\begin{aligned}\Delta(\eta', \zeta)&=|\eta'|^6\begin{vmatrix} 1 & ib_1 & -1 & ib_2 \\ -ia_1 & 1 & -ia_2 & -1 \\ 2i\mu_1a_1 & -\mu_1(b_1^2+1) & 2i\mu_2b_2 & \mu_2(b_2^2+1) \\ \mu_1(b_1^2+1) & 2i\mu_1b_1 & -\mu_2(b_2^2+1) & 2i\mu_2b_2 \end{vmatrix} \\ &=|\eta'|^6D(z),\end{aligned}$$

where

$$\begin{aligned}D(z)=&\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}+\frac{\mu_2z}{c_{s_2}^2}\right)^2+4(\mu_1-\mu_2)^2a_1a_2b_1b_2 \\ &-a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2z}{c_{s_2}^2}\right)^2-a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}\right)^2 \\ &-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}(a_1b_2+a_2b_1)z^2.\end{aligned} \tag{3.2}$$

By (2.8) and (2.17), the propagation speeds of shear and pressure waves should satisfy

$$c_{s_1}<c_{p_1}, \qquad c_{s_2}<c_{p_2},$$

so there are six cases

$$c_{s_1}<c_{p_1}\leqq c_{s_2}<c_{p_2},$$

$$c_{s_1}\leqq c_{s_2}\leqq c_{p_1}\leqq c_{p_2},$$

$$c_{s_1}\leqq c_{s_2}<c_{p_2}\leqq c_{p_1},$$

$$c_{s_2}<c_{p_2}\leqq c_{s_1}<c_{p_1},$$

$$c_{s_2}\leqq c_{s_1}\leqq c_{p_2}\leqq c_{p_1},$$

$$c_{s_2}\leqq c_{s_1}<c_{p_1}\leqq c_{p_2},$$

to consider. From now on, we have only to consider the standard case:

$$c_{s_1}<c_{p_1}<c_{s_2}<c_{p_2}, \tag{3.3}$$

since the other cases can be treated similarly.

Now, we examine the zeros of $D(z)$. For a_1 (resp. a_2, b_1, b_2), we make a branch cut on the real axis of the z-plane between the point $c_{p_1}^2$ and ∞ (resp. $c_{p_2}^2$ and ∞, $c_{s_1}^2$ and ∞, $c_{s_2}^2$ and ∞) as in Figure 3. To determine the number of roots of $D(z)=0$, we make the path γ in the z-plane as in Figure 4. We let $A=c_{s_1}^2$, $B=c_{p_1}^2$, $C=c_{s_2}^2$, $D=c_{p_2}^2$, and take four points A', B', C', D' on the path γ to be near A, B, C, D, respectively. Moreover we take a real number R to be large enough, and a point R' on the path γ to be near R.

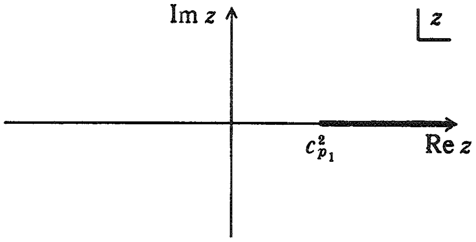

Figure 3. Branch cut between branch points $c^2_{p_1}$ and ∞ in the z-plane

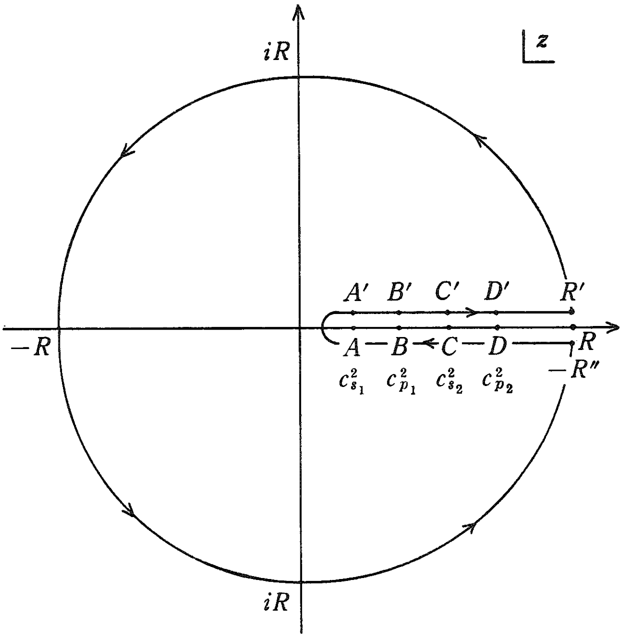

Figure 4. Path γ in the z plane

We discuss the image of $D(z)/z^3$ when z goes on the path above. When we change z to the complex conjugate of z, $D(z)$ changes to its complex conjugate; that is $D(\bar{z})=\overline{D(z)}$. Therefore we can only consider the contour of the upper half plane. $D(A)$ denotes the image of A by $D(z)$.

1. If z is on A', we may consider that the sign of $D(A')$ is the same as that of $D(A)$, so $b_1=0$, and a_1, a_2, $b_2<0$, then we have

$$D(A)=\operatorname{Re} D(A)=\left(\mu_1-2\mu_2+\frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2-a_2b_2(\mu_1-2\mu_2)^2-a_1b_2\mu_1\mu_2\frac{c_{s_1}^2}{c_{s_2}^2}\gtreqless 0\,.$$

2. If z goes from A' to B', we may examine that z goes from A to B, so b_1 is pure imaginary such that $\operatorname{Im} b_1>0$, and a_1, a_2, $b_2<0$, then $\operatorname{Re} D(z)$ and $\operatorname{Im} D(z)$ may take positive and negative values or zero. In fact, we have

$$\begin{aligned}\operatorname{Re} D(z)=&\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}+\frac{\mu_2 z}{c_{s_2}^2}\right)^2\\&-a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}\right)^2-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}a_1b_2z^2\gtreqless 0\,,\end{aligned}$$

$$\begin{aligned}&\operatorname{Im} D(z)\\&=\frac{1}{i}\left(4(\mu_1-\mu_2)^2a_1a_2b_1b_2-a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}\right)^2-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}a_2b_1z^2\right)\gtreqless 0\,.\end{aligned}$$

We shall show that there is no $z\in(c_{s_1}^2,\ c_{p_1}^2)$ such that

$$\operatorname{Re} D(z)=0 \quad\text{and}\quad \operatorname{Im} D(z)=0\,,$$

more precisely, we shall show that

$$\operatorname{Re} D(z)=0 \quad\text{and}\quad \operatorname{Im} D(z)=0 \Longrightarrow z>c_{s_2}^2+c_{p_2}^2\,.$$

For this purpose we consider $D(z)=D(a_1, a_2)$ as a function of a_1 and a_2 by regarding a_1 and a_2 as parameters. As in Figure 5, the equation $\operatorname{Im} D(a_1, a_2)$

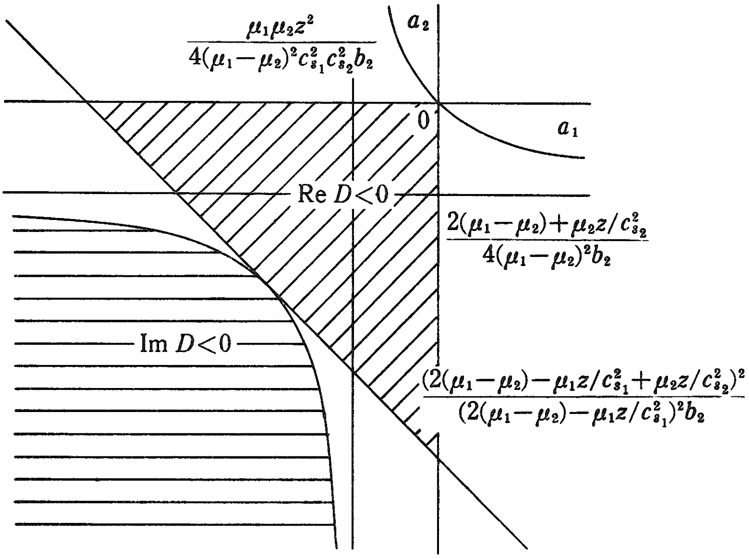

Figure 5. Behavior of $\operatorname{Re} D(z)$ and $\operatorname{Im} D(z)$ for variations of the parameters a_1 and a_2

$=0$ represents an equilateral hyperbola passing through the origin and with asymptotes parallel to the axes, intersecting in the third quadrant. And the equation $\operatorname{Re} D(a_1, a_2)=0$ represents a tangent to one of the branches of the hyperbola at the point (a_1, a_2) where

$$
\begin{aligned}
a_1&=\frac{2(\mu_1-\mu_2)-(\mu_1 z/c_{s_1}^2)+(\mu_2 z/c_{s_2}^2)}{2(\mu_1-\mu_2)b_2},\\
a_2&=\frac{2(\mu_1-\mu_2)-(\mu_1 z/c_{s_1}^2)+(\mu_2 z/c_{s_2}^2)}{2(\mu_1-\mu_2)b_2}\cdot\frac{2(\mu_1-\mu_2)+(\mu_2 z/c_{s_2}^2)}{2(\mu_1-\mu_2)-(\mu_1 z/c_{s_1}^2)}.
\end{aligned}
\tag{3.4}
$$

In fact, we have in the second of (3.4)

$$
a_2 b_2=\frac{2(\mu_1-\mu_2)-(\mu_1 z/c_{s_1}^2)+(\mu_2 z/c_{s_2}^2)}{2(\mu_1-\mu_2)-(\mu_1 z/c_{s_1}^2}\cdot\frac{2(\mu_1-\mu_2)+(\mu_2 z/c_{s_2}^2)}{2(\mu_1-\mu_2)}>1,
\tag{3.5}
$$

so

$$
a_2 b_2=\sqrt{1-\frac{z}{c_{p_2}^2}}\sqrt{1-\frac{z}{c_{s_2}^2}}>1,
$$

from this inequality and $z>0$, we get

$$
z>c_{s_2}^2+c_{p_2}^2.
$$

If we go back to the situation that a_1, a_2, b_1, and b_2 are functions of z, there is no solution $z\,(c_{s_1}^2<z<c_{p_1}^2)$ which satisfies (3.4). Since $\operatorname{Im} D(a_1, a_2)$ is negative in the lower region of the branch of the curve $\operatorname{Im} D(a_1, a_2)=0$ which does not pass the origin, and $\operatorname{Re} D(a_1, a_2)$ is negative in the upper region of the line $\operatorname{Re} D(a_1, a_2)=0$, $D(a_1(z), a_2(z))$ never goes into the third quadrant.

3. If z is on B', we may consider that the sign of $D(B')$ is the same as that of $D(B)$, so b_1 is pure imaginary such that $\operatorname{Im} b_1>0$, $a_1=0$, and a_2, $b_2<0$, then we have

$$
\operatorname{Re} D(B)=\Bigl(2(\mu_1-\mu_2)-\frac{\mu_1 c_{p_1}^2}{c_{s_1}^2}+\frac{\mu_2 c_{p_1}^2}{c_{s_2}^2}\Bigr)^2-a_2 b_2\Bigl(2(\mu_1-\mu_2)-\frac{\mu_1 c_{p_1}^2}{c_{s_1}^2}\Bigr)^2\gtreqless 0,
$$

$$
\operatorname{Im} D(B)=\frac{1}{i}\Bigl(-\frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2}a_2 b_1 c_{p_1}^4\Bigr)>0.
$$

4. If z goes from B' to C', we may examine that z goes from B to C, so a_1, b_1 are pure imaginary such that $\operatorname{Im} a_1$, $\operatorname{Im} b_1>0$ and a_2, $b_2<0$, then we have

$$
\begin{aligned}
\operatorname{Re} D(z)=&\Bigl(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}+\frac{\mu_2 z}{c_{s_2}^2}\Bigr)^2+4(\mu_1-\mu_2)^2 a_1 a_2 b_1 b_2\\
&-a_1 b_1\Bigl(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}\Bigr)^2-a_2 b_2\Bigl(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}\Bigr)^2\gtreqless 0,
\end{aligned}
$$

$$
\operatorname{Im} D(z)=\frac{1}{i}\Bigl(-\frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2}(a_1 b_2+a_2 b_1)z^2\Bigr)>0.
$$

5. If z is on C', we may consider that the sign of $D(C')$ is the same as that of $D(C)$, so a_1, b_1 are pure imaginary such that $\mathrm{Im}\, a_1$, $\mathrm{Im}\, b_1 > 0$, $b_2 = 0$ and $a_2 < 0$, then we have

$$\mathrm{Re}\, D(C) = \left(2\mu_1 - \frac{\mu_1 c_{s_2}^2}{c_{s_1}^2} - \mu_2\right)^2 - a_1 b_1 (2\mu_1 - \mu_2)^2 > 0,$$

$$\mathrm{Im}\, D(C) = \frac{1}{i}\left(-\frac{\mu_1\mu_2}{c_{s_1}^2} a_2 b_1 c_{s_2}^2\right) > 0.$$

6. If z goes from C' to D', we may examine that z goes from C to D, so a_1, b_1, b_2 are pure imaginary such that $\mathrm{Im}\, a_1$, $\mathrm{Im}\, b_1$, $\mathrm{Im}\, b_2 > 0$, and $a_2 < 0$, then we have

$$\mathrm{Re}\, D(z) = \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_1 b_1 \left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 > 0,$$

$$\mathrm{Im}\, D(z) = \frac{1}{i}\left(4(\mu_1 - \mu_2)^2 a_1 a_2 b_1 b_2 - a_2 b_2 \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 - \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2} a_2 b_1 z^2\right) > 0.$$

7. If z is on D', we may consider that the sign of $D(D')$ is the same as that of $D(D)$, so a_1, b_1, b_2 are pure imaginary such that $\mathrm{Im}\, a_1$, $\mathrm{Im}\, b_1$, $\mathrm{Im}\, b_2 > 0$, and $a_2 = 0$, then we have

$$\begin{aligned} D(D) = \mathrm{Re}\, D(D) &= \left(2(\mu_1 - \mu_2) - \frac{\mu_1 c_{p_2}^2}{c_{s_1}^2} + \frac{\mu_2 c_{p_2}^2}{c_{s_2}^2}\right)^2 \\ &\quad - a_1 b_1 \left(2(\mu_1 - \mu_2) + \frac{\mu_2 c_{p_2}^2}{c_{s_2}^2}\right)^2 - \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2} a_1 b_2 c_{p_2}^4 > 0. \end{aligned}$$

8. If z goes from D' to R', we may examine that z goes from D to R, so a_1, a_2, b_1, and b_2 are all pure imaginary such that $\mathrm{Im}\, a_1$, $\mathrm{Im}\, a_2$, $\mathrm{Im}\, b_1$, $\mathrm{Im}\, b_2 > 0$, then we have

$$\begin{aligned} D(z) &= \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1 - \mu_2)^2 a_1 a_2 b_1 b_2 \\ &\quad - a_1 b_1 \left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2 b_2 \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 \\ &\quad - \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2} (a_1 b_2 + a_2 b_1) z^2 > 0. \end{aligned}$$

9. If z goes from R' to $-R'$, along the contour which the radius R is very large, then we have

$$D(z) = \left(\frac{\mu_2^2}{c_{p_1} c_{s_1} c_{s_2}^4} + \frac{\mu_1^2}{c_{p_2} c_{s_2} c_{s_1}^4} + \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^3 c_{p_1}} + \frac{\mu_1\mu_2}{c_{s_1}^3 c_{s_2}^2 c_{p_2}}\right) z^3 + O(z^2),$$

and so

$$\frac{D(z)}{z^3} = \left(\frac{\mu_2^2}{c_{p_1} c_{s_1} c_{s_2}^4} + \frac{\mu_1^2}{c_{p_2} c_{s_2} c_{s_1}^4} + \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^3 c_{p_1}} + \frac{\mu_1\mu_2}{c_{s_1}^3 c_{s_2}^2 c_{p_2}}\right) + O\left(\frac{1}{z}\right), \quad z \to \infty,$$

where

$$\frac{\mu_2^2}{c_{p_1}c_{s_1}c_{s_2}^4}+\frac{\mu_1^2}{c_{p_2}c_{s_2}c_{s_1}^4}+\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^3c_{p_1}}+\frac{\mu_1\mu_2}{c_{s_1}^3c_{s_2}^2c_{p_2}}$$

is real positive.

As mentioned above, there are two qualitatively different cases to be considered; in the first case $D(A)$ is negative, and in the second case $D(A)$ is positive.

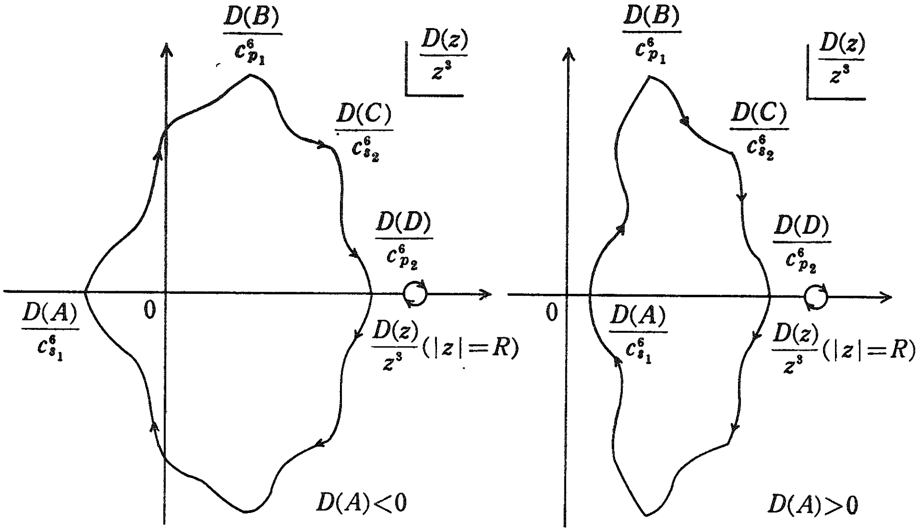

Figure 6. Path of the image point in the $D(z)/z^3$-plane

Now, we have the following asymptotic expansions for sufficiently small z:

$$a_1=\sqrt{1-\frac{z}{c_{p_1}^2}}=1-\frac{z}{2c_{p_1}^2}-\frac{z^2}{8c_{p_1}^4}+O(z^3), \tag{3.6}$$

$$a_2=\sqrt{1-\frac{z}{c_{p_2}^2}}=1-\frac{z}{2c_{p_2}^2}-\frac{z^2}{8c_{p_2}^4}+O(z^3), \tag{3.7}$$

$$b_1=\sqrt{1-\frac{z}{c_{s_1}^2}}=1-\frac{z}{2c_{s_1}^2}-\frac{z^2}{8c_{s_1}^4}+O(z^3), \tag{3.8}$$

$$b_2=\sqrt{1-\frac{z}{c_{s_2}^2}}=1-\frac{z}{2c_{s_2}^2}-\frac{z^2}{8c_{s_2}^4}+O(z^3). \tag{3.9}$$

By the asymptotic expansions (3.6)-(3.9), it follows that

$$D(z)=z^2\left[\frac{(\mu_1-\mu_2)^2}{c_{p_1}^2c_{p_2}^2}+\frac{\mu_1^2-\mu_2^2}{c_{p_1}^2c_{s_2}^2}-\frac{\mu_1^2-\mu_2^2}{c_{p_2}^2c_{s_1}^2}-\frac{(\mu_1+\mu_2)^2}{c_{s_1}^2c_{s_2}^2}+O(z)\right],\quad z\to+0. \tag{3.10}$$

Since the quantity in brackets in (3.10) is equal to

$$\mu_1^2\left(\frac{1}{c_{p_2}^2}+\frac{1}{c_{s_2}^2}\right)\left(\frac{1}{c_{p_1}^2}-\frac{1}{c_{s_1}^2}\right)+\mu_2^2\left(\frac{1}{c_{p_1}^2}+\frac{1}{c_{s_1}^2}\right)\left(\frac{1}{c_{p_2}^2}-\frac{1}{c_{s_2}^2}\right)$$
$$-2\mu_1\mu_2\left(\frac{1}{c_{p_1}^2c_{p_2}^2}+\frac{1}{c_{s_1}^2c_{s_2}^2}\right),$$

it is always negative and never zero. So the order of pole of $D(z)/z^3$ at zero is one. Therefore, from principle of the argument:

$$N-P=C(0),$$

it follows that if $D(A)$ is negative, then the zeros of $D(z)$ do not exist, and if $D(A)$ is positive, there exists only one real zero with order 1 of $D(z)$ on $[0,\ c_{s_1}^2]$. Indeed, there is no $z\in \boldsymbol{C}$ such that

$$\operatorname{Re} D(z)=0, \qquad \operatorname{Im} D(z)=0$$

outside the interval $[0,\ c_{s_1}^2]$. Here N, P denote, respectively, the number of zeros and the number of poles of $D(z)/z^3$ in the complex plane being countered with their proper multiplicities, and $C(0)$ the quotient by 2π of the variation of the argument of $D(z)/z^3$ when z described the closed path γ.

