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THE STABLE TOPOLOGY OF MODULI SPACES OF PERIODIC INSTANTONS

By

Hiromichi MATSUNAGA

1. Introduction and statement of the result

Let M be a smooth 4-manifold which admits an open subset K with one end N and an open submanifold W_0 with two ends N_-, N_+. $W_1, W_2, \cdots$ denote copies of W_0. The 4-manifold M will be called end-periodic if it admits a decomposition $M = K \cup_N W_0 \cup_N W_1 \cup \cdots$, where $N \subset K$ is identified with the end N_- of W_0 and the end N_+ of W_0 is identified with the end N_- of W_1 and so on. Let Y be the compact oriented 4-manifold which is obtained from W_0 by identifying the two ends. The manifold Y has a Z-cover $\tilde{Y} = \cdots_N W_{-1} \cup_N W_0 \cup_N W_1 \cdots$ with projection $\pi : \tilde{Y} \to Y$. A geometric object on M, a vector bundle, a connection, a differential operator, a Riemannian metric etc. will be called end-periodic if its restriction on End$M$= $W_0 \cup_N W_1 \cdots$ is the pull back by π of an object on Y. By making choose a smooth function $s : W_0 \to [0,1]$ such that $s|N_- = 0$ and $s|N_+ = 1$, we obtain a smooth step function t on M such that $t(x) = n + s(x)$ if $x \in W_n$.

Let $P \to M$ be an end-periodic principal $SU(2)$-bundle, and A_0 be an end-periodic connection on P which is gauge equivalent over End$M$ to the product connection on End$M$ $\times SU(2)$. Then by the lemma 7.1 in [7]

$$l = (1/8\pi^2)\int_M tr(F_{A_0} \wedge F_{A_0})$$

is an integer, where tr( ) is the trace on the adjoint representation of the group $SU(2)$. Let $E \to M$ be an end-periodic vector bundle which is associated to the principal bundle $P \to M$. Put $L^2{}_{\mathrm{loc}}(E) = \{\text{section } u ; u \in L^2(E|_A)$ for every measurable $A \subset\subset M\}$, where we assume that the set A has a finite measure, and denote by $\|\cdot\|_{A_0}$ the norm by the covariant derivative $\nabla_{A_0} : C_0^\infty(E) \to C_0^\infty(E \otimes T^*M)$ of compactly supported smooth sections, further $\nabla_{A_0}^{(j)}$ denotes the j-times iterated derivative $\nabla_{A_0} \cdots \nabla_{A_0}$. For $\delta > 0$, put

$$\mathscr{A}_k(\delta) = \{A_0 + a ; a \in L^2{}_{5,\mathrm{loc}}(adP \otimes T^*M) \text{ with norm } \|a\|_{A_0} < \infty\},$$

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where $\|a\|_{A_0}=\int_M^{e^{t\delta}}\sum_{j=0}^5 \|\nabla_{A_0}^{(j)}a\|^2$ and define the small gauge group $\mathscr{G}_k'(\delta)=\{h\in L^2_{6,\mathrm{loc}}(\mathrm{AutP});\|\nabla_{A_0}h\|_{A_0}<\infty$, and tends to the identity at infinity$\}$,where we have used the adjoint representation $\mathrm{ad}:SU(2)/Z_2\to\mathrm{End}(\mathrm{su}(2))$ and the embedding $C^\infty(P\times_{\mathrm{ad}}SU(2)/Z_2)\to C^\infty(P\times_{\mathrm{ad}}\mathrm{End}(su(2))$.

Let $\mathscr{A}_k*(\delta)\subset\mathscr{A}_k(\delta)$ denote the subset of irreducible connections, and g_0 be an end-periodic metric on the tangent bundle TM and $\mathscr{C}$ be the set of asymptotically periodic metrics ((6.1) in [7]). Consider a $\mathscr{G}_k$ equivariant map

$$\mathscr{P}:\mathscr{A}_k(\delta)\times\mathscr{C}\ni(A,\phi)\to P_-(g_0)(\phi^{-1})^*F_A\in L^2_{4,\mathrm{loc}}(\mathrm{ad}P\otimes P_-\Lambda^2T^*M),$$

where P_- denotes the projection to the anti-self dual part. Let $\pi':\mathscr{M}_k'=\mathscr{P}^{-1}(0)/\mathscr{G}_k'\to\mathscr{C}$ be the projection. Put $\mathscr{M}(\phi)_k'=\pi'^{-1}(\phi)$. According to the lemmas 5.3, 5.8 and 8.4 in [7], there exists a positive number $\delta_*>0$ such that for any $\delta,0<\delta<\delta_*,\mathscr{M}_k'(\phi)\cap(\mathscr{A}_k^*(\delta)/\mathscr{G}_k'(\delta))$ is a smooth manifold. $\Omega^3{}_k$ denotes the 3-fold iterated loop space of mappings of degree k. In this paper we consider the case of the manifold $M=S^1\times R^3$ which has been considered as an end-periodic manifold, $M'=S^1\times D^3{}_{3/2}{}^{\cup}(S^1\times S^2\times(1,3))^{\cup}(S^1\times S^2\times(2,4))^{\cup}\cdots$ (Proposition 1 in [1]). Now we have the following result which is proved in Appendix.