Hereafter we denote the real zero of $D(z)$ by c_{St}^2. Then the zero of $\Delta(\eta', \zeta)$ is $c_{St}^2|\eta'|^2$ and is the origin of the Stoneley wave propagating along the interface $x_3=0$ in the elastic space R^3, and c_{St} is its speed.

In conclusion, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta', \zeta)$ defined in (2.22) (the existence of the Stoneley waves) are given as follows: Let $D(z)$ be the polynomial of z defined by (3.2). Then $D(z)=\Delta(\eta', z|\eta'|^2)/|\eta'|^6$ is independent of $\eta'\neq 0$. For

$$D(c_{s_1}^2)=\left(\mu_1-2\mu_2+\frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2+\sqrt{\frac{c_{s_1}^2}{c_{p_2}^2}-1}\sqrt{\frac{c_{s_1}^2}{c_{s_2}^2}-1}(\mu_1-2\mu_2)^2$$
$$+\sqrt{\frac{c_{s_1}^2}{c_{p_1}^2}-1}\sqrt{\frac{c_{s_1}^2}{c_{s_2}^2}-1}\mu_1\mu_2\frac{c_{s_1}^2}{c_{s_2}^2}, \tag{3.11}$$

we obtain

(i) $D(c_{s_1}^2)>0 \Longrightarrow$ The zero $\zeta=c_{St}^2|\eta'|^2$ of $\Delta(\eta', \zeta)$ in ζ exists in $[0,\ c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of Theorem 6.5 that $c_{St}\neq 0$.

(ii) $D(c_{s_1}^2)=0 \Longrightarrow c_{St}=c_{s_1}$ and we shall consider this case under some restricted conditions (cf. Lemma 6.4).

(iii) $D(c_{s_1}^2)<0 \Longrightarrow \Delta(\eta', \zeta)$ has no zero.

REMARK. The minimum speed is either c_{s_1} or c_{s_2}, as is seen from the six cases mentioned above. If $c_{s_2}<c_{s_1}$, then we must replace $D(c_{s_1}^2)$ by $D(c_{s_2}^2)$.

§ 4. Generalized Eigenfunctions of $A_1(\eta')$

In this section, we give a family of generalized eigenfunctions for $A_1(\eta')$ by using the Green function $G_1(x_3, y_3, \eta'; \zeta)$ given in Section 2.

We define $\psi_{1j}(x_3, \eta; \zeta)$ $(j \in M = M_1 \cup M_2 = \{s_1, p_1, s_2, p_2\})$ as follows:

$$\psi_{1j}(x_3, \eta; \zeta) = \begin{cases} \psi^I_{1j}(x_3, \eta; \zeta), & x_3 < 0, \\ \psi^{II}_{1j}(x_3, \eta; \zeta), & x_3 > 0, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} &\psi^I_{1j}(x_3, \eta; \zeta) = F^{-1}_{y_3}[G^I_1(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta) - \zeta)P_j(\eta)\rho_1^{-1}, && x_3 < 0, \\ &\psi^{II}_{1j}(x_3, \eta; \zeta) = F^{-1}_{y_3}[G^{II}_1(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta) - \zeta)P_j(\eta)\rho_2^{-1}, && x_3 > 0, \end{aligned} \tag{4.2}$$

and

$$F^{-1}_{y_3}[G^m_1(x_3, y_3, \eta'; \zeta)] = \frac{1}{\sqrt{2\pi}} \int e^{iy_3\xi} G^m_1(x_3, y_3, \eta'; \zeta)\, dy_3 \quad (m = I, II).$$

Here $\lambda_j(\eta)$ are the eigenvalues of $A_1(\eta')$ given by (2.9) and (2.17), and $P_j(\eta)$ are the mutually orthogonal projections of $A_1(\eta')$ given by (2.12), (2.13) and (2.17).

The motivation for these particular definitions (4.1) and (4.2) is shown in Section 6 (Lemma 6.1) below.

LEMMA 4.1. *Let ζ be non-real. Then we have for $j \in M$*

$$(A^I_1(\eta') - \zeta I)\psi^I_{1j}(x_3, \eta; \zeta) = \frac{1}{\sqrt{2\pi}} e^{ix_3\xi}(\lambda_j(\eta) - \zeta)P_j(\eta), \tag{4.3}$$

$$(A^{II}_1(\eta') - \zeta I)\psi^{II}_{1j}(x_3, \eta; \zeta) = \frac{1}{\sqrt{2\pi}} e^{ix_3\xi}(\lambda_j(\eta) - \zeta)P_j(\eta), \tag{4.4}$$

$$\psi^I_{1j}(x_3, \eta; \zeta)|_{x_3=0} = \psi^{II}_{1j}(x_3, \eta; \zeta)|_{x_3=0}, \tag{4.5}$$

$$B^I_1(\eta')\psi^I_{1j}(x_3, \eta; \zeta)_{x_3=0} = B^{II}_1(\eta')\psi^{II}_{1j}(x_3, \eta; \zeta)|_{x_3=0}. \tag{4.6}$$

PROOF. Let $\phi \in C^\infty_0(\boldsymbol{R}_-, \boldsymbol{C}^2)$ and $\psi \in C^\infty_0(\boldsymbol{R}, \boldsymbol{C}^2)$. Then

$$\begin{aligned} &\langle (A^I_1(\eta') - \zeta I)F^{-1}_{y_3}[G^I_1(x_3, y_3, \eta'; \zeta)](\xi), \phi(x_3)\psi(\xi)\rangle_{x_3, \xi} \\ &= \langle F^{-1}_{x_3}[G^I_1(x_3, y_3, \eta'; \zeta)](\xi), (A^I_1(\eta') - \bar{\zeta} I)\phi(x_3)\psi(\xi)\rangle_{x_3, \xi} \\ &= \langle G^I_1(x_3, y_3, \eta'; \zeta), (A^I_1(\eta') - \bar{\zeta} I)\phi(x_3)F^{-1}_{\xi}[\psi](y_3)\rangle_{x_3, y_3} \\ &= \langle\langle G^I_1(x_3, y_3, \eta'; \zeta), (A^I_1(\eta') - \bar{\zeta} I)\phi(x_3)\rangle_{x_3}, F^{-1}_{\xi}[\psi](y_3)\rangle_{y_3} \\ &= \langle\langle (A^I_1(\eta') - \zeta I)G^I_1(x_3, y_3, \eta'; \zeta), \phi(x_3)\rangle_{x_3}, F^{-1}_{\xi}[\psi](y_3)\rangle_{y_3} \\ &= \langle\langle \delta(x_3 - y_3)I, \phi(x_3)\rangle_{x_3}, F^{-1}_{\xi}[\psi](y_3)\rangle_{y_3} \end{aligned}$$

$$
\begin{aligned}
&=\int_{R_-}\phi(x_3)F_\xi^{-1}[\psi](x_3)\rho_1^{-1}dx_3\\
&=\int_{R_-}\Big(\frac{1}{\sqrt{2\pi}}\int e^{ix_3\xi}\phi(x_3)\psi(\xi)d\xi\Big)dx_3\\
&=\Big\langle\frac{1}{\sqrt{2\pi}}e^{ix_3\xi}I,\ \phi(x_3)\psi(\xi)\Big\rangle_{x_3,\xi},
\end{aligned}
$$

where, for example,

$$\langle f,\ g\rangle_{x_3}=\int_{R_-}f\cdot g\rho_1 dx_3, \qquad f\cdot g=\sum_{i=1}^{3}f_i\bar{g}_i.$$

In view of (4.2), the last equality implies the formula (4.3).

The formula (4.4) can be proved in the same way. The interface conditions (4.5) and (4.6) are obvious because $G_1^I(x_3,\ y_3,\ \eta'\,;\ \zeta)$ and $G_1^{II}(x_3,\ y_3,\ \eta'\,;\ \zeta)$ satisfy the interface conditions. □

Now we shall introduce the generalized or improper eigenfunctions $\psi_{1j}(x_3,\ \eta)$ $(j\in M)$ of $A_1(\eta')$, making use of the expressions (2.23) and (2.24) of the Green functions G_1^I and G_2^{II}.

From the elementary formulas:

$$F_{y_3}^{-1}[H(-y_3)e^{-i\tau y_3}](\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{i(\xi-\tau)y_3}dy_3=\frac{1}{\sqrt{2\pi}}\frac{1}{i(\xi-\tau)},$$

$$F_{y_3}^{-1}[H(y_3)e^{i\tau y_3}](\xi)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{i(\xi+\tau)y_3}dy_3=-\frac{1}{\sqrt{2\pi}}\frac{1}{i(\xi+\tau)},$$

for $\tau\in C$, $\operatorname{Im}\tau\geqq 0$, it follows that

$$
\begin{aligned}
(4.7)\qquad & F_{y_3}^{-1}[G_1^I(x_3,\ y_3,\ \eta'\,;\ \zeta)](\xi)=\frac{1}{2\sqrt{2\pi}}\\
&\times\Big[-\Big\{\frac{\Delta_{p_1}^1(\eta',\ \zeta)}{\Delta(\eta',\ \zeta)}e^{-i\tau_{p_1}x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}(|\eta'|\tau_{p_1})\\
&\quad+\frac{\Delta_{s_1}^1(\eta',\ \zeta)}{\Delta(\eta',\ \zeta)}e^{-i\tau_{p_1}x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}(-\tau_{s_1}|\eta'|)\Big\}\\
&\quad+\Big\{e^{-i\tau_{p_1}x_3}\frac{1}{\xi+\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}(|\eta'|-\tau_{p_1})\\
&\quad+\frac{\Delta_{p_2}^1(\eta',\ \zeta)}{\Delta(\eta',\ \zeta)}e^{-i\tau_{p_1}x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}(|\eta'|-\tau_{p_2})\\
&\quad+\frac{\Delta_{s_2}^1(\eta',\ \zeta)}{\Delta(\eta',\ \zeta)}e^{-i\tau_{p_1}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}\\
&\quad-\Big\{\frac{\Delta_{p_1}^2(\eta',\ \zeta)}{\Delta(\eta',\ \zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}\tau_{s_1}\\ |\eta'|\end{pmatrix}(|\eta'|\tau_{p_1})
\end{aligned}
$$

$$+\frac{\Delta_{s_1}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(-\tau_{s_1}|\eta'|)\Big\}$$

$$+\Big\{e^{-i\tau_{s_1}x_3}\frac{1}{\xi+\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_1}|\eta'|)$$

$$+\frac{\Delta_{p_2}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(|\eta'|-\tau_{p_2})$$

$$+\frac{\Delta_{s_2}^2(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}$$

$$+\Big\{e^{i\xi x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_1}\end{pmatrix}(|\eta'|\tau_{p_1})$$

$$+e^{i\xi x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_1}\\|\eta'|\end{pmatrix}(-\tau_{s_1}|\eta'|)\Big\}$$

$$-\Big\{e^{i\xi x_3}\frac{1}{\xi+\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(|\eta'|-\tau_{p_1})$$

$$+e^{i\xi x_3}\frac{1}{\xi+\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_1}|\eta'|)\Big\}\Big],\qquad x_3<0,$$

$$(4.8)\qquad F_{y_3}^{-1}[G_1^{II}(x_3,\ y_3,\ \eta'\,;\ \zeta)](\xi)=\frac{1}{2\sqrt{2\pi}}$$

$$\times\Big[-\Big\{\frac{\Delta_{p_1}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\tau_{p_1})$$

$$+\frac{\Delta_{s_1}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(-\tau_{s_1}|\eta'|)$$

$$+e^{i\tau_{p_2}x_3}\frac{1}{\xi-\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\tau_{p_2})\Big\}$$

$$+\Big\{\frac{\Delta_{p_2}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|-\tau_{p_2})$$

$$+\frac{\Delta_{s_2}^3(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}$$

$$-\Big\{\frac{\Delta_{p_1}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}-\tau_{s_2}\\|\eta'|\end{pmatrix}(|\eta'|\tau_{p_1})$$

$$+\frac{\Delta_{s_1}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_2}\\|\eta'|\end{pmatrix}(-\tau_{s_1}|\eta'|)$$

$$+e^{i\tau_{s_2}x_3}\frac{1}{\xi-\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\|\eta'|\end{pmatrix}(-\tau_{s_2}|\eta'|)\Big\}$$

$$+\Big\{\frac{\Delta_{p_2}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\|\eta'|\end{pmatrix}(|\eta'|-\tau_{p_2})$$

$$+\frac{\Delta_{s_2}^4(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\|\eta'|\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}$$

$$
\begin{aligned}
&+\Big\{e^{i\xi x_3}\frac{1}{\xi-\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}(|\eta'|\tau_{p_2})\\
&+e^{i\xi x_3}\frac{1}{\xi-\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}(-\tau_{s_2}|\eta'|)\Big\}\\
&-\Big\{e^{i\xi x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_2}\end{pmatrix}(|\eta'|-\tau_{p_2})\\
&+e^{i\xi x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}\tau_{s_2}\\ |\eta'|\end{pmatrix}(\tau_{s_2}|\eta'|)\Big\}\Big], \qquad x_3>0.
\end{aligned}
$$

Let us take the limits of the expressions (4.1) for $\zeta\to\lambda_j(\eta)\pm i0$. First, we note the following formulas as $\zeta\to\lambda_{p_1}(\eta)\pm i0$:

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\tau_{p_1}=\lim_{\eta\to\lambda_{p_1}(\eta)\pm i0}\sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2}=\pm|\xi|,
$$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\tau_{s_1}=\xi_{s_1}(\eta',\lambda_{p_1})=\pm\sqrt{\frac{c_{p_1}^2(|\eta'|^2+\xi^2)}{c_{s_1}^2}-|\eta'|^2},
$$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\tau_{p_2}=\xi_{p_2}(\eta',\lambda_{p_1})=\begin{cases}\pm\sqrt{\dfrac{c_{p_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}-|\eta'|^2} & (c_{p_1}^2|\eta|^2>c_{p_2}^2|\eta'|^2)\\ i\sqrt{|\eta'|^2-\dfrac{c_{p_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}} & (c_{p_1}^2|\eta|^2<c_{p_2}^2|\eta'|^2),\end{cases}
$$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\tau_{s_2}=\xi_{s_2}(\eta',\lambda_{p_1})=\begin{cases}\pm\sqrt{\dfrac{c_{p_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}-|\eta'|^2} & (c_{p_1}^2|\eta|^2>c_{s_2}^2|\eta'|^2)\\ i\sqrt{|\eta'|^2-\dfrac{c_{p_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}} & (c_{p_1}^2|\eta|^2<c_{s_2}^2|\eta'|^2).\end{cases}
$$

Moreover, we have for $\xi>0$

$$
\begin{aligned}
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\frac{\lambda_{p_1}(\eta)-\zeta}{\xi\mp\tau_{p_1}(\eta',\zeta)}&=\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\frac{c_{p_1}^2(|\eta'|^2+\xi^2)-\zeta}{\xi\mp\sqrt{(\zeta/c_{p_1}^2)-|\eta'|^2}}\\
&=\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\frac{(c_{p_1}^2(|\eta'|^2+\xi^2)-\zeta)(\xi\pm\sqrt{(\zeta/c_{p_1}^2)-|\eta'|^2}}{\xi^2-((\zeta/c_{p_1}^2)-|\eta'|^2}\\
&=\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}c_{p_1}^2\left(\xi\pm\sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2}\right)\\
&=2c_{p_1}^2\xi,
\end{aligned}
$$

and also for $\xi<0$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\frac{\lambda_{p_1}(\eta)-\zeta}{\xi\pm\tau_{p_1}(\eta',\zeta)}=\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0}\frac{(c_{p_1}^2(|\eta'|^2+\xi^2)-\zeta)(\xi\mp\sqrt{(\zeta/c_{p_1})-|\eta'|^2}}{\xi^2-((\zeta/c_{p_1}^2)-|\eta'|^2)}
$$

$$= \lim_{\zeta \to \lambda_{p_1}(\eta) \pm i0} c_{p_1}^2 \Big(\xi \mp \sqrt{\frac{\zeta}{c_{p_1}^2} - |\eta'|^2}\Big)$$

$$= 2c_{p_1}^2 \xi .$$

If $\xi > 0$, then we have

$$\begin{cases} \psi_{1p_1}(x_3, \eta ; \zeta) \longrightarrow \psi^{+}_{1p_1}(x_3, \eta) & \text{as} \quad \zeta \longrightarrow \lambda_{p_1}(\eta) + i0 , \\ \psi_{1p_1}(x_3, \eta ; \zeta) \longrightarrow x^{-}_{1p_1}(x_3, \eta) & \text{as} \quad \zeta \longrightarrow \lambda_{p_1}(\eta) - i0 . \end{cases}$$

Here the limit functions $\psi^{\pm}_{1p_1}(x_3, \eta)$ are given respectively by the following:

$$\psi^{\pm}_{1p_1}(x_3, \eta) = \begin{cases} \psi^{\pm I}_{1p_1}(x_3, \eta), & x_3 < 0 , \\ \psi^{\pm II}_{1p_1}(x_3, \eta), & x_3 > 0 , \end{cases}$$

(4.9) $\psi^{+I}_{1p_1}(x_3, \eta)$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{|\eta|^2} \frac{1}{\rho_1} \Big[- \frac{\Delta^1_{p_1}(\eta', \lambda_{p_1})}{\Delta(\eta', \lambda_{p_1})} e^{-i\xi x_3} \begin{pmatrix} |\eta'|^2 & \xi |\eta'| \\ -\xi |\eta'| & -\xi^2 \end{pmatrix}$$

$$- \frac{\Delta^2_{p_1}(\eta', \lambda_{p_1})}{\Delta(\eta', \lambda_{p_1})} e^{-i\xi_{s_1}(\eta', \lambda_{p_1}) x_3} \begin{pmatrix} \xi_{s_1}(\eta', \lambda_{p_1}) |\eta'| & \xi_{s_1}(\eta', \lambda_{p_1}) \xi \\ |\eta'|^2 & \xi |\eta'| \end{pmatrix}$$

$$+ e^{i\xi x_3} \begin{pmatrix} |\eta'|^2 & \xi |\eta'| \\ \xi |\eta'| & \xi^2 \end{pmatrix} \Big],$$

(4.10) $\psi^{+II}_{1p_1}(x_3, \eta)$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{|\eta|^2} \frac{1}{\rho_2} \Big[- \frac{\Delta^3_{p_1}(\eta', \lambda_{p_1})}{\Delta(\eta', \lambda_{p_1})} e^{i\xi_{p_2}(\eta', \lambda_{p_1}) x_3} \begin{pmatrix} |\eta'|^2 & \xi |\eta'| \\ \xi_{p_2}(\eta', \lambda_{p_1}) |\eta'| & \xi_{p_2}(\eta', \lambda_{p_1}) \xi \end{pmatrix}$$

$$- \frac{\Delta^4_{p}(\eta', \lambda_{p_1})}{\Delta(\eta', \lambda_{p_1})} e^{i\xi_{s_2}(\eta', \lambda_{p_1}) x_3} \begin{pmatrix} -\xi_{s_2}(\eta', \lambda_{p_1}) |\eta'| & -\xi_{s_2}(\eta', \lambda_{p_1}) \xi \\ |\eta'|^2 & \xi |\eta'| \end{pmatrix} \Big],$$

(4.11) $\psi^{-I}_{1p_1}(x_3, \eta) = \psi^{-II}_{1p_1}(x_3, \eta) = O_{2\times 2}$,

where $O_{2\times 2}$ denotes the 2×2 zero matrix.

Next, we note the following formulas as $\zeta \to \lambda_j(\eta) \pm i0$ ($j \in \{s_1, p_2, s_2\}$).