PROPOSITION. The manifold M' admits an end-periodic metric.

Then the main result in this article is

THEOREM. There exists a map $\mathscr{M}_k'(\phi)\to\Omega^3{}_k(S^3)$ which induces a surjection of homology groups

$$\mathrm{H}_q(\mathscr{M}_k'(\phi))\to\mathrm{H}_q(\Omega^3{}_k(S^3))\quad\text{for }q\leqq[k/2].$$

In the previous paper [1] we have discussed the moduli space of self-dual, asymptotically periodic instantons. There we have used the gauge group $\mathscr{G}_k(\delta)=\{h\in L^2_{6,\mathrm{loc}}(\mathrm{Aut}P);\|\nabla_{A_0}h\|_{A_0}<\infty\}$ instead of the small gauge group $\mathscr{G}_k'(\delta)$. Let $\pi:\mathscr{M}_k=\mathscr{P}^{-1}(0)/\mathscr{G}_k(\delta)\cap(\mathscr{A}_k^*(\delta))/\mathscr{G}_k\ )\to\mathscr{C}$ be the projection and put $\mathscr{M}_k(\phi)=\pi^{-1}(\phi)$. Then we have a principal SO(3)-bundle $\mathscr{M}_k'(\phi)\cap(\mathscr{A}_k^*(\delta))/\mathscr{G}_k'(\delta))\to\mathscr{M}_k(\phi)$.

We prove the main theorem in the sections 2 and 3. Our main tools are periodic instantons due to Harrington-Shepard, Atiyah-Jones diagram and Taubes' existence theorem ([3], [2], [8]).

I am grateful to Doctor Yamaguchi K. for his indication of the usefulness of the proposition (A.1) in [6], and wish to thank the referee for his kind advices.

2. Deformation of Harrington-Shepard's periodic instantons

We abbreviate hyperbolic functions as follows:

$$\text{ch} = \cosh, \text{ and } \text{sh} = \sinh.$$

Let r be the distance from the source to a point in R^3 and $\tau \in [0, 2\pi]$.
Then Harrington-Shepard's periodic solution is given by

$$\phi = 1 + \frac{1}{r} \cdot \frac{\text{sh}r}{\text{ch}r - \cos\tau} \quad ([3]).$$

Let t be the smooth step function in the selection 1, and f be a smooth cut off function such that $f \mid K - N = 1$ and $\{\text{support } f\} \subset K$. We put

$$\tilde{\phi}(\delta) = 1 + \frac{1}{r} \cdot \frac{\text{sh}r}{\text{ch}r - \cos\tau} \cdot e^{-t\delta} \quad \text{for } \delta > 0$$

$$\hat{\phi} = 1 + \frac{1}{r} \cdot \frac{\text{sh}r}{\text{ch}r - \cos\tau} \cdot f(r)$$

Then $\hat{\phi}$ is an end-periodic function and $\tilde{\phi} = \hat{\phi} + (\tilde{\phi} - \hat{\phi})$. We put $\nabla_{x_i} = \nabla_i$ for i=1,2,3. By a direct calculation

$$\nabla_i \log \tilde{\phi} = \frac{e^{-t\delta}}{\tilde{\phi}} \cdot \frac{x_i}{r^2(\text{ch}r - \cos\tau)} \cdot \left(-\frac{\text{sh}r}{r} + \frac{1 - \text{ch}r \cos\tau}{\text{ch}r \cos\tau} - t' \delta \text{sh}r \right)$$

We denote by G_i the factor $\frac{x_i}{r^2(\text{ch}r - \cos\tau)}$ and by $G^\#$ the factor $\left(-\frac{\text{sh}r}{r} + \frac{1 - \text{ch}r \cos\tau}{\text{ch}r \cos\tau} - t' \delta \text{sh}r \right)$. By further calculations

$$\nabla_r \log \tilde{\phi} = -\frac{1}{\tilde{\phi}} \cdot \frac{\sin\tau \, \text{sh}r}{r(\text{ch}r - \cos\tau)^2} \cdot e^{-t\delta}$$