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{p_1} = \xi_{p_1}(\eta', \lambda_{s_1}) = \begin{cases} \pm \sqrt{\dfrac{c_{s_1}^2(|\eta'|^2 + \xi^2)}{c_{p_1}^2} - |\eta'|^2} & (c_{s_1}^2 |\eta|^2 > c_{p_1}^2 |\eta'|^2) \\ i \sqrt{|\eta'|^2 - \dfrac{c_{s_1}^2(|\eta'|^2 + \xi^2)}{c_{p_1}^2}} & (c_{s_1}^2 |\eta|^2 < c_{p_1}^2 |\eta'|^2), \end{cases}$$

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{s_1} = \pm |\xi| ,$$

$$\lim_{\zeta\to\lambda_{s_1}(\eta)\pm i0}\tau_{p_2}=\xi_{p_2}(\eta',\lambda_{s_1})=\begin{cases}\pm\sqrt{\dfrac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}-|\eta'|^2} & (c_{s_1}^2|\eta|^2>c_{p_2}^2|\eta'|^2)\\ i\sqrt{|\eta'|^2-\dfrac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}} & (c_{s_1}^2|\eta|^2<c_{p_2}^2|\eta'|^2),\end{cases}$$

$$\lim_{\zeta\to\lambda_{s_1}(\eta)\pm i0}\tau_{s_2}=\xi_{s_2}(\eta',\lambda_{s_1})=\begin{cases}\pm\sqrt{\dfrac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}-|\eta'|^2} & (c_{s_1}^2|\eta|^2>c_{s_2}^2|\eta'|^2)\\ i\sqrt{|\eta'|^2-\dfrac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}} & (c_{s_1}^2|\eta^2|<c_{s_2}^2|\eta'|^2),\end{cases}$$

$$\lim_{\zeta\to\lambda_{p_2}(\eta)\pm i0}\tau_{p_1}=\xi_{p_1}(\eta',\lambda_{p_2})=\pm\sqrt{\frac{c_{p_2}^2(|\eta'|^2+\xi^2)}{c_{p_1}^2}-|\eta'|^2},$$

$$\lim_{\zeta\to\lambda_{p_2}(\eta)\pm i0}\tau_{s_1}=\xi_{s_1}(\eta',\lambda_{p_2})=\pm\sqrt{\frac{c_{p_2}^2(|\eta'|^2+\xi^2)}{c_{s_1}^2}-|\eta'|^2},$$

$$\lim_{\zeta\to\lambda_{p_2}(\eta)\pm i0}\tau_{p_2}=\pm|\xi|,$$

$$\lim_{\zeta\to\lambda_{p_2}(\eta)\pm i0}\tau_{s_2}=\xi_{s_2}(\eta',\lambda_{p_2})=\pm\sqrt{\frac{c_{p_2}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}-|\eta'|^2},$$

$$\lim_{\zeta\to\lambda_{s_2}(\eta)\pm i0}\tau_{p_1}=\xi_{p_1}(\eta',\lambda_{s_2})=\pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{p_1}^2}-|\eta'|^2},$$

$$\lim_{\zeta\to\lambda_{s_2}(\eta)\pm i0}\tau_{s_1}=\xi_{s_1}(\eta',\lambda_{s_2})=\pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^3)}{c_{s_1}^2}-|\eta'|^2},$$

$$\lim_{\zeta\to\lambda_{s_2}(\eta)\pm i0}\tau_{p_2}=\xi_{p_2}(\eta',\lambda_{s_2})=\begin{cases}+\sqrt{\dfrac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}-|\eta'|^2} & (c_{s_2}^2|\eta^2|>c_{p_2}^2|\eta'|^2)\\ i\sqrt{|\eta'|^2-\dfrac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}} & (c_{s_2}^2(|\eta^2|<c_{p_2}^2|\eta'|^2),\end{cases}$$

$$\lim_{\zeta\to\lambda_{s_2}(\eta)\pm i0}\tau_{s_2}=\pm|\xi|.$$

If $\xi>0$, then we have

$$\begin{cases}\phi_{1s_1}(x_3,\eta\,;\zeta)\to\phi_{1s_1}^+(x_3,\eta) & \text{as } \zeta\to\lambda_{s_1}(\eta)+i0,\\ \phi_{1s_1}(x_3,\eta\,;\zeta)\to\phi_{1s_1}^-(x_3,\eta) & \text{as } \zeta\to\lambda_{s_1}(\eta)-i0,\end{cases}$$

$$\begin{cases}\phi_{1p_2}(x_3,\eta\,;\zeta)\to\phi_{1p_2}^+(x_3,\eta) & \text{as } \zeta\to\lambda_{p_2}(\eta)+i0,\\ \phi_{1p_2}(x_3,\eta\,;\zeta)\to\phi_{1p_2}^-(x_3,\eta) & \text{as } \zeta\to\lambda_{p_2}(\eta)-i0,\end{cases}$$

$$\begin{cases}\phi_{1s_2}(x_3,\eta\,;\zeta)\to\phi_{1s_2}^+(x_3,\eta) & \text{as } \zeta\to\lambda_{s_2}(\eta)+i0,\\ \phi_{1s_2}(x_3,\eta\,;\zeta)\to\phi_{1s_2}^-(x_3,\eta) & \text{as } \zeta\to\lambda_{s_2}(\eta)-i0.\end{cases}$$

Here the limit functions $\phi_{1j}^\pm(x_3,\eta)$, $j\in\{s_1, p_2, s_2\}$, are given respectively by the following:

$$\psi_{1j}^{\pm}(x_3,\ \eta)=\begin{cases}\psi_{1j}^{\pm I}(x_3,\ \eta), & x_3<0,\\ \psi_{1j}^{\pm II}(x_3,\ \eta), & x_3>0, \qquad j\in\{s_1,\ p_2,\ s_2\},\end{cases}$$

(4.12)

$$\begin{aligned}&\psi_{1s_1}^{+I}(x_3,\ \eta)\\ &=\frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\Big[-\frac{\Delta_{s_1}^1(\eta',\ \lambda_{s_1})}{\Delta(\eta',\ \lambda_{s_1})}e^{-i\xi_{p_1}(\eta',\lambda_{s_1})x_3}\begin{pmatrix}-\xi|\eta'| & |\eta'|^2\\ \xi_{p_1}(\eta',\ \lambda_{s_1})\xi & -\xi_{p_1}(\eta',\ \lambda_{s_1})|\eta'|\end{pmatrix}\\ &\qquad-\frac{\Delta_{s_1}^2(\eta',\ \lambda_{s_1})}{\Delta(\eta',\ \lambda_{s_1})}e^{-i\xi x_3}\begin{pmatrix}-\xi^2 & \xi|\eta'|\\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\\ &\qquad+e^{i\xi x_3}\begin{pmatrix}\xi^2 & -\xi|\eta'|\\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\Big],\end{aligned}$$

(4.13)

$$\begin{aligned}&\psi_{1s_1}^{+II}(x_3,\ \eta)\\ &=\frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_2}\Big[-\frac{\Delta_{s_1}^3(\eta',\ \lambda_{s_1})}{\Delta(\eta',\ \lambda_{s_1})}e^{i\xi_{p_2}(\eta',\lambda_{s_1})x_3}\begin{pmatrix}-\xi|\eta'| & |\eta'|^2\\ -\xi_{p_2}(\eta',\ \lambda_{s_1})\xi & \xi_{p_2}(\eta',\ \lambda_{s_1})|\eta'|\end{pmatrix}\\ &\qquad-\frac{\Delta_{s_1}^4(\eta',\ \lambda_{s_1})}{\Delta(\eta',\ \lambda_{s_1})}e^{i\xi_{s_2}(\eta',\lambda_{s_1})x_3}\begin{pmatrix}\xi_{s_2}(\eta',\ \lambda_{s_1})\xi & -\xi_{s_2}(\eta',\ \lambda_{s_1})|\eta'|\\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\Big],\end{aligned}$$

(4.14) $\psi_{1p_2}^{+I}(x_3,\ \eta)=\psi_{1p_2}^{+II}(x_3,\ \eta)=O_{2\times 2}$,

(4.15) $\psi_{1s_2}^{+I}(x_3,\ \eta)=\psi_{1s_2}^{+II}(x_3,\ \eta)=O_{2\times 2}$,

(4.16) $\psi_{1s_1}^{-I}(x_3,\ \eta)=\psi_{1s_1}^{-II}(x_3,\ \eta)=O_{2\times 2}$,

(4.17)

$$\begin{aligned}&\psi_{1p_2}^{-I}(x_3,\ \eta)\\ &=\frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\Big[-\frac{\Delta_{p_2}^1(\eta',\ \lambda_{p_2})}{\Delta(\eta',\ \lambda_{p_2})}e^{-i\xi_{p_1}(\eta',\lambda_{p_2})x_3}\begin{pmatrix}|\eta'|^2 & \xi|\eta'|\\ -\xi_{p_1}(\eta',\ \lambda_{p_2})|\eta'| & -\xi_{p_1}(\eta',\ \lambda_{p_2})\xi\end{pmatrix}\\ &\qquad-\frac{\Delta_{p_2}^2(\eta',\ \lambda_{p_2})}{\Delta(\eta',\ \lambda_{p_2})}e^{-i\xi_{s_1}(\eta',\lambda_{p_2})x_3}\begin{pmatrix}\xi_{s_1}(\eta',\ \lambda_{p_2})|\eta'| & \xi_{s_1}(\eta',\ \lambda_{p_2})\xi\\ |\eta'| & \xi|\eta'|\end{pmatrix}\Big],\end{aligned}$$

(4.18)

$$\begin{aligned}&\psi_{1p_2}^{-II}(x_3,\ \eta)\\ &=\frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_2}\Big[-\frac{\Delta_{p_2}^3(\eta',\ \lambda_{p_2})}{\Delta(\eta',\ \lambda_{p_2})}e^{-i\xi x_3}\begin{pmatrix}|\eta'|^2 & \xi|\eta'|\\ -\xi|\eta'| & -\xi^2\end{pmatrix}\\ &\qquad-\frac{\Delta_{p_2}^4(\eta',\ \lambda_{p_2})}{\Delta(\eta',\ \lambda_{p_2})}e^{i\xi_{s_2}(\eta',\lambda_{p_2})x_3}\begin{pmatrix}-\xi_{s_2}(\eta',\ \lambda_{p_2})|\eta'| & -\xi_{s_2}(\eta',\ \lambda_{p_2})\xi\\ |\eta'|^2 & \xi|\eta'|\end{pmatrix}\\ &\qquad+e^{i\xi x_3}\begin{pmatrix}|\eta'|^2 & \xi|\eta'|\\ \xi|\eta'| & \xi^2\end{pmatrix}\Big],\end{aligned}$$

(4.19)

$$\psi^{-I}_{1s_2}(x_3,\ \eta)$$
$$=\frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\Big[-\frac{\Delta^1_{s_1}(\eta',\ \lambda_{s_2})}{\Delta(\eta',\ \lambda_{s_2})}e^{-i\xi_{p_1}(\eta',\lambda_{s_2})x_3}\begin{pmatrix}-\xi|\eta'| & |\eta'|^2\\ \xi_{p_1}(\eta',\ \lambda_{s_1})\xi & -\xi_{p_1}(\eta',\ \lambda_{s_2})|\eta'|\end{pmatrix}$$
$$-\frac{\Delta^2_{s_2}(\eta',\ \lambda_{s_2})}{\Delta(\eta',\ \lambda_{s_2})}e^{-i\xi_{s_1}(\eta',\lambda_{s_2})x_3}\begin{pmatrix}-\xi_{s_1}(\eta',\ \lambda_{s_2})\xi & \xi_{s_1}(\eta',\ \lambda_{s_2})|\eta'|\\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\Big],$$

(4.20)

$$\psi^{-II}_{1s_2}(x_3,\ \eta)$$
$$=\frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_2}\Big[-\frac{\Delta^3_{s_2}(\eta',\ \lambda_{s_2})}{\Delta(\eta',\ \lambda_{s_2})}e^{i\xi_{p_2}(\eta',\lambda_{s_2})x_3}\begin{pmatrix}-\xi|\eta'| & |\eta'|^2\\ -\xi_{p_2}(\eta',\ \lambda_{s_2})\xi & \xi_{p_2}(\eta',\ \lambda_{s_2})|\eta'|\end{pmatrix}$$
$$-\frac{\Delta^4_{s_2}(\eta',\ \lambda_{s_2})}{\Delta(\eta',\ \lambda_{s_2})}e^{-i\xi x_3}\begin{pmatrix}-\xi^2 & \xi|\eta'|\\ -|\eta'| & |\eta'|^2\end{pmatrix}$$
$$-e^{i\xi x_3}\begin{pmatrix}\xi^2 & -\xi|\eta'|\\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\Big].$$

If $\xi<0$, then the eigenfunctions $\{\psi^+_{1j}(x_3,\ \eta)\}_{j\in M}$ and $\{\psi^-_{1j}(x_3,\ \eta)\}_{j\in M}$ coincide with the eigenfunctions $\{\psi^-_{1j}(x_3,\ \eta)\}_{j\in M}$ and $\{\psi^+_{1j}(x_3,\ \eta)\}_{j\in M}$ in the case $\xi>0$, respectively.

From Lemma 4.1, we get for $j\in M$,

$$A^I_1(\eta')\psi^{\pm I}_{1j}(x_3,\ \eta)=\lambda_j(\eta)\psi^{\pm I}_{1j}(x_3,\ \eta),\qquad x_3<0,$$
$$A^{II}_1(\eta')\psi^{\pm II}_{1j}(x_3,\ \eta)=\lambda_j(\eta)\psi^{\pm II}_{1j}(x_3,\ \eta),\qquad x_3>0,$$
$$\psi^{\pm I}_{1j}(x_3,\ \eta)|_{x_3=0}=\psi^{\pm II}_{1j}(x_3,\ \eta)|_{x_3=0},$$
$$B^I_1(\eta')\psi^{\pm I}_{1j}(x_3,\ \eta)|_{x_3=0}=B^{II}_1(\eta')\psi^{\pm II}_{1j}(x_3,\ \eta)|_{x_3=0}.$$

This shows that $\psi^\pm_{1j}(x_3,\ \eta)$ $(j\in M)$ are generalized eigenfunctions for $A_1(\eta')$.

Next we define $\psi^{St}_{1j}(x_3,\ \eta\ ;\ \zeta)$ $(j\in M)$ as follows:

$$\psi^{St}_{1j}(x_3,\ \eta\ ;\ \zeta)=\begin{cases}\psi^{St\,I}_{1j}(x_3,\ \eta\ ;\ \zeta), & x_3<0,\\ \psi^{St\,II}_{1j}(x_3,\ \eta\ ;\ \zeta), & x_3>0,\end{cases}$$
$$\psi^{St\,I}_{1j}(x_3,\ \eta\ ;\ \zeta)=\frac{\zeta-c^2_{St}|\eta'|^2}{\zeta-\lambda_j(\eta)}\psi^I_{1j}(x_3,\ \eta\ ;\ \zeta),\qquad x_3<0,$$
$$\psi^{St\,II}_{1j}(x_3,\ \eta\ ;\ \zeta)=\frac{\zeta-c^2_{St}|\eta'|^2}{\zeta-\lambda_j(\eta)}\psi^{II}_{1j}(x_3,\ \eta\ ;\ \zeta),\qquad x_3>0.$$

From the expression of $\psi_{1j}(x_3,\ \eta\ ;\ \zeta)(j\in M)$, we see that the limits

$$\psi^{St\,I}_{1j}(x_3,\ \eta)=\lim_{\zeta\to c^2_{St}|\eta'|^2}\psi^{St\,I}_{1j}(x_3,\ \eta\ ;\ \zeta)$$
$$\psi^{St\,II}_{1j}(x_3,\ \eta)=\lim_{\zeta\to c^2_{St}|\eta'|^2}\psi^{St\,II}_{1j}(x_3,\ \eta\ ;\ \zeta)$$

exist. Moreover, by Lemma 4.1, $\psi_{1j}^{St\,I}(x_3,\ \eta)$ and $\psi_{1j}^{St\,II}(x_3,\ \eta)$ satisfy for $j\in M$

$$\begin{aligned}
&A_1^I(\eta')\psi_{1j}^{St\,I}(x_3,\ \eta)=c_{St}^2|\eta'|^2\psi_{1j}^{St\,I}(x_3,\ \eta), \qquad x_3<0,\\
&A_1^{II}(\eta')\psi_{1j}^{St\,II}(x_3,\ \eta)=c_{St}^2|\eta'|^2\psi_{1j}^{St\,II}(x_3,\ \eta), \qquad x_3>0,\\
&\psi_{1j}^{St\,I}(x_3,\ \eta)|_{x_3=0}=\psi_{1j}^{St\,II}(x_3,\ \eta)|_{x_3=0},\\
&B_1^I(\eta')\psi_{1j}^{St\,I}(x_3,\ \eta)|_{x_3=0}=B_1^{II}(\eta')\psi_{1j}^{St}(x_3,\ \eta)|_{x_3=0}.
\end{aligned}$$

This shows that $\psi_{1j}^{St}(x_3,\ \eta)\,(j\in M)$ are generalized eigenfunctions for $A_1(\eta')$ corresponding to the Stoneley wave. Let us give an expression of $\psi_{1j}^{St}(x_3,\ \eta)$ for each $j\in M$. If suffices to consider the case where the Lopatinski determinant $\Delta(\eta',\ \zeta)$ has real zero. If the Lopatinski determinant has no zero, we consider $\psi_{1j}^{St}(x_3,\ \eta)\equiv 0$ $(j\in M)$. Put

$$\Delta(\eta',\ \zeta)=(\zeta-c_{St}^2|\eta'|^2)\Delta_0(\eta',\ \zeta),$$

and noting that $c_{St}\leqq c_{s_1}<c_{p_1}<c_{s_2}<c_{p_2}$, we have

$$\begin{aligned}
&\lim_{\zeta\to c_{St}^2|\eta'|^2}\tau_{p_1}=\lim_{\zeta\to c_{St}^2|\eta'|^2}\sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2}=i\sqrt{|\eta'|^2-\frac{c_{St}^2}{c_{p_1}^2}|\eta'|^2}=i\xi_{p_1}^{St},\\
&\lim_{\zeta\to c_{St}^2|\eta'|^2}\tau_{s_1}=\lim_{\zeta\to c_{St}^2|\eta'|^2}\sqrt{\frac{\zeta}{c_{s_1}^2}-|\eta'|^2}=i\sqrt{|\eta'|^2-\frac{c_{St}^2}{c_{s_1}^2}|\eta'|^2}=i\xi_{s_1}^{St},\\
&\lim_{\zeta\to c_{St}^2|\zeta'|^2}\tau_{p_2}=\lim_{\zeta\to c_{St}^2|\eta'|^2}\sqrt{\frac{\zeta}{c_{p_2}^2}-|\eta'|^2}=i\sqrt{|\eta'|^2-\frac{c_{St}^2}{c_{p_2}^2}|\eta'|^2}=i\xi_{p_2}^{St},\\
&\lim_{\zeta\to c_{St}^2|\eta'|^2}\tau_{s_2}=\lim_{\zeta\to c_{St}^2|\eta'|^2}\sqrt{\frac{\zeta}{c_{s_2}^2}-|\eta'|^2}=i\sqrt{|\eta'|^2-\frac{c_{St}^2}{c_{s_2}^2}|\eta'|^2}=i\xi_{s_2}^{St},
\end{aligned}$$

where $\xi_{p_1}^{St}$, $\xi_{s_1}^{St}$, $\xi_{p_2}^{St}$, and $\xi_{s_2}^{St}$ are all real. So we have for $x_3<0$

(4.21)

$$\begin{aligned}
\psi_{1j}^{St\,I}(x_3,\ \eta)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{c_{St}^2|\eta'|^2}\frac{1}{\rho_1}\frac{1}{i}
&\times\Bigg[\frac{\Delta_{p_1}^1(\eta',\ c_{St}^2|\eta'|^2)}{\Delta_0(\eta',\ c_{St}^2|\eta'|^2)}e^{\xi_{p_1}^{St}x_3}\frac{1}{\xi_{p_1}^{St}}\frac{1}{\xi-i\xi_{p_1}^{St}}\begin{pmatrix}|\eta'|^2 & i\xi_{p_1}^{St}|\eta'|\\ -i\xi_{p_1}^{St}|\eta'| & \xi_{p_1}^{St2}\end{pmatrix}\\
&+\frac{\Delta_{s_1}^1(\eta',\ c_{St}^2|\eta'|^2)}{\Delta_0(\eta',\ c_{St}^2|\eta'|^2)}e^{\xi_{p_1}^{St}x_3}\frac{1}{\xi_{s_1}^{St}}\frac{1}{\xi-i\xi_{s_1}^{St}}\begin{pmatrix}-i\xi_{s_1}^{St}|\eta'| & |\eta'|^2\\ -\xi_{p_1}^{St}\xi_{s_1}^{St} & -i\xi_{p_1}^{St}|\eta'|\end{pmatrix}\\
&-\frac{\Delta_{p_2}^1(\eta',\ c_{St}^2|\eta'|^2)}{\Delta_0(\eta',\ c_{St}^2|\eta'|^2)}e^{\xi_{p_2}^{St}x_3}\frac{1}{\xi_{p_2}^{St}}\frac{1}{\xi-i\xi_{p_2}^{St}}\begin{pmatrix}|\eta'|^2 & -i\xi_{p_2}^{St}|\eta'|\\ -i\xi_{p_1}^{St}|\eta'| & -\xi_{p_1}^{St}\xi_{p_2}^{St}\end{pmatrix}\\
&-\frac{\Delta_{s_2}^1(\eta',\ c_{St}^2|\eta'|^2)}{\Delta_0(\eta',\ c_{St}^2|\eta'|^2)}e^{\xi_{p_1}^{St}x_3}\frac{1}{\xi_{s_2}^{St}}\frac{1}{\xi-i\xi_{s_2}^{St}}\begin{pmatrix}i\xi_{s_2}^{St}|\eta'| & |\eta'|^2\\ \xi_{p_1}^{St}\xi_{s_2}^{St} & -i\xi_{p_1}^{St}|\eta'|\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Delta^2_{p_1}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix} i\xi^{St}_{s_1}|\eta'| & -\xi^{St}_{s_1}\xi^{St}_{p_1} \\ |\eta'|^2 & i\xi^{St}_{p_1}|\eta'| \end{pmatrix} \\
&+\frac{\Delta^2_{s_1}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix} \xi^{St2}_{s_1} & i\xi^{St}_{s_1}|\eta'| \\ -i\xi^{St}_{s_1}|\eta'| & |\eta'|^2 \end{pmatrix} \\
&-\frac{\Delta^2_{p_2}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix} i\xi^{St}_{s_1}|\eta'| & \xi^{St}_{s_1}\xi^{St}_{p_2} \\ |\eta'|^2 & -i\xi^{St}_{p_2}|\eta'| \end{pmatrix} \\
&-\frac{\Delta^2_{s_2}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix} -\xi^{St}_{s_1}\xi^{St}_{s_2} & i\xi^{St}_{s_1}|\eta'| \\ i\xi^{St}_{s_2}|\eta'| & |\eta'|^2 \end{pmatrix}\Bigg]P_j(\eta),
\end{aligned}
$$

and for $x_3>0$

(4.22)

$$
\psi^{St\,II}_{1j}(x_3,\ \eta)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{c^2_{St}|\eta'|^2}\frac{1}{\rho_2}\frac{1}{i}
$$

$$
\begin{aligned}
&\times\Bigg[\frac{\Delta^3_{p_1}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix} |\eta'|^2 & i\xi^{St}_{p_1}|\eta'| \\ i\xi^{St}_{p_2}|\eta'| & -\xi^{St}_{p_1}\xi^{St}_{p_2} \end{pmatrix} \\
&+\frac{\Delta^3_{s_1}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix} -i\xi^{St}_{s_1}|\eta'| & |\eta'|^2 \\ \xi^{St}_{s_1}\xi^{St}_{p_2} & i\xi^{St}_{p_2}|\eta'| \end{pmatrix} \\
&-\frac{\Delta^3_{p_2}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix} |\eta'|^2 & i\xi^{St}_{p_2}|\eta'| \\ i\xi^{St}_{p_2}|\eta'| & -\xi^{St2}_{p_2} \end{pmatrix} \\
&-\frac{\Delta^3_{s_2}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix} i\xi^{St}_{s_2}|\eta'| & |\eta'|^2 \\ -\xi^{St}_{p_2}\xi^{St}_{s_2} & i\xi^{St}_{p_2}|\eta'| \end{pmatrix} \\
&+\frac{\Delta^4_{p_1}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix} -i\xi^{St}_{s_2}|\eta'| & \xi^{St}_{s_2}\xi^{St}_{p_1} \\ |\eta'|^2 & i\xi^{St}_{p_1}|\eta'| \end{pmatrix} \\
&+\frac{\Delta^4_{s_1}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix} -\xi^{St}_{s_1}\xi^{St}_{s_2} & -i\xi^{St}_{s_2}|\eta'| \\ -i\xi^{St}_{s_1}|\eta'| & |\eta'|^2 \end{pmatrix} \\
&-\frac{\Delta^4_{p_2}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix} -i\xi^{St}_{s_2}|\eta'| & -\xi^{St}_{s_2}\xi^{St}_{p_2} \\ |\eta'|^2 & -i\xi^{St}_{p_2}|\eta'| \end{pmatrix} \\
&-\frac{\Delta^4_{s_2}(\eta',\ c^2_{St}|\eta'|^2)}{\Delta_0(\eta',\ c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix} \xi^{St}_{s_2} & -i\xi^{St}_{s_2}|\eta'| \\ i\xi^{St}_{s_2}|\eta'| & |\eta'|^2 \end{pmatrix}\Bigg]P_j(\eta),
\end{aligned}
$$

where $j\in M=\{p_1,\ s_1,\ p_2,\ s_2\}$.

In conclusion, $\{\psi^{\pm}_{1j}(x_3,\ \eta)\}_{j\in M}$ are generalized eigenfunctions corresponding to the roots of the characteristic equation of $A_1(\eta')$. $\{\psi^{St}_{1j}(x_3,\ \eta)\}_{j\in M}$ are generalized eigenfunctions corresponding to the zero of the Lopatinski determinant of $A_1(\eta')$. (4.9)-(4.20) and (4.21)-(4.22) are explicit formulas of these generalized eigenfunctions for $A_1(\eta')$.

§5. Generalized Eigenfunctions of $A_2(\eta')$

In this section, we give an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta')-\zeta I(\zeta\notin \boldsymbol{R})$ by the same method as getting $G_1(x_3, y_3, \eta'; \zeta)$, in order to define generalized eigenfunctions for the operator $A_2(\eta')$.

Consider the following interface problem:

$$\text{(5.1)}\quad \begin{aligned}&(A_2^I(\eta', D)-\zeta)w^I(x_3, \eta'; \zeta)=f(\eta', x_3), \qquad x_3<0,\\ &(A_2^{II}(\eta', D)-\zeta)w^{II}(x_3, \eta'; \zeta)=f(\eta', x_3), \qquad x_3>0,\end{aligned}$$

$$\text{(5.2)}\quad w^I(x_3, \eta'; \zeta)|_{x_3=0}=w^{II}(x_3, \eta'; \zeta)|_{x_3=0},$$

$$\text{(5.3)}\quad B_2^I(\eta')w^I(x_3, \eta'; \zeta)|_{x_3=0}=B_2^{II}(\eta')w^{II}(x_3, \eta'; \zeta)|_{x_3=0},$$

where $f(\cdot, x_3)\in C_0^\infty(\boldsymbol{R}\backslash\{0\})$. Let us seek the solutions $w^I(x_3, \eta'; \zeta)$ and $w^{II}(x_3, \eta'; \zeta)$ in the form

$$\begin{aligned}&w^I(x_3, \eta'; \zeta)=E_2^I(x_3, \eta'; \zeta)-K_2^I(x_3, \eta'; \zeta),\\ &w^{II}(x_3, \eta'; \zeta)=E_2^{II}(x_3, \eta'; \zeta)-K_2^{II}(x_3, \eta'; \zeta).\end{aligned}$$

The expressions of E_2^I and E_2^{II} corresponding to (2.16) and (2.18) in Section 2 are given in the following form:

$$\begin{aligned}E_2^I(x_3, \eta'; \zeta)=\frac{i}{2}\Big(&\int_{-\infty}^{x_3}\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}f(\eta', y_3)dy_3\\ &+\int_{x_3}^{\infty}\frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}f(\eta', y_3)dy_3\Big), \qquad x_3<0,\\ E_2^{II}(x_3, \eta'; \zeta)=\frac{i}{2}\Big(&\int_{-\infty}^{x_3}\frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}f(\eta', y_3)dy_3\\ &+\int_{x_3}^{\infty}\frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}f(\eta', y_3)dy_3\Big), \qquad x_3>0,\end{aligned}$$

where

$$\tau_{s_1}=\sqrt{\frac{\zeta}{c_{s_1}^2}-|\eta'|^2}, \qquad \operatorname{Im}\tau_{s_1}\geqq 0, \qquad \tau_{s_2}=\sqrt{\frac{\zeta}{c_{s_2}^2}-|\eta'|^2}, \qquad \operatorname{Im}\tau_{s_2}\geqq 0.$$

On the other hand, put

$$\begin{aligned}&K_2^I(x_3, \eta'; \zeta)=\alpha e^{-i\tau_{s_1}x_3}, \qquad x_3<0,\\ &K_2^{II}(x_3, \eta'; \zeta)=\beta e^{i\tau_{s_2}x_3}, \qquad x_3>0,\end{aligned}$$

where α and β are determined so that w^I and w^{II} satisfy the interface conditions (5.2) and (5.3). Then the equations on α and β can be written in the matrix form as follows:

$$\begin{pmatrix} 1 & -1 \\ \rho_1 c_{s_1}^2 \tau_{s_1} & \rho_2 c_{s_2}^2 \tau_{s_2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{i}{2} \begin{pmatrix} h_1(\eta', \zeta) \\ h_2(\eta', \zeta) \end{pmatrix},$$

where

$$h_1(\eta', \zeta) = \int_{-\infty}^{0} \frac{e^{-i\tau_{s_1} y_3}}{c_{s_1}^2 \tau_{s_1}} f(\eta', y_3) dy_3 + \int_{0}^{\infty} \frac{e^{i\tau_{s_1} y_3}}{c_{s_1}^2 \tau_{s_1}} f(\eta', y_3) dy_3$$
$$- \int_{-\infty}^{0} \frac{e^{-i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3) dy_3 - \int_{0}^{\infty} \frac{e^{i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3) dy_3,$$

$$h_2(\eta', \zeta) = \rho_1 \Big(-\int_{-\infty}^{0} e^{-i\tau_{s_1} y_3} f(\eta', y_3) dy_3 + \int_{0}^{\infty} e^{i\tau_{s_1} y_3} f(\eta', y_3) dy_3 \Big)$$
$$+ \rho_2 \Big(\int_{-\infty}^{0} e^{-i\tau_{s_2} y_3} f(\eta', y_3) dy_3 - \int_{0}^{\infty} e^{i\tau_{s_2} y_3} f(\eta', y_3) dy_3 \Big).$$

The Lopatinski determinant for the problem (5.1), (5.2), and (5.3)

$$\Delta'(\eta', \zeta) = \rho_1 c_{s_1}^2 \tau_{s_1} + \rho_2 c_{s_2}^2 \tau_{s_2}$$

has no zero with respect to ζ for $|\eta'| \neq 0$. Therefore

(5.4) $w^I(x_3, \eta'; \zeta) = E_2^I(x_3, \eta'; \zeta) - K_2^I(x_3, \eta'; \zeta)$

$$= \frac{i}{2} \Big(\int_{-\infty}^{x_3} \frac{e^{i(x_3 - y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} f(\eta', y_3) dy_3 + \int_{x_3}^{\infty} \frac{e^{-i(x_3 - y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} f(\eta', y_3) dy_3 \Big)$$
$$- \frac{1}{\Delta'(\eta', \zeta)} \frac{i}{2} e^{-i\tau_{s_1} x_3}$$
$$\times \Big[\rho_2 c_{s_2}^2 \tau_{s_2} \Big(\int_{-\infty}^{0} \frac{e^{-i\tau_{s_1} y_3}}{c_{s_1}^2 \tau_{s_1}} f(\eta', y_3) dy_3 + \int_{0}^{\infty} \frac{e^{i\tau_{s_1} y_3}}{c_{s_1}^2 \tau_{s_1}} f(\eta', y_3) dy_3 \Big)$$
$$+ \rho_1 \Big(-\int_{-\infty}^{0} e^{-i\tau_{s_1} y_3} f(\eta', y_3) dy_3 + \int_{0}^{\infty} e^{i\tau_{s_1} y_3} f(\eta', y_3) dy_3 \Big)$$
$$- 2\rho_2 \int_{0}^{\infty} e^{i\tau_{s_2} y_3} f(\eta', y_3) dy_3 \Big], \qquad x_3 < 0,$$

(5.5) $w^{II}(x_3, \eta'; \zeta) = E_2^{II}(x_3, \eta'; \zeta) - K_2^{II}(x_3, \eta'; \zeta)$

$$= \frac{i}{2} \Big(\int_{-\infty}^{x_3} \frac{e^{i(x_3 - y_3)\tau_{s_2}}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3) dy_3 + \int_{x_3}^{\infty} \frac{e^{-i(x_3 - y_3)\tau_{s_2}}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3) dy_3 \Big)$$
$$- \frac{1}{\Delta'(\eta', \zeta)} \frac{i}{2} e^{i\tau_{s_2} x_3}$$
$$\times \Big[\rho_1 c_{s_1}^2 \tau_{s_1} \Big(\int_{-\infty}^{0} \frac{e^{-i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3) dy_3 + \int_{0}^{\infty} \frac{e^{i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3) dy_3 \Big)$$
$$+ \rho_2 \Big(\int_{-\infty}^{0} e^{-i\tau_{s_2} y_3} f(\eta', y_3) dy_3 - \int_{0}^{\infty} e^{i\tau_{s_2} y_3} f(\eta', y_3) dy_3 \Big)$$

$$-2\rho_1\int_{-\infty}^{0} e^{-i\tau_{s_1}y_3} f(\eta', y_3)dy_3\Big], \qquad x_3>0.$$

So, the Green function of $A_2(\eta')-\zeta$ is given by the following form:

$$G_2(x_3, y_3, \eta'; \zeta)=\begin{cases} G_2^I(x_3, y_3, \eta'; \zeta), & x_3<0, \\ G_2^{II}(x_3, y_3, \eta'; \zeta), & x_3>0, \end{cases}$$

where

$$\begin{aligned} &G_2^I(x_3, y_3, \eta'; \zeta) \\ &\quad =\frac{i}{2}\Big[H(x_3-y_3)\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}+H(y_3-x_3)\frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}} \\ &\quad -\frac{1}{\Delta'(\eta', \zeta)}e^{-i\tau_{s_1}x_3}\Big\{\rho_2 c_{s_2}^2\tau_{s_2}\Big(H(-y_3)\frac{e^{-i\tau_{s_1}y_3}}{c_{s_1}^2\tau_{s_1}}+H(y_3)\frac{e^{i\tau_{s_1}y_3}}{c_{s_1}^2\tau_{s_1}}\Big) \\ &\quad +\rho_1(-H(-y_3)e^{-i\tau_{s_1}y_3}+H(y_3)e^{i\tau_{s_1}y_3})-2\rho_2 H(y_3)e^{i\tau_{s_2}y_3}\Big\}\Big], \qquad x_3<0, \end{aligned}$$

$$\begin{aligned} &G_2^{II}(x_3, y_3, \eta'; \zeta) \\ &\quad =\frac{i}{2}\Big[H(x_3-y_3)\frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}+H(y_3-x_3)\frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}} \\ &\quad -\frac{1}{\Delta'(\eta', \zeta)}e^{i\tau_{s_2}x_3}\Big\{\rho_1 c_{s_1}^2\tau_{s_1}\Big(H(-y_3)\frac{e^{-i\tau_{s_2}y_3}}{c_{s_2}^2\tau_{s_2}}+H(y_3)\frac{e^{i\tau_{s_2}y_3}}{c_{s_2}^2\tau_{s_2}}\Big) \\ &\quad +\rho_2(H(-y_3)e^{-i\tau_{s_2}y_3}-H(y_3)e^{i\tau_{s_2}y_3})-2\rho_1 H(-y_3)e^{-i\tau_{s_1}y_3}\Big\}\Big], \qquad x_3>0. \end{aligned}$$

Now we define $\psi_{2k}(x_3, \eta; \zeta)\,(k\in N=\{s_1, s_2\})$ by

$$\psi_{2k}(x_3, \eta; \zeta)=\begin{cases} \psi_{2k}^I(x_3, \eta; \zeta), & x_3<0, \\ \psi_{2k}^{II}(x_3, \eta; \zeta), & x_3>0, \end{cases} \tag{5.6}$$

and

$$\begin{aligned} &\psi_{2k}^I(x_3, \eta; \zeta)=F_{y_3}^{-1}[G_2^I(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_k(\eta)-\zeta)\rho_1^{-1}, \qquad x_3<0, \\ &\psi_{2k}^{II}(x_3, \eta; \zeta)=F_{y_3}^{-1}[G_2^{II}(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_k(\eta)-\zeta)\rho_2^{-1}, \qquad x_3>0, \end{aligned} \tag{5.7}$$

where $\lambda_k(\eta)(k\in N=\{s_1, s_2\})$ are the eigenvalues of $A_2(\eta')$ which have concrete expressions $\lambda_{s_1}(\eta)\,(=c_{s_1}^2|\eta|^2)$, $\lambda_{s_2}(\eta)\,(=c_{s_2}^2|\eta|^2)$.

The motivation for these particular definitions (5.6) and (5.7) is shown in Section 6 (Lemma 6.1) below. Then we can see that the limit

$$\psi_{2k}^{\pm}(x_3, \eta)=\lim_{\zeta\to\lambda_k(\eta)\pm i0}\psi_{2k}(x_3, \eta; \zeta)$$

exist. Moreover applying Lemma 4.1 to $A_2(\eta')$, $\psi_{2k}^I(x_3, \eta)$ and $\psi_{2k}^{II}(x_3, \eta)$ satisfy the equations

$$A_2^I(\eta')\psi_{2k}^{\pm I}(x_3,\ \eta)=\lambda_k(\eta)\psi_{2k}^{\pm I}(x_3,\ \eta),\qquad x_3<0,$$

$$A_2^{II}(\eta')\psi_{2k}^{\pm II}(x_3,\ \eta)=\lambda_k(\eta)\psi_{2k}^{\pm II}(x_3,\ \eta),\qquad x_3>0,$$

$$\psi_{2k}^{\pm I}(x_3,\ \eta)|_{x_3=0}=\psi_{2k}^{\pm II}(x_3,\ \eta)|_{x_3=0},$$

$$B_2^I(\eta')\psi_{2k}^{\pm I}(x_3,\ \eta)|_{x_3=0}=B_2^{II}(\eta')\psi_{2k}^{\pm II}(x_3,\ \eta)|_{x_3=0},$$

where $k\in N=\{s_1,\ s_2\}$.

We note the following relations:

$$\lim_{\zeta\to\lambda_{s_1}(\eta)\pm i0}\tau_{s_1}=\pm|\xi|,$$

$$\lim_{\zeta\to\lambda_{s_1}(\eta)\pm i0}\tau_{s_2}=\xi_{s_2}(\eta',\ \lambda_{s_1})=\begin{cases}\pm\sqrt{\dfrac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}-|\eta'|^2} & (c_{s_1}^2|\eta|^2>c_{s_2}^2|\eta'|^2)\\ i\sqrt{|\eta'|^2-\dfrac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}} & (c_{s_1}^2|\eta|^2<c_{s_2}^2|\eta'|^2),\end{cases}$$

$$\lim_{\zeta\to\lambda_{s_2}(\eta)\pm i0}\tau_{s_1}=\xi_{s_1}(\eta',\ \lambda_{s_2})=\pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{s_1}^2}-|\eta'|^2},$$

$$\lim_{\zeta\to\lambda_{s_2}(\eta)\pm i0}\tau_{s_2}=\pm|\xi|.$$

We have for $\xi>0$

$$\lim_{\zeta\to\lambda_k(\eta)\pm i0}\frac{\lambda_k(\eta)-\zeta}{\xi\mp\tau_k(\eta',\ \zeta)}=2c_k^2\xi,$$

and also for $\xi<0$,

$$\lim_{\zeta\to\lambda_k(\eta)\pm i0}\frac{\lambda_k(\eta)-\zeta}{\xi\pm\tau_k(\eta',\ \zeta)}=2c_k^2\xi,$$

where $c_k(k\in N)$ are defined by

$$c_{s_1}^2=\frac{\mu_1}{\rho_1},\qquad c_{s_2}^2=\frac{\mu_2}{\rho_2},$$

and $\tau_k(\eta',\ \zeta)$ are defined by

$$\tau_{s_1}=\sqrt{\frac{\zeta}{c_{s_1}^2}-|\eta'|^2},\qquad \operatorname{Im}\tau_{s_1}\geqq 0,\qquad \tau_{s_2}=\sqrt{\frac{\zeta}{c_{s_2}^2}-|\eta'|^2},\qquad \operatorname{Im}\tau_{s_2}\geqq 0.$$

If $\xi>0$, then we have the following expressions:

$$(5.8)\quad \psi_{2s_1}^{+I}(x_3,\ \eta)=\frac{1}{\sqrt{2\pi}}\frac{1}{\rho_1}\left[e^{i\xi x_3}+\frac{\rho_1c_{s_1}^2\xi-\rho_2c_{s_2}^2\xi_{s_2}(\eta',\ \lambda_{s_1})}{\Delta'(\eta',\ \lambda_{s_1})}e^{-i\xi x_3}\right],$$

$$(5.9)\qquad \psi^{+II}_{2s_1}(x_3,\ \eta)=\frac{1}{\sqrt{2\pi}}\frac{1}{\rho_2}\frac{2\rho_1 c^2_{s_1}\xi}{\Delta'(\eta',\ \lambda_{s_1})}e^{i\xi_{s_2}(\eta',\lambda_{s_1})x_3},$$

$$(5.10)\qquad \psi^{+I}_{2s_2}(x_3,\ \eta)=\psi^{+II}_{2s_2}(x_3,\ \eta)=0,$$

$$(5.11)\qquad \psi^{-I}_{2s_1}(x_3,\ \eta)=\psi^{-II}_{2s_1}(x_3,\ \eta)=0,$$

$$(5.12)\qquad \psi^{-I}_{2s_2}(x_3,\ \eta)=-\frac{1}{\sqrt{2\pi}}\frac{1}{\rho_1}\frac{2\rho_2 c_{s2}^2\xi}{\Delta'(\eta',\ \lambda_{s_2})}e^{-i\xi_{s_1}(\eta',\lambda_{s_2})x_3},$$

$$(5.13)\qquad \psi^{-II}_{2s_2}(x_3,\ \eta)=\frac{1}{\sqrt{2\pi}}\frac{1}{\rho_2}\left[e^{i\xi x_3}-\frac{\rho_1 c^2_{s_1}\xi_{s_1}(\eta',\ \lambda_{s_2})+\rho_2 c^2_{s_2}\xi}{\Delta'(\eta',\ \lambda_{s_2})}e^{-i\xi x_3}\right].$$

If $\xi<0$, then the eigenfunctions $\{\psi^+_{2k}(x_3,\ \eta)\}_{k\in N}$ and $\{\psi^-_{2k}(x_3,\ \eta)\}_{k\in N}$ coincide with the eigenfunctions $\{\psi^-_{2k}(x_3,\ \eta)\}_{k\in N}$ and $\{\psi^+_{2k}(x_3,\ \eta)\}_{k\in N}$ in the case $\xi>0$, respectively.

In conclusion, $\{\psi^{\pm}_{2k}(x_3,\ \eta)\}_{k\in N}$ are generalized eigenfunctions corresponding to the roots of the characteristic equation of $A_2(\eta')$. (5.8)-(5.13) are the explicit formulas of these generalized eigenfunctions for $A_2(\eta')$. Since the Lopatinski determinant of $A_2(\eta')$ has no zero, we need not consider other eigenfunctions.

§6. Construction of the Spectral Family of A

In this section, we construct the spectral family of A by means of the generalized eigenfunctions of $A_1(\eta')$ and $A_2(\eta')$ defined in Section 4 and Section 5, respectively. Then we define the Fourier transforms of $f\in\mathcal{H}$ with respect to these generalized eigenfunctions of A and we prove the corresponding Parseval formula (Theorem 6.5). The key lemma is Lemma 6.4 below, which justifies to pass to the limit under the integral sign over $\boldsymbol{R}^3$.

Using $\psi_{1j}(x_3,\ \eta\ ;\ \zeta)\ (j\in M)$ defined in Section 4, we define $\psi_{1j}(x,\ \eta\ ;\ \zeta)(j\in M)$ by

$$(6.1)\qquad \psi_{1j}(x,\ \eta\ ;\ \zeta)=\begin{cases}\psi^I_{1j}(x,\ \eta\ ;\ \zeta), & x\in\boldsymbol{R}^3_-,\\ \psi^{II}_{1j}(x,\ \eta\ ;\ \zeta), & x\in\boldsymbol{R}^3_+,\end{cases}$$

where

$$\psi^I_{1j}(x,\ \eta\ ;\ \zeta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi^I_{1j}(x_3,\ \eta\ ;\ \zeta)\oplus O_{1\times 1}),$$

$$\psi^{II}_{1j}(x,\ \eta\ ;\ \zeta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi^{II}_{1j}(x_3,\ \eta\ ;\ \zeta)\oplus O_{1\times 1}).$$

Here $O_{n\times n}$ denotes the $n\times n$ zero matrix, and $\eta=(\eta', \xi)=(\eta_1, \eta_2, \xi)$. Using $\psi_{2k}(x_3, \eta\,;\,\zeta)$ $(k\in N)$ defined in Section 5, we define $\psi_{2k}(x, \eta\,;\,\zeta)$ $(k\in N)$ by

$$(6.2)\qquad \psi_{2k}(x, \eta\,;\,\zeta)=\begin{cases} \psi^{I}_{2k}(x, \eta\,;\,\zeta), & x\in \boldsymbol{R}^3_-, \\ \psi^{II}_{2k}(x, \eta\,;\,\zeta), & x\in \boldsymbol{R}^3_+, \end{cases}$$

where

$$\psi^{I}_{2k}(x, \eta\,;\,\zeta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi^{I}_{2k}(x_3, \eta\,;\,\zeta)),$$

$$\psi^{II}_{2k}(x, \eta\,;\,\zeta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi^{II}_{2k}(x_3, \eta\,;\,\zeta)).$$

Further we define for $f\in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{C}^3)$

$$(6.3)\qquad \hat{f}_{1j}(\eta\,;\,\zeta)=\int_{\boldsymbol{R}^3}\psi_{1j}(x, \eta\,;\,\zeta)^* f(x)\rho(x_3)dx, \qquad j\in M,$$

$$(6.4)\qquad \hat{f}_{2k}(\eta\,;\,\zeta)=\int_{\boldsymbol{R}^3}\psi_{2k}(x, \eta\,;\,\zeta)^* f(x)\rho(x_3)dx, \qquad k\in N.$$

Next lemma shows the motivation of definitions (4.1), (4.2), (5.6) and (5.7).

LEMMA 6.1. *Let* $f\in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{C}^3)$. *Then we have*

$$(6.5)\qquad (\mathrm{UC})^{-1}F_x[R(\bar{\zeta})f]=\sum_{j\in M}\frac{\hat{f}_{1j}(\eta\,;\,\zeta)}{\lambda_j(\eta)-\bar{\zeta}}+\sum_{k\in N}\frac{\hat{f}_{2k}(\eta\,;\,\zeta)}{\lambda_k(\eta)-\bar{\zeta}},$$

in the distribution sense.

PROOF. Let $\phi\in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{C}^3)$. We denote by $\langle\ ,\ \rangle_{\eta_1,\eta_2,\xi}$ the duality between $\mathcal{S}$ and $\mathcal{S}'$ where $\mathcal{S}$ is the space of rapidly decreasing C^∞ functions and $\mathcal{S}'$ the space of temperate distributions. F_x denotes the Fourier transformation with respect to $x=(x_1, x_2, x_3)$. Then from the Parseval equality

$$\begin{aligned}&\langle(\mathrm{UC})^{-1}(F_{x\to(\eta',\xi)}\phi)(\eta), (\mathrm{UC})^{-1}(F_{x\to(\eta',\xi)}[R(\bar{\zeta})f])(\eta)\rangle_{\eta_1,\eta_2,\xi}\\ &=\langle\phi(x), R(\bar{\zeta})f(x)\rangle_{x_1,x_2,x_3}\\ &=\langle R(\zeta)\phi(x), f(x)\rangle_{x_1,x_2,x_3}\\ &=\langle(A-\zeta)^{-1}\phi(x), f(x)\rangle_{x_1,x_2,x_3}\cdot(*)\end{aligned}$$

From (1.10), it follows that

$$\begin{aligned}(6.6)\quad &(A-\zeta)^{-1}\phi(x)\\ &=F^{-1}_{\eta'\to x'}[\mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'\to\eta'}\phi)(\eta', x_3)](x)\end{aligned}$$

$$=F^{-1}_{\eta'\to x'}[\mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'\to\eta'}\psi)(\eta',\ x_3)](x)$$

$$+F^{-1}_{\eta'\to x'}[\mathrm{UC}(O_{2\times2}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'\to\eta'}\psi)(\eta',\ x_3)](x),$$

so that from (4.1), (4.2), (5.6), (5.7), (6.1)-(6.4) and the Parseval equality

$$(*)=\langle F^{-1}_{\eta'}[\mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'}\psi)](x),\ f(x)\rangle_{x_1,x_2,x_3}$$

$$=\langle \mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',\ x_3),\ (F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3}$$

$$+\langle \mathrm{UC}(O_{2\times2}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',\ x_3),\ (F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3}$$

$$=\langle \mathrm{UC}(G_1(x_3,\ y_3,\ \eta'\,;\ \zeta)\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',\ y_3),$$

$$(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3,y_3}$$

$$+\langle \mathrm{UC}(O_{2\times2}\oplus G_2(x_3,\ y_3,\ \eta'\,;\ \zeta))(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',\ y_3),$$

$$(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3,y_3}$$

$$=\langle\langle \mathrm{UC}(G_1(x_3,\ y_3,\ \eta'\,;\ \zeta)\oplus O_{1\times1})(\mathrm{UC})^{-1},$$

$$(F_{x'}\psi)(\eta',\ y_3)\rangle_{y_3},\ (F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3}$$

$$+\langle\langle \mathrm{UC}(O_{2\times2}\oplus G_2(x_3,\ y_3,\ \eta'\,;\ \zeta))(\mathrm{UC})^{-1},$$

$$(F_{x'}\psi)(\eta',\ y_3)\rangle_{y_3},\ (F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3}$$

$$=\langle\langle \mathrm{UC}(F^{-1}_{y_3}[G_1(x_3,\ y_3,\ \eta'\,;\ \zeta)\oplus O_{1\times1}])(\xi)(\mathrm{UC})^{-1},$$

$$(F_{y_3}F_{x'}\psi)(\eta)\rangle_{\xi},\ (F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3}$$

$$+\langle\langle \mathrm{UC}(F^{-1}_{y_3}[O_{2\times2}\oplus G_2(x_3,\ y_3,\ \eta'\,;\ \zeta)])(\xi)(\mathrm{UC})^{-1},$$

$$(F_{y_3}F_{x'}\psi)(\eta)\rangle_{\xi},\ (F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,x_3}$$

$$=\langle \mathrm{UC}(F^{-1}_{y_3}[G_1(x_3,\ y_3,\ \eta'\,;\ \zeta)\oplus O_{1\times1}])(\xi)(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),$$

$$(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$+\langle \mathrm{UC}(F^{-1}_{y_3}[O_{2\times2}\oplus G_2(x_3,\ y_3,\ \eta'\,;\ \zeta)])(\xi)(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),$$

$$(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$=\sum_{j\in M}\langle \mathrm{UC}(F^{-1}_{y_3}[G_1(x_3,\ y_3,\ \eta'\,;\ \zeta)\oplus O_{1\times1}])(\xi)(\lambda_j(\eta)-\zeta)(P_j(\eta)\oplus O_{1\times1})\rho(x_3)^{-1}$$

$$\rho(x_3)(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\lambda_j(\eta)-\bar{\zeta})^{-1}(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$+\sum_{k\in N}\langle \mathrm{UC}(F^{-1}_{y_3}[O_{2\times2}\oplus G_2(x_3,\ y_3,\ \eta'\,;\ \zeta)])(\xi)(\lambda_k(\eta)-\zeta)\rho(x_3)^{-1}\rho(x_3)$$

$$(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\lambda_k(\eta)-\bar{\zeta})^{-1}(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$=\sum_{j\in M}\langle \mathrm{UC}(\psi_{1j}(x_3,\ \eta\,;\ \zeta)\oplus O_{1\times1})\rho(x_3)$$

$$
\begin{aligned}
&\qquad (\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\lambda_j(\eta)-\bar{\zeta})^{-1}(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}\\
&+\sum_{k\in N}\langle \mathrm{UC}(O_{2\times 2}\oplus\psi_{2k}(x_3,\ \eta\ ;\ \zeta))\rho(x_3)\\
&\qquad (\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\lambda_k(\eta)-\bar{\zeta})^{-1}(F_{x'}f)(\eta',\ x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}\\
&=\sum_{j\in M}\Big\langle(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\psi_{1j}(x_3,\ \eta\ ;\ \zeta)^*\oplus O_{1\times 1})\rho(x_3)(\mathrm{UC})^{-1}\\
&\qquad\times(\lambda_j(\eta)-\bar{\zeta})^{-1}\frac{1}{2\pi}\int e^{-i(x_1\eta_1+x_2\eta_2)}f(x)dx'\Big\rangle_{\eta_1,\eta_2,\xi,x_3}\\
&+\sum_{k\in M}\Big\langle(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (O_{1\times 1}\oplus\psi_{2k}(x_3,\ \eta\ ;\ \zeta)^*)\rho(x_3)(\mathrm{UC})^{-1}\\
&\qquad\times(\lambda_k(\eta)-\bar{\zeta})^{-1}\frac{1}{2\pi}\int e^{-i(x_1\eta_1+x_2\eta_2)}f(x)dx'\Big\rangle_{\eta_1,\eta_2,\xi,x_3}\\
&=\sum_{j\in M}\Big\langle(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\lambda_j(\eta)-\bar{\zeta})^{-1}\int\psi_{1j}(x,\ \eta\ ;\ \zeta)^*f(x)\rho(x_3)dx\Big\rangle_{\eta_1,\eta_2,\xi}\\
&+\sum_{k\in N}\Big\langle(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\ (\lambda_k(\eta)-\bar{\zeta})^{-1}\int\psi_{2k}(x,\ \eta\ ;\ \zeta)^*f(x)\rho(x_3)dx\Big\rangle_{\eta_1,\eta_2,\xi}\\
&=\Big\langle(\mathrm{UC})^{-1}(F_{y_3\to\xi}F_{x'\to\eta'}\psi)(\eta),\ \sum_{j\in M}\frac{\hat{f}_{1j}(\eta\ ;\ \zeta)}{(\lambda_j(\eta)-\bar{\zeta})}+\sum_{k\in N}\frac{\hat{f}_{2k}(\eta\ ;\ \zeta)}{(\lambda_k(\eta)-\bar{\zeta})}\Big\rangle_{\eta_1,\eta_2,\xi}\\
&=\Big\langle(\mathrm{UC})^{-1}(F_{x_3\to\xi}F_{x'\to\eta'}\psi)(\eta),\ \sum_{j\in M}\frac{\hat{f}_{1j}(\eta\ ;\ \zeta)}{(\lambda_j(\eta)-\bar{\zeta})}+\sum_{k\in N}\frac{\hat{f}_{2k}(\eta\ ;\ \zeta)}{(\lambda_k(\eta)-\bar{\zeta})}\Big\rangle_{\eta_1,\eta_2,\xi}.
\end{aligned}
$$

This completes the proof of (6.5). □

The self-adjoint operator A admits a uniquely determined spectral resolution:

$$A=\int_{-\infty}^{\infty}\lambda d\pi(\lambda)$$

where $\{\pi(\lambda)\}_{-\infty<\lambda<\infty}$ denotes the right-continuous spectral family of A. The representation of $\pi(\lambda)$ is based on the well-known theorem of Stone (see, e.g., [14]):

$$
\begin{aligned}
(6.7)\qquad &\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\\
&=s-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_a^b[R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)]d\lambda,\qquad a<b.
\end{aligned}
$$

From (6.5) and (6.7), we obtain the following.

LEMMA 6.2. *Let* $f\in C_0^\infty(\boldsymbol{R}^3,\ \boldsymbol{C}^3)$ *and* $0<a<b<\infty$. *Then we have*

$$(6.8)\qquad \Big(\Big(\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\Big)f,\ f\Big)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\Big(\sum_{j\in M}\int_a^b d\lambda\int_{\boldsymbol{R}^3}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\eta$$

$$+\sum_{k\in N}\int_a^b d\lambda\int_{\boldsymbol{R}^3}\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\eta\Big)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\Big(\sum_{j\in M}\int_{\boldsymbol{R}^3}d\eta\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda$$

$$+\sum_{k\in N}\int_{\boldsymbol{R}^3}d\eta\int_a^b\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda\Big).$$

PROOF. From (6.5) and the resolvent identity

$$R(\zeta)-R(\zeta')=(\zeta-\zeta')R(\zeta)R(\zeta')=(\zeta-\zeta')R(\zeta')R(\zeta),$$

we have

$$([R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)]f,\ f)$$

$$=(2i\varepsilon R(\lambda\pm i\varepsilon)R(\lambda\mp i\varepsilon)f,\ f)$$

$$=2i\varepsilon(R(\lambda\mp i\varepsilon)f,\ R(\lambda\mp i\varepsilon)f)$$

$$=2i\varepsilon((\mathrm{UC})^{-1}F[R(\lambda\mp i\varepsilon)f],\ (\mathrm{UC})^{-1}F[R(\lambda\mp i\varepsilon)f])$$

$$=2i\varepsilon\Big(\sum_{j\in M}\Big(\frac{\hat{f}_{1j}(\cdot\,;\,\lambda\pm i\varepsilon)}{\lambda_j(\eta)-(\lambda\mp i\varepsilon)},\ \frac{\hat{f}_{1j}(\cdot\,;\,\lambda\pm i\varepsilon)}{\lambda_j(\eta)-(\lambda\mp i\varepsilon)}\Big)$$

$$+\sum_{k\in N}\Big(\frac{\hat{f}_{2k}(\cdot\,;\,\lambda\pm i\varepsilon)}{\lambda_k(\eta)-(\lambda\mp i\varepsilon)},\ \frac{\hat{f}_{2k}(\cdot\,;\,\lambda\pm i\varepsilon)}{\lambda_k(\eta)-(\lambda\mp i\varepsilon)}\Big)\Big)$$

$$=2i\Big(\sum_{j\in M}\int_{\boldsymbol{R}^3}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\eta$$

$$+\sum_{k\in N}\int_{\boldsymbol{R}^3}\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\eta\Big),$$

and hence by (6.7) for any interval $(a,\ b)\subset\boldsymbol{R}_+$

$$\Big(\Big(\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\Big)f,\ f\Big)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_a^b([R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)]f,\ f)d\lambda$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^b d\lambda\Big(\sum_{j\in M}\int_{\boldsymbol{R}^3}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\eta$$

$$+\sum_{k\in N}\int_a^b d\lambda\int_{\boldsymbol{R}^3}\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\eta\Big)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\Big(\sum_{j\in M}\int_{\boldsymbol{R}^3}d\eta\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda$$

$$+\sum_{j\in M}\int_{R^3}d\eta\int_a^b\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta\,;\,\lambda\pm i\varepsilon)|^2d\lambda\Big),$$

since from (6.5) $\sum_{j\in M}(\hat{f}_{1j}(\eta\,;\,\zeta)/\lambda_j(\eta)-\bar{\zeta})+\sum_{k\in N}(\hat{f}_{2k}(\eta\,;\,\zeta)/\lambda_k(\eta)-\bar{\zeta})$ is a continuous L^2-valued function of ζ for $\mathrm{Im}\,\zeta\neq 0$. □

Using the generalized eigenfunctions $\psi^{\pm}_{1j}(x_3,\eta)$ (given by (4.9)-(4.20)) and $\psi^{St}_{1j}(x_3,\eta)$ (given by (4.21)-(4.22)) $(j\in M)$ for $A_1(\eta')$ and $\psi^{\pm}_{2k}(x_3,\eta)$ (given by (5.8)-(5.13)) $(k\in N)$ for $A_2(\eta')$, we define the generalized eigenfunctions $\psi^{\pm}_{1j}(x,\eta)$, $\psi^{St}_{1j}(x,\eta)\,(j\in M)$ and $\psi^{\pm}_{2k}(x,\eta)$ $(k\in N)$ for A by

$$\psi^{\pm}_{1j}(x,\eta)=\begin{cases}\psi^{\pm I}_{1j}(x,\eta), & x\in \boldsymbol{R}^3_-,\\ \psi^{\pm II}_{1j}(x,\eta), & x\in \boldsymbol{R}^3_+,\end{cases}\tag{6.9}$$

where

$$\psi^{\pm I}_{1j}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi^{\pm I}_{1j}(x_3,\eta)\oplus O_{1\times 1}),\qquad x\in \boldsymbol{R}^3_-,$$

$$\psi^{\pm II}_{1j}(x,\eta)=\frac{1}{2\pi}e^{i(x_1y_1+x_2\eta_2)}\mathrm{UC}(\psi^{\pm II}_{1j}(x_3,\eta)\oplus O_{1\times 1}),\qquad x\in \boldsymbol{R}^3_+,$$

$$\psi^{St}_{1j}(x,\eta)=\begin{cases}\psi^{St\,I}_{1j}(x,\eta), & x\in \boldsymbol{R}^3_-,\\ \psi^{St\,II}_{1j}(x,\eta), & x\in \boldsymbol{R}^3_+,\end{cases}\tag{6.10}$$

where

$$\psi^{St\,I}_{1j}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi^{St\,I}_{1j}(x_3,\eta)\oplus O_{1\times 1}),\qquad x\in \boldsymbol{R}^3_-,$$

$$\psi^{St\,II}_{1j}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi^{St\,II}_{1j}(x_3,\eta)\oplus O_{1\times 1}),\qquad x\in \boldsymbol{R}^3_+,$$

and

$$\psi^{\pm}_{2k}(x,\eta)=\begin{cases}\psi^{\pm I}_{2k}(x,\eta), & x\in \boldsymbol{R}^3_-,\\ \psi^{\pm II}_{2k}(x,\eta), & x\in \boldsymbol{R}^3_+,\end{cases}\tag{6.11}$$

where

$$\psi^{\pm I}_{2k}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi^{\pm I}_{2k}(x_3,\eta)),\qquad x\in \boldsymbol{R}^3_-,$$

$$\psi^{\pm II}_{2k}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi^{\pm II}_{2k}(x_3,\eta)),\qquad x\in \boldsymbol{R}^3_+.$$

Here we consider the case where the Stoneley wave exists, i.e., $D(c^2_{s_1})>0$ if $c_{s_1}<c_{s_2}$ and that $D(c^2_{s_2})>0$ if $c_{s_2}<c_{s_1}$ as shown in Section 3. Note that there is no term $\psi^{St}_{1j}(x,\eta)\,(j\in M)$, if Stoneley wave does not exist.

Then we easily have the following proposition.

PROPOSITION 6.3. *Let* $x\in \boldsymbol{R}^3$ *and* $\eta\in \boldsymbol{R}^3$ $(\eta\neq 0)$. *Then we have*:
(1) $\psi^{\pm}_{1j}(x,\eta)\,(j\in M)$ *belong to* $\mathcal{H}_{\mathrm{loc}}=L^2_{\mathrm{loc}}(\boldsymbol{R}^3, \boldsymbol{C}^3, \rho(x_3)dx)$ *and*

$$A\varphi_{1j}^{\pm}(x,\ \eta)=\lambda_j(\eta)\varphi_{1j}^{\pm}(x,\ \eta),$$

$$\varphi_{1j}^{\pm I}(x,\ \eta)|_{x_3=0}=\varphi_{1j}^{\pm II}(x,\ \eta)|_{x_3=0},$$

$$\sigma_{i3}(\varphi_{1j}^{\pm I}(x,\ \eta))|_{x_3=0}=\sigma_{i3}(\varphi_{1j}^{\pm II}(x,\ \eta))|_{x_3=0}.$$

(2) *$\varphi_{1j}^{St}(x,\ \eta)$ $(j\in M)$ belong to $\mathcal{H}_{\mathrm{loc}}$ and*

$$A\varphi_{1j}^{St}(x,\ \eta)=c_{St}^2|\eta'|^2\varphi_{1j}^{St}(x,\ \eta),$$

$$\varphi_{1j}^{St\,I}(x,\ \eta)|_{x_3=0}=\varphi_{1j}^{St\,II}(x,\ \eta)|_{x_3=0},$$

$$\sigma_{i3}(\varphi_{1j}^{St\,I}(x,\ \eta))|_{x_3=0}=\sigma_{i3}(\varphi_{1j}^{St\,II}(x,\ \eta))|_{x_3=0}.$$

(3) *$\varphi_{2k}^{\pm}(x,\ \eta)$ $(k\in N)$ belong to $\mathcal{H}_{\mathrm{loc}}$ and*

$$A\varphi_{2k}^{\pm}(x,\ \eta)=\lambda_k(\eta)\varphi_{2k}^{\pm}(x,\ \eta),$$

$$\varphi_{2k}^{\pm I}(x,\ \eta)|_{x_3=0}=\varphi_{2k}^{\pm II}(x,\ \eta)|_{x_3=0},$$

$$\sigma_{i3}(\varphi_{2k}^{\pm I}(x,\ \eta))|_{x_3=0}=\sigma_{i3}(\varphi_{2k}^{\pm II}(x,\ \eta))|_{x_3=0}.$$

Proposition 6.3 means that $\{\varphi_{1j}^{+}(x,\ \eta),\ \varphi_{1j}^{St}(x,\ \eta),\ \varphi_{2k}^{+}(x,\ \eta)\}_{j\in M,\,k\in N}$ and $\{\varphi_{1j}^{-}(x,\ \eta),\ \varphi_{1j}^{St}(x,\ \eta),\ \varphi_{2k}^{-}(x,\ \eta)\}_{j\in M,\,k\in N})$ are two families of generalized eigenfunctions for the operator A. One is a family of outgoing eigenfunctions, and the other is a family of incoming eigenfunctions. We shall show later the completeness of each family.

In order to obtain the desired representation of the spectral family, it remains to pass to the limit under the integral sign over $\boldsymbol{R}^3$ in (6.8) and evaluate the limit of the integral over $(a,\ b)$. For $f\in\mathcal{H}$, we define the Fourier components with respect to the generalized eigenfunctions $\varphi_{1j}^{\pm}(x,\ \eta)$, $\varphi_{1j}^{St}(x,\ \eta)(j\in M)$ and $\varphi_{2k}^{\pm}(x,\ \eta)(k\in N)$ for A by

$$\hat{f}_{1j}^{\pm}(\eta)=\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|x|\leq R}\varphi_{1j}^{\pm}(x,\ \eta)^{*}f(x)\rho(x_3)dx,\qquad j\in M,\tag{6.12}$$

$$\hat{f}_{1j}^{St}(\eta)=\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|x|\leq R}\varphi_{1j}^{St}(x,\ \eta)^{*}f(x)\rho(x_3)dx,\qquad j\in M,\tag{6.13}$$

$$\hat{f}_{2k}^{\pm}(\eta)=\underset{R\to\infty}{\mathrm{l.\,i.\,m.}}\int_{|x|\leq R}\varphi_{2k}^{\pm}(x,\ \eta)^{*}f(x)\rho(x_3)dx,\qquad k\in N.\tag{6.14}$$

Then the mapping $f\mapsto(\hat{f}_{1j}^{\pm},\ \hat{f}_{1j}^{St},\ \hat{f}_{2k}^{\pm})$ may be considered as the generalized Fourier transform of f.

The following lemma gives the representation of the spectral family of A by means of the generalized eigenfunctions of A.

LEMMA 6.4. *We assume that $D(c_{s_1}^2)>0$ if $c_{s_1}<c_{s_2}$ and that $D(c_{s_2}^2)>0$ if $c_{s_2}<$*

c_{s_1}. *Let* $f\in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{C}^3)$ *and* $0<a<b<\infty$. *Then we have*

$$\left(\left(\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\right)f,\ f\right) \tag{6.15}$$

$$=\sum_{j\in M}\left(\int_{a\leqq\lambda_j(\eta)\leqq b}|\hat{f}_{1j}^{\pm}(\eta)|^2d\eta+\int_{a\leqq c_{St}^2|\eta'|^2\leqq b}|\hat{f}_{1j}^{St}(\eta)|^2d\eta\right)$$

$$+\sum_{k\in N}\int_{a\leqq\lambda_k(\eta)\leqq b}|\hat{f}_{2k}^{\pm}(\eta)|^2d\eta\,.$$

REMARK. Under the following two conditions (i.e., Stoneley wave does not exist), the formula (6.15) holds without the second terms on the right hand side:

(i) If $c_{s_1}<c_{s_2}$, then either $D(c_{s_1}^2)<0$ or $D(c_{s_1}^2)=0$ and the expression (6.23) below does not vanish.

(ii) If $c_{s_2}<c_{s_1}$, then either $D(c_{s_2}^2)<0$ or $D(c_{s_2}^2)=0$ and the expression (6.23), exchanged with c_{s_1} and c_{s_2}, does not vanish.

PROOF. The essential part of the proof of this lemma is to justify the passage to the limit under the integral sign over $\boldsymbol{R}^3$ in (6.8).

First of all, from the definition of $\psi_{1j}(x, \eta\,;\,\zeta)\,(j\in M)$

$$\hat{f}_{1j}(\eta\,;\,\zeta)=\int_{\boldsymbol{R}^3}\psi_{1j}(x,\eta\,;\,\zeta)^*f(x)\rho(x_3)dx \tag{6.16}$$

$$=\frac{1}{2\pi}\int_{\boldsymbol{R}_-^3}e^{-i(x_1\eta_1+x_2\eta_2)}(\psi_{1j}^{I}(x_3,\eta\,;\,\zeta)^*\oplus O_{1\times1})(\mathrm{UC})^{-1}f(x)\rho_1dx$$

$$+\frac{1}{2\pi}\int_{\boldsymbol{R}_+^3}e^{-i(x_1\eta_1+x_2\eta_2)}(\psi_{1j}^{II}(x_3,\eta\,;\,\zeta)^*\oplus O_{1\times1})(\mathrm{UC})^{-1}f(x)\rho_2dx$$

$$=\int_{\boldsymbol{R}_-}(\psi_{1j}^{I}(x_3,\eta\,;\,\zeta)^*\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'}f)(\eta', x_3)\rho_1dx_3$$

$$+\int_{\boldsymbol{R}_+}(\psi_{1j}^{II}(x_3,\eta\,;\,\zeta)^*\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'}f)(\eta', x_3)\rho_2dx_3\,.$$

By the expressions (4.2), (4.7), and (4.8), the integrands in the last two integrals of the right-hand side of (6.16) are summable with respect to x_3. Furthermore, using the inequality $|\alpha||\beta|\leqq(|\alpha|^2+|\beta|^2)/2$, we have

$$|\hat{f}_{1j}(\eta\,;\,\zeta)|^2\leqq C\sum_{l=1}^{4}\sum_{m\in M}\left(\left|\frac{1}{\tau_m(\eta)\zeta}\frac{\lambda_m(\eta)-\zeta}{\xi-\tau_m(\eta)}\right|^2\right. \tag{6.17}$$

$$\left.+\left|\frac{\Delta_m^l(\eta',\zeta)}{\Delta(\eta',\zeta)}\frac{1}{\tau_m(\eta)\zeta}\frac{\lambda_m(\eta)-\zeta}{\xi-\tau_m(\eta)}\right|^2\right)|g(\eta')|^2,$$

where C is a positive constant and $g(\eta')$ is a rapidly decreasing function with respect to η'.

Next, we consider justifying the passage to the limit under the integral sign over $\boldsymbol{R}^3$. We separate the integral as follows:

$$\begin{aligned}&\int_{\boldsymbol{R}^3}\left(\int_a^b \frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda\right)d\eta\\ &=\int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}}\int_a^b d\lambda d\eta+\int_{\{|\eta|<3R\}\cap\{|\xi|<\delta\}}\int_a^b d\lambda d\eta+\int_{\{|\eta|>3R\}}\int_a^b d\lambda d\eta\\ &=I^1_{Rj}(\varepsilon)+I^2_{Rj}(\varepsilon)+I^3_{Rj}(\varepsilon).\end{aligned}$$

We divide the proof into three steps. Without loss of generality, we can assume that $c_{s_1}<c_{s_2}$, replacing c_{s_1} by c_{s_2} when $c_{s_2}<c_{s_1}$.

Step 1. First we consider $I^1_{Rj}(\varepsilon)$. Since

$$c^2_{St}|\eta'|^2<c^2_{s_1}(|\eta'|^2+\xi^2)=\lambda_{s_1}(\eta)$$

for $|\xi|>\delta$, the limits $\hat{f}_{1j}(\eta\,;\,\lambda\pm i0)$ exist when $\lambda=\lambda_j(\eta)$ and are continuous in η. So, for any ε such that $0<\varepsilon\leqq\varepsilon_0$

$$\begin{aligned}&\int_{\lambda_j(\eta)-\delta_1}^{\lambda_j(\eta)+\delta_1}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda\\ &\leqq c_1\int_{\lambda_j(\eta)-\delta_1}^{\lambda_j(\eta)+\delta_1}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}d\lambda\\ &\leqq c_1\int_{-(\delta_1/\varepsilon)}^{\delta_1/\varepsilon}\frac{1}{x^2+1}dx\\ &\leqq c_1\pi\,,\end{aligned}$$

where c_1 is a positive constant and independent of ε and η. Since

$$\begin{aligned}\hat{f}^{St}_{1j}(\eta\,;\,\zeta)&=\int_{\boldsymbol{R}^3}\psi^{St}_{1j}(x,\,\eta\,;\,\zeta)^* f(x)\rho(x_3)dx\\ &=\frac{\zeta-c^2_{St}|\eta'|^2}{\zeta-\lambda_j(\eta)}\hat{f}_{1j}(\eta\,;\,\zeta),\end{aligned}$$

the limits $\hat{f}^{St}_{1j}(\eta\,;\,\lambda\pm i0)$ exists when $\lambda=c^2_{St}|\eta'|^2$ and are continuous in η. So

$$\begin{aligned}&\int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda\\ &=\int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2}\frac{\varepsilon}{(c^2_{St}|\eta'|^2-\lambda)^2+\varepsilon^2}|\hat{f}^{St}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2 d\lambda\\ &\leqq c_2\int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2}\frac{\varepsilon}{(c^2_{St}|\eta'|^2-\lambda)^2+\varepsilon^2}d\lambda\\ &\leqq c_2\pi\,,\end{aligned}$$

where c_2 is a positive constant and independent of ε and η. It follows from the well-known formula of Cauchy (see, e.g., [15]) that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^b\frac{\varepsilon}{(\lambda-\mu)^2+\varepsilon^2}f(\lambda)d\lambda=\chi_{(a,b)}(\mu)f(\mu),$$

that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}\,|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2d\lambda\\
&\quad=\chi_{(a,b)}(\lambda_j(\eta))\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_j(\eta)-\delta_1}^{\lambda_j(\eta)+\delta_1}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}\,|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2d\lambda\\
&\qquad+\chi_{(a,b)}(c_{St}^2|\eta'|^2)\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{c_{St}^2|\eta'|^2-\delta_2}^{c_{St}^2|\eta'|^2+\delta_2}\frac{\varepsilon}{(c_{St}^2|\eta'|^2-\lambda)^2+\varepsilon^2}\,|\hat{f}_{1j}^{St}(\eta\,;\,\lambda\pm i\varepsilon)|^2d\lambda\\
&\quad=\chi_{(a,b)}(\lambda_j(\eta))|\hat{f}_{1j}(\eta\,;\,\lambda_j(\eta)\pm i0)|^2\\
&\qquad+\chi_{(a,b)}(\lambda_j(\eta))\lim_{\varepsilon\downarrow 0}\max_{\lambda\in(\lambda_j(\eta)-\delta_1,\,\lambda_j(\eta)+\delta_1)}|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)-\hat{f}_{1j}(\eta\,;\,\lambda\pm i0)|^2\\
&\qquad+\chi_{(a,b)}(c_{St}^2|\eta'|^2)|\hat{f}_{1j}^{St}(\eta\,;\,c_{St}^2|\eta'|^2\pm i0)|^2\\
&\qquad+\chi_{(a,b)}(c_{St}^2|\eta'|^2)\lim_{\varepsilon\downarrow 0}\max_{\lambda\in(c_{St}^2|\eta'|^2-\delta_2,\,c_{St}^2|\eta'|^2+\delta_2)}|\hat{f}_{1j}^{St}(\eta\,;\,\lambda\pm i\varepsilon)-\hat{f}_{1j}^{St}(\eta\,;\,\lambda\pm i0)|^2\\
&\quad=\chi_{(a,b)}(\lambda_j(\eta))|\hat{f}_{1j}^{\pm}(\eta)|^2+\chi_{(a,b)}(c_{St}^2|\eta'|^2)|\hat{f}_{1j}^{St}(\eta)|^2.
\end{aligned}$$

By the Lebesgue bounded convergence theorem, we have

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}I_{Rj}^1(\varepsilon)&=\int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}}\left(\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}\,|\hat{f}_{1j}(\eta\,;\,\lambda\pm i\varepsilon)|^2d\lambda\right)d\eta\\
&=\int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}\cap\{a\leqq\lambda_j(\eta)\leqq b\}}|\hat{f}_{1j}^{\pm}(\eta)|^2d\eta\\
&\quad+\int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}\cap\{a\leqq c_{St}^2|\eta'|^2\leqq b\}}|\hat{f}_{1j}^{St}(\eta)|^2d\eta.
\end{aligned}$$

Step 2. Next we consider the $I_{Rj}^2(\varepsilon)$. The principal difficulty in interchanging limit and integration occurs when the zeros of the denominators of $\varepsilon/|\zeta-\lambda_j(\eta)|^2$ and the zero of the Lopatinski determinant $\Delta(\eta',\zeta)$ nearly coincide, that is, when $c_{St}=c_{s_1}$. This is the case where $D(c_{s_1}^2)=0$.

1) Consider the first term of the right-hand side of (6.17). If sign ξ is the same as sign τ_{s_1}, then

$$\frac{1}{\tau_{s_1}(\xi-\tau_{s_1})}=\frac{2}{(\xi-\tau_{s_1})(\xi+\tau_{s_1})}+\frac{1}{\tau_{s_1}(\xi+\tau_{s_1})},$$

so we have

$$\frac{1}{|\tau_{s_1}|\,|\xi-\tau_{s_1}|}\leqq\frac{2}{|\lambda_{s_1}(\eta)-\zeta|}+\frac{1}{|\tau_{s_1}|^2},$$

hence

$$\lim_{\varepsilon\downarrow 0}\int_a^b \frac{\varepsilon}{|\zeta-\lambda_{s_1}(\eta)|^2}\left|\frac{1}{\tau_{s_1}\zeta}\,\frac{\lambda_{s_1}(\eta)-\zeta}{\xi-\tau_{s_1}}\right|^2 |g(\eta')|^2 d\lambda$$

$$\leqq c'\lim_{\varepsilon\downarrow 0}\left(\int_a^b \frac{1}{\lambda^2}\,\frac{\varepsilon}{(\lambda_{s_1}(\eta)-\lambda)^2+\varepsilon^2}\,|g(\eta')|^2 d\lambda\right.$$

$$\left.+\int_a^b \frac{1}{\lambda^2}\,\frac{\varepsilon}{(c_{s_1}^2|\eta'|^2-\lambda)^2+\varepsilon^2}\,|g(\eta')|^2 d\lambda\right)$$

$$\leqq c_3\pi\,.$$

If sign ξ is different from sign τ_{s_1}, then

$$\frac{1}{|\tau_1|\,|\xi-\tau_{s_1}|}\leqq\frac{1}{|\tau_{s_1}|^2}\,,$$

hence

$$\lim_{\varepsilon\downarrow 0}\int_a^b \frac{\varepsilon}{|\zeta-\lambda_{s_1}(\eta)|^2}\left|\frac{1}{\tau_{s_1}\zeta}\,\frac{\lambda_{s_1}(\eta)-\zeta}{\xi-\tau_{s_1}}\right|^2 |g(\eta')|^2 d\lambda\leqq c_4\pi\,.$$

Here c_3, c_4 are positive constants and independent of ε and η.

2) Consider the second term of the right-hand side of (6.17). By the change of variable $z=\zeta/|\eta'|^2$, we have

$$\frac{\Delta_{s_1}^l(\eta',\zeta)}{\Delta(\eta',\zeta)}=\frac{D_{s_1}^l(z)}{D(z)}\,,\qquad l=1,\ 2,\ 3,\ 4\,,$$

since $\Delta(\eta',\zeta)=|\eta'|^6D(z)$ and $\Delta_{s_1}^l(\eta',\zeta)=|\eta'|^6D_{s_1}^l(z)$, where

$$D_{s_1}^1(z)=2ib_1\left[2(\mu_1-\mu_2)a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}\right)\right.$$
$$\left.-\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s1}^2}+\frac{\mu_2z}{c_{2_2}^s}\right)\left(2(\mu_1-\mu_2)+\frac{\mu_2z}{c_{s_2}^2}\right)\right],\tag{6.18}$$

$$D_{s_1}^2(z)=\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}+\frac{\mu_2z}{c_{s_2}^2}\right)^2-4(\mu_1-\mu_2)^2a_1a_2b_1b_2$$
$$+a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2z}{c_{s_2}^2}\right)^2-a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}\right)^2$$
$$-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}(a_1b_2-a_2b_1)z^2\,,\tag{6.19}$$

$$D_{s_1}^3(z)=2ib_1\left[2a_1b_2\mu_1(\mu_2-\mu_1)\left(4-\frac{z}{c_{s_1}^2}\right)\right.$$
$$\left.-\frac{\mu_1z}{c_{s_1}^2}\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}+\frac{\mu_2z}{c_{s_2}^2}\right)\right],\tag{6.20}$$

$$D_{s_1}^4(z)=-2b_1\frac{\mu_1z}{c_{s_1}^2}(a_1+a_2)\left(2(\mu_1-\mu_2)-\frac{\mu_1z}{c_{s_1}^2}\right).\tag{6.21}$$

We have the following asymptotic formulas for a_1, b_1, a_2 and b_2 as functions of $t=\sqrt{z-c_{s_1}^2}$, respectively,

$$a_1=\frac{1}{c_{p_1}}\sqrt{-t^2+c_{p_1}^2-c_{s_1}^2}=\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\left(1-\frac{t^2}{2(c_{p_1}^2-c_{s_1}^2)}+O(t^4)\right) \quad (t\to 0),$$

$$b_1=\frac{1}{c_{s_1}}it \quad (t\to 0),$$

$$a_2=\frac{1}{c_{p_2}}\sqrt{-t^2+c_{p_2}^2-c_{s_1}^2}=\frac{\sqrt{c_{p_2}^2-c_{s_1}^2}}{c_{p_2}}\left(1-\frac{t^2}{2(c_{p_2}^2-c_{s_1}^2)}+O(t^4)\right) \quad (t\to 0),$$

$$b_2=\frac{1}{c_{s_2}}\sqrt{-t^2+c_{s_2}^2-c_{s_1}^2}=\frac{\sqrt{c_{s_2}^2-c_{s_1}^2}}{c_{s_2}}\left(1-\frac{t^2}{2(c_{s_2}^2-c_{s_1}^2)}+O(t^4)\right) \quad (t\to 0).$$

We consider $D(z)$ as a function of t. Then we obtain the expansion of $\tilde{D}(t)=D(z(t))$ in powers of t. By assumption $c_{St}=c_{s_1}$, we have that $D(c_{s_1}^2)$, defined by (3.11), is equal to zero. This means the constant term in $\tilde{D}(t)$ is equal to zero; that is,

$$(6.22)\qquad \left(\mu_1-2\mu_2+\frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2-\frac{\sqrt{c_{p_2}^2-c_{s_1}^2}}{c_{p_2}}\frac{\sqrt{c_{s_2}^2-c_{s_1}^2}}{c_{s_2}}(\mu_1-2\mu_2)^2 \\ -\mu_1\mu_2\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\frac{\sqrt{c_{s_2}^2-c_{s_1}^2}}{c_{s_2}}\frac{c_{s_1}^2}{c_{s_2}^2}=0.$$

The coefficient of t in $\tilde{D}(t)$ is given in the form

$$(6.23)\qquad 4(\mu_1-\mu_2)^2\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\frac{\sqrt{c_{p_2}^2-c_{s_1}^2}}{c_{p_2}}\frac{\sqrt{c_{s_2}^2-c_{s_1}^2}}{c_{s_2}} \\ -\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\left(2(\mu_1-\mu_2)+\frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2-\mu_1\mu_2\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\frac{c_{s_1}^2}{c_{s_2}^2}.$$

From now on, we shall consider only the case where this coefficient of t in $\tilde{D}_1(t)$ is not equal to 0.

On the other hand, we can see that for (6.18)-(6.21)

$$(6.24)\qquad \tilde{D}_{s_1}^l(t)=D_{s_1}^l(z(t))=\text{const.}\times t+O(t^2) \quad \text{as } t\to 0 \quad (l=1, 2, 3, 4),$$

hence the functions $D_{s_1}^l(z)/D(z)$ are bounded when z varies near $c_{s_1}^2$. Thus

$$\lim_{\varepsilon\downarrow 0}\int_a^b\frac{\varepsilon}{|\zeta-\lambda_{s_1}(\eta)|^2}\left|\frac{\Delta_{s_1}^l(\eta';\zeta)}{\Delta(\eta';\zeta)}\frac{1}{\tau_{s_1}\zeta}\frac{\lambda_{s_1}(\eta)-\zeta}{\xi-\tau_{s_1}}\right|^2|g(\eta')|^2d\lambda\leqq c_5\pi,$$

where c_5 is a positive constant and independent of ε and η. This means that in the case where $D(c_{s_1}^2)=0$, the Lopatinski determinant has no zeros under the condition that the expression (6.23) does not equal 0. Therefore, by the Lebesgue bounded convergence theorem, we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}I_{R_j}^2(\varepsilon)=\int_{\{|\eta|<3R\}\cap\{|\xi|<\delta\}}c|g(\eta')|^2d\eta$$

$$\leqq\int_{|\eta'|<3R} c\delta|g(\eta')|^2 d\eta'$$

$$\to 0 \qquad \text{as} \quad \delta\to 0.$$

Step 3. Finally we consider $I^3_{R_j}(\varepsilon)$. Divide the domain of integration as follows:

$$I^3_{R_j}(\varepsilon)=\int_{\{|\eta|>3R\}\cap\{|\eta'|>2R\}}\int_a^b d\lambda d\eta+\int_{\{|\eta|>3R\}\cap\{|\eta'|<2R\}}\int_a^b d\lambda d\eta. \tag{6.25}$$

1) Let us consider the first term of the right-hand side of (6.25). Take R such that $R>\max\{1/3c_j^2, 1/4c_{St}^2\}$. Since $R>(1/3c_j^2)$, we have

$$|\xi\pm\tau_j|\geqq|\xi\pm iR|=\sqrt{\xi^2+R^2},$$

and

$$\left|\frac{1}{\tau_j\zeta}\frac{1}{\xi-\tau_j}\right|^2\leqq\left|\frac{1}{Ra}\frac{1}{\sqrt{\xi^2+R^2}}\right|^2\leqq\frac{1}{R}\frac{k}{\xi^2+R^2}\quad\left(\frac{1}{Ra}\leqq k\right).$$

Since $g(\eta')$ is a rapidly decreasing function of η', it follows that

$$\int_{\{|\eta|>3R\}\cap\{|\eta'|>2R\}}\int_a^b\frac{\varepsilon}{|\zeta-\lambda_j(\eta)|^2}\left|\frac{1}{\tau_j\zeta}\frac{\lambda_j(\eta)-\zeta}{\xi-\tau_j}\right|^2|g(\eta')|^2 d\lambda d\eta$$

$$\leqq\int_{\{|\eta|>3R\}\cap\{|\eta'|>2R\}}\int_a^b\frac{\varepsilon k}{R(\xi^2+R^2)}|g(\eta')|^2 d\lambda d\eta$$

$$\leqq\varepsilon\int_{|\eta'|>2R}\left(\int_R\frac{k}{(\xi^2+R^2)}d\xi\right)|g(\eta')|^2 d\eta'$$

$$\leqq\varepsilon\frac{c_6}{R}\pi,$$

where c_6 is a positive constant and independent of ε and η. Since $R>(1/4c_{St}^2)$, the zero of $\Delta(\eta', \zeta)$ does not exist in $0<\lambda<R(\zeta=\lambda+i\varepsilon)$.

2) Consider the second term of the right-hand side of (6.25). Since $|\xi|>R$ and we have, taking $R>(3/c_j^2)$,

$$|\xi-\tau_j|^2\geqq\left|\frac{\xi^2-(\xi^2/2)}{\xi+\sqrt{(\lambda/c_j^2)-|\eta'|^2}}\right|^2\geqq\left|\frac{\xi^2/2}{\xi+\sqrt{R^2/2}}\right|^2\geqq c\xi^2,$$

it follows that

$$\int_{\{|\eta|>3R\}\cap\{|\eta'|<2R\}}\int_a^b\frac{\varepsilon}{|\zeta-\lambda_j(\eta)|^2}\left|\frac{1}{\tau_j\zeta}\frac{\lambda_j(\eta)-\zeta}{\xi-\tau_j}\right|^2|g(\eta')|^2 d\lambda d\eta$$

$$\leqq\int_{\{|\eta|>3R\}\cap\{|\eta'|<2R\}}\frac{c_j^2}{c\xi^2}\frac{1}{a^2}\left(\int_a^b\frac{\varepsilon}{\lambda-c_j^2|\eta'|^2}d\lambda\right)|g(\eta')|^2 d\eta$$

$$\leqq\int_{|\eta'|<2R}c'\varepsilon\frac{1}{R^2}\log R|g(\eta')|^2 d\eta'$$

$$\leqq\varepsilon\frac{c_7}{R},$$

where c_7 is a positive constant and independent of ε and η. The terms having $\Delta(\eta', \zeta)$ as the denominator may be estimated in the same way as in $I^2_{R_j}(\varepsilon)$.

Thus we have

$$\lim I^3_{Rj}(\varepsilon) \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty .$$

The second term in the right hand side of (6.15) can be handled in the same way. The proof of Lemma 6.4 is now complete. □

We can easily extend the equation (6.15) for all $f \in \mathcal{H}$ and obtain $\pi(a) = \pi(a-)$, $a \neq 0$.

THEOREM 6.5. *We assume that* $D(c^2_{s_1}) > 0$ *if* $c_{s_1} < c_{s_2}$ *and that* $D(c^2_{s_2}) > 0$ *if* $c_{s_2} < c_{s_1}$. *Let* $f, g \in \mathcal{H}$ *and* $0 < a < b < \infty$. *Then we have*

$$(6.26) \quad ((\pi(b) - \pi(a))f, g)$$
$$= \sum_{j \in M} \left(\int_{a \leq \lambda_j(\eta) \leq b} \hat{f}^{\pm}_{1j}(\eta) \cdot \hat{g}^{\pm}_{1j}(\eta) d\eta + \int_{a \leq c^2_{St} |\eta'|^2 \leq b} \hat{f}^{St}_{1j}(\eta) \cdot \hat{g}^{St}_{1j}(\eta) d\eta \right)$$
$$+ \sum_{k \in N} \int_{a \leq \lambda_k(\eta) \leq b} \hat{f}^{\pm}_{2k}(\eta) \cdot \hat{g}^{\pm}_{2k}(\eta) d\eta ,$$

and the Parseval formula

$$(6.27) \qquad (f, g) = \sum_{j \in M} \left(\int_{R^3} \hat{f}^{\pm}_{1j}(\eta) \cdot \hat{g}^{\pm}_{1j}(\eta) d\eta + \int_{R^3} \hat{f}^{St}_{1j}(\eta) \cdot \hat{g}^{St}_{1j}(\eta) d\eta \right)$$
$$+ \sum_{k \in N} \int_{R^3} \hat{f}^{\pm}_{2k}(\eta) \cdot \hat{g}^{\pm}_{2k}(\eta) d\eta .$$

RROOF. It suffices to prove that $0 \neq \sigma_p(A)$, where $\sigma_p(A)$ denotes the point spectrum of A. As to $A_1(\eta')$ the characteristic polynomials associated with $A^I_1(\eta', D)$ and $A^{II}_1(\eta', D)$ are $c^2_{p_1} c^2_{s_1} (\xi - |\eta'|)^2 (\xi + |\eta'|)^2$ and $c^2_{p_2} c^2_{s_2} (\xi - |\eta'|)^2 (\xi + |\eta'|)^2$, respectively, so we have

$$v^I = \begin{pmatrix} \alpha_1 \\ -i\alpha_1 \end{pmatrix} x_3 e^{|\eta'| x_3} + \begin{pmatrix} \alpha_2 \\ i \dfrac{c^2_{p_1} + c^2_{s_1}}{(c^2_{p_1} - c^2_{s_1})|\eta'|} \alpha_1 - i\alpha_2 \end{pmatrix} e^{|\eta'| x_3}, \qquad x_3 < 0 ,$$

$$v^{II} = \begin{pmatrix} \alpha_3 \\ i\alpha_3 \end{pmatrix} x_3 e^{-|\eta'| x_3} + \begin{pmatrix} \alpha_4 \\ i \dfrac{c^2_{p_2} + c^2_{s_2}}{(c^2_{p_2} - c^2_{s_2})|\eta'|} \alpha_3 + i\alpha_4 \end{pmatrix} e^{-|\eta'| x_3}, \qquad x_3 > 0 .$$

Since u^I and u^{II} should satisfy the interface conditions, we have

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ \dfrac{c^2_{p_1} + c^2_{s_1}}{(c^2_{p_1} - c^2_{s_1})|\eta'|} & -1 & -\dfrac{c^2_{p_2} + c^2_{s_2}}{(c^2_{p_2} - c^2_{s_2})|\eta'|} & -1 \\ \dfrac{\rho_1 c^4_{s_1}}{c^2_{p_1} - c^2_{s_1}} & -\rho_1 c^2_{s_1} |\eta'| & -\dfrac{\rho_2 c^4_{s_2}}{c^2_{p_2} - c^2_{s_2}} & -\rho_2 c^2_{s_2} |\eta'| \\ \dfrac{\rho_1 c^2_{s_1} c^2_{p_1}}{c^2_{p_1} - c^2_{s_1}} & \rho_1 c^2_{s_1} |\eta'| & -\dfrac{\rho_2 c^2_{s_2} c^2_{p_2}}{c^2_{p_2} - c^2_{s_2}} & -\rho_2 c^2_{s_2} |\eta'| \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0 .$$

The determinant of the 4×4 matrix above

$$\Delta=(\rho_1 c_{s_1}^2)^2(c_{p_2}^2+c_{s_2}^2)+(\rho_2 c_{s_2}^2)^2(c_{p_1}^2+c_{s_1}^2)(c_{s_2}^2-c_{p_2}^2)$$
$$-2\rho_1 c_{s_1}^2 \rho_2 c_{s_2}^2 (c_{p_1}^2 c_{p_2}^2+c_{s_1}^2 c_{s_2}^2)$$

is negative, so $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ have only the trivial solution (0, 0, 0, 0). As to $A_2(\eta')$, $0\neq\sigma_p(A_2(\eta'))$, because the Lopatinski determinant for the problem (5.1), (5.2) and (5.3) has no zero with respect to ζ for $|\eta'|\neq 0$ as showed in Section 5.

§7. Eigenfunction Expansions for A

In this section, we prove the eigenfunction expansion theorem for A. To this end, we use the representation of the spectral family of A developed in Section 6. Throughout this section, we assume that $D(c_{s_1}^2)>0$ if $c_{s_1}<c_{s_2}$ and that $D(c_{s_2}^2)>0$ if $c_{s_2}<c_{s_1}$. Note that, under the following two conditions (i.e., Stoneley wave does not exist), the theorems in this section hold without the terms corresponding to the Stoneley waves:

(i) If $c_{s_1}<c_{s_2}$, then either $D(c_{s_1}^2)<0$ or $D(c_{s_1}^2)=0$ and the expression (6.23) does not vanish.

(ii) If $c_{s_1}<c_{s_1}$, then either $D(c_{s_2}^2)<0$ or $D(c_{s_2}^2)=0$ and the expression (6.23), exchanged with c_{s_1} and c_{s_2}, does not vanish.

Let us begin with definition of mappings needed to formulate and prove the expansion theorem.

LEMMA 7.1. *We define the mappings by*

$$\Phi_{1j}^{\pm}:\quad \mathcal{H}\ni f \longrightarrow \hat{f}_{1j}^{\pm}(\eta)\in L^2(\boldsymbol{R}_+^3,\ \boldsymbol{C}^3)(\xi>0)\ \in L^2(\boldsymbol{R}_-^3,\ \boldsymbol{C})(\xi<0), \qquad j\in M,$$

$$\Phi_{1j}^{St}:\quad \mathcal{H}\ni f \longrightarrow \hat{f}_{1j}^{St}(\eta)\in L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3), \qquad j\in M,$$

$$\Phi_{2k}^{\pm}:\quad \mathcal{H}\ni f \longrightarrow \hat{f}_{2k}^{\pm}(\eta)\in L^2(\boldsymbol{R}_+^3,\ \boldsymbol{C})(\xi>0)\ \in L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)(\xi<0), \qquad k\in N,$$

and put for $f\in H$

$$\Phi^{\pm} f=\Big(\sum_{j\in M}\Phi_{1j}^{\pm} f,\ \sum_{j\in M}\Phi_{1j}^{St} f,\ \sum_{k\in N}\Phi_{2k}^{\pm} f\Big).$$

Then there exist a family of operators $Q_j^{\pm}(\eta)$, $Q_j^{St}(\eta)$, $Q_k^{\pm}(\eta)$, *and we have*

$$Q_j^{\pm}(\eta)\Phi_{1j}^{\pm}=\Phi_{1j}^{\pm}, \qquad j\in M, \tag{7.1}$$

$$\Phi_{1j}^{\pm *}\Phi_{1l}^{\pm}=0, \qquad if \quad j\neq l, \tag{7.2}$$

$$Q_j^{St}(\eta)\Phi_{1j}^{St}=\Phi_{1j}^{St}, \qquad j\in M, \tag{7.3}$$

$$\Phi_{1j}^{St *}\Phi_{1l}^{St}=0, \qquad if \quad j\neq l, \tag{7.4}$$

$$Q_k^{\pm}(\eta)\Phi_{2k}^{\pm}=\Phi_{2k}^{\pm}, \qquad k\in N, \tag{7.5}$$

$$\Phi_{2k}^{\pm *}\Phi_{3i}^{\pm}=0, \qquad \textit{if} \quad k\neq i, \tag{7.6}$$

where $\sum_{j\in M}Q_j^{\pm}(\eta)=\sum_{j\in M}Q_j^{St}(\eta)=\sum_{k\in N}Q_k^{\pm}(\eta)=I.$

Moreover $\Phi^{\pm}$ *is an isometry; that is,*

$$\Phi^{\pm *}\Phi^{\pm}=I_{\mathscr{H}}, \qquad \Phi^{\pm}\Phi^{\pm *}=P^{\pm}, \tag{7.7}$$

where $P^{\pm}$ *is the orthogonal projection in* $L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)\oplus L^2(\boldsymbol{R}^3, \boldsymbol{C}^3)\oplus L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)$.

PROOF. The formulas (7.1)-(7.6) follow immediately from the definition of $\Phi_{1j}^{\pm}$, Φ_{1j}^{St} and $\Phi_{2k}^{\pm}$, while the formula (7.7) follows from Theorem 6.5. □

The first half of next theorem expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

THEOREM 7.2. *Let* $f\in\mathscr{H}$.

(1) *The following expansion formula holds:*

$$\begin{aligned} f(x)=&\sum_{j\in M}\underset{R\to\infty}{\text{l. i. m.}}\int_{|\eta|\leq R}(\psi_{1j}^{\pm}(x, \eta)\hat{f}_{1j}^{\pm}(\eta)+\psi_{1j}^{St}(x, \eta)\hat{f}_{1j}^{St}(\eta))d\eta \\ &+\sum_{k\in N}\underset{R\to\infty}{\text{l. i. m.}}\int_{|\eta|\leq R}\psi_{2k}^{\pm}(x, \eta)\hat{f}_{2k}^{\pm}(\eta)d\eta . \end{aligned} \tag{7.8}$$

(2) $f\in D(A)$ *if and only if we have*

$$\lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \in Q_j^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3), \qquad j\in M,$$

$$c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta)\in Q_j^{St}(\eta)L^2(\boldsymbol{R}^3, \boldsymbol{C}^3), \qquad j\in M,$$

$$\lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta)\in Q_k^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3), \qquad k\in N.$$

Moreover, in this case, we have the following formulas:

$$\begin{aligned} Af(x)=&\sum_{j\in M}\underset{R\to\infty}{\text{l. i. m.}}\int_{|\eta|\leq R}(\lambda_j(\eta)\psi_{1j}^{\pm}(x, \eta)\hat{f}_{1j}^{\pm}(\eta)+c_{St}^2|\eta'|^2\psi_{1j}^{St}(x, \eta)\hat{f}_{1j}^{St}(\eta))d\eta \\ &+\sum_{k\in N}\underset{R\to\infty}{\text{l. i. m.}}\int_{|\eta|\leq R}\lambda_k(\eta)\psi_{2k}^{\pm}(x, \eta)\hat{f}_{2k}^{\pm}(\eta)d\eta , \end{aligned} \tag{7.9}$$

and

$$\widehat{(Af)}_{1j}^{\pm}(\eta)=\lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \qquad j\in M, \tag{7.10}$$

$$\widehat{(Af)}_{1j}^{St}(\eta)=c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta), \qquad j\in M, \tag{7.11}$$

$$\widehat{(Af)}_{2k}^{\pm}(\eta)=\lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta), \qquad k\in N. \tag{7.12}$$

PROOF. Let $f\in\mathscr{H}$ have a compact support, and $g\in C_0^{\infty}(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)$. We have

$$\begin{aligned}(f,\ \Phi_{1j}^{\pm *}g)_{\mathcal{H}}&=(\Phi_{1j}^{\pm}f,\ g)_{L^2(\boldsymbol{R}_{\pm}^3,\boldsymbol{C}^3)}\\&=(\psi_{1j}^{\pm *}(x,\ \eta)f(x),\ g(\eta))_{\mathcal{H}\times L^2(\boldsymbol{R}_{\pm}^3,\boldsymbol{C}^3)}\\&=(f(x),\ \psi_{1j}^{\pm}(x,\ \eta)g(\eta))_{\mathcal{H}\times L^2(\boldsymbol{R}_{\pm}^3,\boldsymbol{C}^3)}\\&=\int_{\boldsymbol{R}^3}f(x)\cdot\left(\int_{\boldsymbol{R}_{\pm}^3}\psi_{1j}^{\pm}(x,\ \eta)g(\eta)d\eta\right)\rho(x_3)dx,\end{aligned}$$

and so

$$\Phi_{1j}^{\pm *}g(x)=\int_{\boldsymbol{R}_{\pm}^3}\psi_{1j}^{\pm}(x,\ \eta)g(\eta)d\eta. \tag{7.13}$$

By virtue of the boundedness of $\Phi_{1j}^{\pm}$ and $\Phi_{1j}^{\pm *}$, we find that (7.13) holds for all $g\in L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3)$, where the integrals are taken in the sense of the limit in the mean.

Similarly, we can verify that

$$\Phi_{1j}^{St *}g(x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\leq R}\psi_{1j}^{St}(x,\ \eta)g(\eta)d\eta,\qquad \text{for}\quad g\in L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3),$$

$$\Phi_{2k}^{\pm *}g(x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\leq R}\psi_{2k}^{\pm}(x,\ \eta)g(\eta)d\eta,\qquad \text{for}\quad g\in L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3).$$

Thus (7.8) follows from (7.2), (7.4), (7.6), and (7.7).

Next we prove the diagonal representation of A. From Theorem 6.5, we have

$$\begin{aligned}(\pi(\lambda)f,\ g)=&\sum_{j\in M}\left(\int_{\lambda_j(\eta)\leq\lambda}\hat{f}_{1j}^{\pm}(\eta)\cdot\hat{g}_{1j}^{\pm}(\eta)d\eta+\int_{c_{St}^2|\eta'|^2\leq\lambda}\hat{f}_{1j}^{St}(\eta)\cdot\hat{g}_{1j}^{St}(\eta)d\eta\right)\\&+\sum_{k\in N}\int_{\lambda_k(\eta)\leq\lambda}\hat{f}_{2k}^{\pm}(\eta)\cdot\hat{g}_{2k}^{\pm}(\eta)d\eta\end{aligned}$$

for $f,\ g\in\mathcal{H}$. It is well known that $f\in D(A)$ if and only if

$$\int_{-\infty}^{\infty}\lambda^2d(\pi(\lambda)f,\ f)<\infty,$$

(e.g., [4]). Thus it is easy to see that $f\in D(A)$ if and only if

$$\hat{f}_{1j}^{\pm}(\eta),\ \lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta)\in Q_j^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3),\qquad j\in M,$$

$$\hat{f}_{1j}^{St}(\eta),\ c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta)\in Q_j^{St}(\eta)L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3),\qquad j\in M,$$

$$\hat{f}_{2k}^{\pm}(\eta),\ \lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta)\in Q_k^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3),\qquad k\in N.$$

Let $\alpha_r(x)$ be a C^∞ real valued function such that $\alpha_r(x)=1$ for $|x|<r$, $=0$ for $|x|>r+1$. For $f\in D(A)$,

$$(\widehat{Af})_{1j}^{\pm}(\eta)=\underset{r\to\infty}{\mathrm{l.i.m.}}\int_{\boldsymbol{R}^3}\psi_{1j}^{\pm}(x,\ \eta)^*\alpha_r(x)(Af)(x)\rho(x_3)dx$$

$$
\begin{aligned}
&=\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\boldsymbol{R}^3_-}[A^I(\alpha_r(x)\psi^{\pm I}_{1j}(x,\ \eta))]^*f(x)\rho_1dx\\
&\quad+\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\boldsymbol{R}^3_+}[A^{II}(\alpha_r(x)\psi^{\pm II}_{1j}(x,\ \eta))]^*f(x)\rho_2dx\\
&=\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\boldsymbol{R}^3_-}(A^I\psi^{\pm I}_{1j}(x,\ \eta))^*\alpha_r(x)f(x)\rho_1dx\\
&\quad+\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\{x\in\boldsymbol{R}^3_-,\ r\leqq|x|\leqq r+1\}}[(A^I\alpha_r(x))\psi^{\pm I}_{1j}(x,\ \eta)]^*f(x)\rho_1dx\\
&\quad+\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\boldsymbol{R}^3_+}(A^{II}\psi^{\pm II}_{1j}(x,\ \eta))^*\alpha_r(x)f(x)\rho_2dx\\
&\quad+\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\{x\in\boldsymbol{R}^3_+,\ r\leqq|x|\leqq r+1\}}[(A^{II}\alpha_r(x))\psi^{\pm II}_{1j}(x,\ \eta)]^*f(x)\rho_2dx\\
&=\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\boldsymbol{R}^3}(A\psi^{\pm}_{1j}(x,\ \eta))^*\alpha_r(x)f(x)\rho(x_3)dx\\
&=\underset{r\to\infty}{\mathrm{l.\,i.\,m.}}\int_{\boldsymbol{R}^3}\lambda_j(\eta)\psi^{\pm}_{1j}(x,\ \eta)^*\alpha_r(x)f(x)\rho(x_3)dx\\
&=\lambda_j(\eta)\hat{f}^{\pm}_{1j}(\eta),
\end{aligned}
$$

where

$$
Au(x)=\begin{cases} A^Iu(x)=M^Iu(x), & x_3<0,\\ A^{II}u(x)=M^{II}u(x), & x_3>0,\end{cases}
$$

for $u\in D(A)$. This proves (7.10).

Similarly we can show (7.11) and (7.12), and thereby (7.9) follows. The proof of Theorem 7.2 is now complete. □

The following theorem gives an explicit expression of the ranges $R(\Phi^{\pm})$, $R(\Phi^{\pm}_{1j})$, $R(\Phi^{St}_{1j})$ and $R(\Phi^{\pm}_{2k})$.

THEOREM 7.3. (1) *For* $R(\Phi^{\pm})$, *we have*

$$
\begin{aligned}
R(\Phi^{\pm})&=\sum_{j\in M}\oplus Q^{\pm}_j(\eta)L^2(\boldsymbol{R}^3_{\pm},\ \boldsymbol{C}^3)\oplus\sum_{j\in M}\oplus Q^{St}_j(\eta)L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)\\
&\quad\oplus\sum_{k\in N}Q^{\pm}_k(\eta)L^2(\boldsymbol{R}^3_{\pm},\ \boldsymbol{C}^3)\\
&=L^2(\boldsymbol{R}^3_{\pm},\ \boldsymbol{C}^3)\oplus L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)\oplus L^2(\boldsymbol{R}^3_{\pm},\ \boldsymbol{C}^3).
\end{aligned}
\tag{7.14}
$$

(2) *For* $R(\Phi^{\pm}_{1j})$, $R(\Phi^{St}_{1j})$ *and* $R(\Phi^{\pm}_{2k})$, *we have*

$$R(\Phi^{\pm}_{1j})=Q^{\pm}_j(\eta)L^2(\boldsymbol{R}^3_{\pm},\ \boldsymbol{C}^3),\qquad \Phi^{\pm}_{1j}\Phi^{\pm *}_{1l}=0,\quad j\neq l, \tag{7.15}$$

$$R(\Phi^{St}_{1j})=Q^{St}_j(\eta)L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3),\qquad \Phi^{St}_{1j}\Phi^{St *}_{1l}=0,\quad j\neq l, \tag{7.16}$$

$$R(\Phi^{\pm}_{2k})=Q^{\pm}_k(\eta)L^2(\boldsymbol{R}^3_{\pm},\ \boldsymbol{C}^3),\qquad \Phi^{\pm}_{2k}\Phi^{\pm *}_{2i}=0,\quad k\neq i, \tag{7.17}$$

that is, the mappings $\Phi_{1j}^{\pm}$, Φ_{1j}^{St} *and* $\Phi_{2k}^{\pm}$ *are partial isometries.*

This implies that $\Phi^{\pm}$ *are unitary operators in* $\mathcal{H}$, *and that the systems of generalized eigenfunctions* $\{\psi_{1j}^{\pm}, \psi_{1j}^{St}, \psi_{2k}^{\pm}\}_{j\in M,\, k\in N}$ *and* $\{\psi_{1j}^{-}, \psi_{1j}^{St}, \psi_{2k}^{-}\}_{j\in M,\, k\in N}$ *are complete, respectively.*

PROOF. It suffices to prove that:

$$g\in N(\Phi^{\pm *})\cap\Big(\sum_{j\in M}\oplus Q_j^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)$$

$$\oplus\sum_{j\in M}\oplus Q_j^{St}(\eta)L^2(\boldsymbol{R}^3, \boldsymbol{C}^3)\oplus\sum_{k\in N}\oplus Q_k^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)\Big)$$

$$\Longrightarrow g\equiv 0.$$

Let

$$g(\eta)\equiv g_{1s_1}^{\pm}(\eta)\oplus\cdots\oplus g_{1p_2}^{\pm}(\eta)\oplus g_{1s_1}^{St}(\eta)\oplus\cdots\oplus g_{1p_2}^{St}(\eta)\oplus g_{2s_1}^{\pm}(\eta)\oplus g_{2s_2}^{\pm}(\eta)$$

$$\in N(\Phi^{\pm *})\cap\Big(\sum_{j\in M}\oplus Q_j^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)$$

$$\oplus\sum_{j\in M}\oplus Q_j^{St}(\eta)L^2(\boldsymbol{R}^3, \boldsymbol{C}^3)\oplus\sum_{k\in N}\oplus Q_k^{\pm}(\eta)L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3)\Big).$$

Then it follows that

$$0=\Phi^{\pm *}g=\underset{N\to\infty}{\mathrm{l.\,i.\,m.}}\sum_{j\in M}\int_{\boldsymbol{R}_{\pm}^3}\psi_{1j}^{\pm}(x, \eta)g_L(\eta)d\eta$$

$$+\underset{N\to\infty}{\mathrm{l.\,i.\,m.}}\sum_{j\in M}\int_{\boldsymbol{R}^3}\psi_{1j}^{St}(x, \eta)g_L(\eta)d\eta$$

$$+\underset{N\to\infty}{\mathrm{l.\,i.\,m.}}\sum_{k\in N}\int_{\boldsymbol{R}_{\pm}^3}\psi_{2k}^{\pm}(x, \eta)g_L(\eta)d\eta$$

where $g_L(\eta)=g(\eta)$ for $|\eta|<L$, $=0$ for $|\eta|>L$. Hence, for non-real ζ, we have

$$(7.18)\qquad (\mathrm{UC})^{-1}F_{x'}(A-\zeta)^{-1}\Phi^{\pm *}g_L\to 0 \quad\text{in } L^2(\boldsymbol{R}^3, \boldsymbol{C}^3) \text{ as } L\to\infty.$$

Let $f\in L^2(\boldsymbol{R}^3, \boldsymbol{C}^3)$ such that $F_{\eta'}^{-1}f\in C_0^{\infty}(\boldsymbol{R}^3, \boldsymbol{C}^3)$. By (7.18) and (6.6), we have

$$(f, (\mathrm{UC})^{-1}F_{x'}(A-\zeta)^{-1}\Phi^{\pm *}g_L)=\sum_{j\in M}(f, ((A_1(\eta')-\zeta)^{-1}\oplus O_{1\times 1})(\mathrm{UC})^{-1}F_{x'}\Phi_{1j}^{\pm *}g_L)$$

$$+\sum_{j\in M}(f, ((A_1(\eta')-\zeta)^{-1}\oplus O_{1\times 1}(\mathrm{UC})^{-1}F_{x'}\Phi_{1j}^{St *}g_L)$$

$$+\sum_{k\in N}(f, (O_{2\times 2}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}F_{x'}\Phi_{2k}^{\pm *}g_L)$$

$$=\sum_{j\in M}(\Phi_{1j}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})((A_1(\eta')-\bar{\zeta})^{-1}\oplus O_{1\times 1})f, g_L)$$

$$+\sum_{j\in M}(\Phi_{1j}^{St}F_{\eta'}^{-1}(\mathrm{UC})((A_1(\eta')-\bar{\zeta})^{-1}\oplus O_{1\times 1})f, g_L)$$

$$+\sum_{k\in N}(\Phi_{2k}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})(O_{2\times 2}\oplus(A_2(\eta')-\bar{\zeta})^{-1})f, g_L)$$

$$=\sum_{j\in M}((\lambda_j(\eta)-\bar{\zeta})^{-1}\Phi_{1j}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f, g_L)$$

$$+\sum_{j\in M}((c_{St}^2|\eta'|^2-\bar{\zeta})^{-1}\Phi_{1j}^{St}F_{\eta'}^{-1}(\mathrm{UC})f,\ g_L)$$

$$+\sum_{k\in N}((\lambda_k(\eta)-\bar{\zeta})^{-1}\Phi_{2k}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f,\ g_L)$$

$$\to 0 \quad \text{as} \quad L\to\infty .$$

Thus

$$0=\sum_{j\in M}\int_{R_\pm^3}\frac{1}{\lambda_j(\eta)-\bar{\zeta}}\Phi_{1j}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\eta$$

$$+\sum_{j\in M}\int_{R^3}\frac{1}{c_{St}^2|\eta'|^2-\bar{\zeta}}\Phi_{1j}^{St}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\eta$$

$$+\sum_{k\in N}\int_{R_\pm^3}\frac{1}{\lambda_k(\eta)-\bar{\zeta}}\Phi_{2k}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\eta ,$$

and hence we obtain that

$$0=\sum_{j\in M}\int_{R_\pm^3}\frac{\varepsilon}{\pi}\Big(\int_a^b\frac{1}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}\Phi_{1j}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\lambda\Big)d\eta$$

$$+\sum_{j\in M}\int_{R^3}\frac{\varepsilon}{\pi}\Big(\int_a^b\frac{1}{(c_{St}^2|\eta'|^2-\lambda)^2+\varepsilon^2}\Phi_{1j}^{St}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\lambda\Big)d\eta$$

$$+\sum_{k\in N}\int_{R_\pm^3}\frac{\varepsilon}{\pi}\Big(\int_a^b\frac{1}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}\Phi_{2k}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\lambda\Big)d\eta$$

$$=\sum_{j\in M}\Big(\int_{a\leqq\lambda_j(\eta)\leqq b}\Phi_{1j}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\eta+\int_{a\leqq c_{St}^2|\eta'|^2\leqq b}\Phi_{1j}^{St}F_{\eta'}^{-1}(\mathrm{UC})f\cdot g\,d\eta\Big)$$

$$+\sum_{k\in N}\int_{a\leqq\lambda_k(\eta)\leqq b}\Phi_{2k}^{\pm}F_{\eta'}^{-}(\mathrm{UC})f\cdot g\,d\eta$$

$$=\sum_{j\in M}((\Phi_{1j}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f,\ g_{1j}^{\pm}(\Delta))+(\Phi_{1j}^{St}F_{\eta'}^{-1}(\mathrm{UC})f,\ g_{1j}^{St}(\Delta)))$$

$$+\sum_{k\in N}(\Phi_{2k}^{\pm}F_{\eta'}^{-1}(\mathrm{UC})f,\ g_{2k}^{\pm}(\Delta))$$

$$=\sum_{j\in M}((f,\ (\mathrm{UC})^{-1}F_{x'}\Phi_{1j}^{\pm *}g_{1j}^{\pm}(\Delta))+(f,\ (\mathrm{UC})^{-1}F_{x'}\Phi_{1j}^{St*}g_{1j}^{St}(\Delta)))$$

$$+\sum_{k\in N}(f,\ (\mathrm{UC})^{-1}F_{x'}\Phi_{2j}^{\pm *}g_{2k}^{\pm}(\Delta)),$$

where

$g_{1j}^{\pm}(\Delta)=g(\eta)$ for $a\leqq\lambda_j(\eta)\leqq b,$ $=0$ otherwise.

$g_{1j}^{St}(\Delta)=g(\eta)$ for $a\leqq c_{St}^2|\eta'|^2\leqq b,$ $=0$ otherwise.

$g_{2k}^{\pm}(\Delta)=g(\eta)$ for $a\leqq\lambda_k(\eta)\leqq b,$ $=0$ otherwise.

So we have

$$
\begin{aligned}
0= & \sum_{j\in M}((\mathrm{UC})^{-1}F_{x'}\Phi_{1j}^{\pm *}g_{1j}^{\pm}(\Delta)+(\mathrm{UC})^{-1}F_{x'}\Phi_{1j}^{St*}g_{1j}^{St}(\Delta)) \\
& +\sum_{k\in N}(\mathrm{UC})^{-1}F_{x'}\Phi_{2k}^{\pm *}g_{2k}^{\pm}(\Delta) \\
= & \sum_{j\in M}(\mathrm{UC})^{-1}F_{x'}\left[\int_{\boldsymbol{R}_{\pm}^{3}}\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm}(x_3,\ \eta)\oplus O_{1\times 1})g_{1j}^{\pm}(\Delta)d\eta\right] \\
& +\sum_{j\in M}(\mathrm{UC})^{-1}F_{x'}\left[\int_{\boldsymbol{R}^{3}}\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{St}(x_3,\ \eta)\oplus O_{1\times 1})g_{1j}^{St}(\Delta)d\eta\right] \\
& +\sum_{k\in N}(\mathrm{UC})^{-1}F_{x'}\left[\int_{\boldsymbol{R}_{\pm}^{3}}\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3,\ \eta))g_{2k}^{\pm}(\Delta)d\eta\right] \\
= & \sum_{j\in M}\left(\int_{\boldsymbol{R}_{\pm}}(\psi_{1j}^{\pm}(x_3,\ \eta)\oplus O_{1\times 1})g_{1j}^{\pm}(\Delta)d\xi+\int_{\boldsymbol{R}}(\psi_{1j}^{St}(x_3,\ \eta)\oplus O_{1\times 1})g_{1j}^{St}(\Delta)d\xi\right) \\
& +\sum_{k\in N}\int_{\boldsymbol{R}_{\pm}}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3,\ \eta))g_{2k}^{\pm}(\Delta)d\xi \\
= & \sum_{j\in M}\left(\int_{\boldsymbol{R}_{\pm}\cap a\leqq\lambda_j(\eta)\leqq b}(\psi_{1j}^{\pm}(x_3,\ \eta)\oplus O_{1\times 1})g(\eta)d\xi\right. \\
& \left.+\int_{\boldsymbol{R}\cap a\leqq c_{St}^{2}|\eta'|^{2}\leqq b}(\psi_{1j}^{St}(x_3,\ \eta)\oplus O_{1\times 1})g(\eta)d\xi\right) \\
& +\sum_{k\in N}\int_{\boldsymbol{R}_{\pm}\cap a\leqq\lambda_k(\eta)\leqq b}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3,\ \eta))g(\eta)d\xi\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
g(\eta)=0\quad \text{in}\quad & \sum_{j\in M}\oplus Q_{j}^{\pm}(\eta)L^{2}(\boldsymbol{R}_{\pm}^{3},\ \boldsymbol{C}^{3})\oplus\sum_{j\in M}\oplus Q_{j}^{St}(\eta)L^{2}(\boldsymbol{R}^{3},\ \boldsymbol{C}^{3}) \\
& \oplus\sum_{k\in N}\oplus Q_{k}^{\pm}(\eta)L^{2}(\boldsymbol{R}_{\pm}^{3},\ \boldsymbol{C}^{3})\ ,
\end{aligned}
$$

since a and b are arbitrary, and $\psi_{1j}^{\pm}(x_3,\ \eta)$, $\psi_{1j}^{St}(x_3,\ \eta)$ $(j\in M)$ and $\psi_{2k}^{\pm}(x_3,\ \eta)$ $(k\in N)$ are linear independent.

This completes the proof of Theorem 7.3. □

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Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305
Japan