The gauge potential is given by
$\tilde{A}_i = \sqrt{-1}\bar{\sigma}_{ij}\nabla_j(\log\tilde{\phi})$, where $\bar{\sigma}_{ij} = (1/4\sqrt{-1})[\sigma_i, \sigma_j]$ for $i, j = 1,2,3$ and $\bar{\sigma}_{i4} = -\frac{1}{2}\sigma_i$, (c.f.[3] and Jackiw, R., Nohl, C., Rebbi, C., Conformal properties of pseudo particle configurations, Phys. Review D 15, 8 (1977)). To get the curvature we need the following formulas,

$$\nabla_j \nabla_i \log \tilde{\phi} = -\frac{1}{\tilde{\phi}^2} e^{-2t\delta} (G_j \cdot G^\#)(G_i \cdot G^\#) + \frac{1}{\tilde{\phi}} \nabla_j \nabla_i \tilde{\phi}$$

$$\nabla_j \nabla_i \tilde{\phi} = e^{-t\delta} \left\{ \left[-\frac{t'\delta x_j}{r} G_i + \frac{\delta_{ij}}{r^2(\text{ch}r - \cos\tau)} - \frac{2x_i x_j}{r^4(\text{ch}r - \cos\tau)} - \frac{x_i x_j}{r^3} \cdot \frac{\text{sh}r}{(\text{ch}r - \cos\tau)^2} \right] G^\# \right.$$

$$\left. + \left[\frac{x_j \text{sh}r}{r^2} - \frac{x_j \text{ch}r}{r^2} - \frac{\text{sh}r \cos\tau}{\text{ch}r - \cos\tau} \cdot \frac{x_j}{r} - \frac{(1 - \text{ch}r \cos\tau)\text{sh}r}{(\text{ch}r - \cos\tau)^2} \cdot \frac{x_j}{r} - \frac{t''\delta x_j \text{sh}r}{r} - \frac{x_j t'\delta \text{ch}r}{r} \right] G_i \right\}$$

$$\nabla_\gamma \nabla_i \tilde{\phi} = e^{-\iota\delta}\left(\frac{-x_j \sin\tau}{r^2(\mathrm{ch}r - \cos\tau)^2} G^\# + G \cdot \frac{\sin^2 r \sin\tau}{(\mathrm{ch}r - \cos\tau)^2}\right)$$

$$\nabla_\gamma \nabla_\gamma \log\tilde{\phi} = \nabla_\gamma\left(\frac{\nabla_\gamma \tilde{\phi}}{\tilde{\phi}}\right) = -\frac{1}{\tilde{\phi}^2}(\nabla_r \tilde{\phi})^2 + \frac{1}{\tilde{\phi}} \nabla_\gamma \nabla_\gamma \tilde{\phi}$$

$$\nabla_\gamma \nabla_\gamma \tilde{\phi} = e^{-\iota\delta} \cdot \frac{\mathrm{sh}r}{r} \cdot \frac{\mathrm{ch}r \cos\tau - \sin^2\tau - 1}{(\mathrm{ch}r - \cos\tau)^2}$$

Since $\phi \fallingdotseq 0$ as $r \geqq 1$, we obtain approximately the difference between our potential and H-S's in [3]:

$$\nabla_i \log\tilde{\phi} : e^{-\iota\delta} \cdot \frac{x_i t'\delta\, \mathrm{sh}r}{r^2(\mathrm{ch}r - \cos\tau)}$$

$$\nabla_j \nabla_i \log\tilde{\phi} : e^{-2\iota\delta}\{2G_i G^\# G_j (t'\delta\, \mathrm{sh}r) - G_i G_j (t'\delta\, \mathrm{sh}r)^2\} + e^{-\iota\delta} \frac{x_i t'\delta}{r} G_i G^\# + \frac{\delta x_j (t''\mathrm{sh}r + t'\mathrm{ch}r)}{r} G_i$$

Therefore $\tilde{A} = \hat{A} + (\tilde{A} - \hat{A}) \in \mathscr{A}(2\delta)$ for any δ such that $0 < 2\delta < \delta_*$, where $\tilde{A}$ and $\hat{A}$ denote the connections derived from $\tilde{\phi}$ and $\hat{\phi}$.

Now we consider an electric field $E : R \to R^3 \cup \{\infty\}$ which is by definition linear and the field of a single charge has the properties:

1) $E \to 0$ at ∞, 2) $E \to \infty$ at the source, 3) E is spherically symmetric (c.f.[2]). Then we have

LEMMA. The map $(\nabla_i \log\tilde{\phi}) : C_1(R^3) \to \Omega^3{}_1(S^3)$ gives an electric field.

PROOF. As $r \to \infty, \phi \to 1, e^{-\iota\delta} \to 0, t'$ is bounded. Then $\nabla_i \log\tilde{\phi} \to 0$. As $r \to 0$, $\mathrm{sh}r/r \to 1$, $\mathrm{ch}r \to 1$, $e^{-\iota\delta} = 1$, $t' = 0$. Let τ to be zero. Then $(-\mathrm{sh}r/r - 1) \to -2$. By the fact $(x_1/r^2)^2 + (x_2/r^2)^2 + (x_3/r^2)^2 \to \infty$ we have $\|(\nabla_i \log\tilde{\phi})\| \to \infty$. Now clearly $(\nabla_i \log\tilde{\phi})$ is spherically symmetric in R^3. Thus, we obtain the lemma.

Next we consider homotopic deformation,

$$\tilde{\phi}_{(s)}(\delta) = 1 + \frac{s}{r} + \frac{(1-s)\mathrm{sh}r}{r(\mathrm{ch}r \cos\tau)} \cdot e^{-\iota\delta}, \quad 0 \leqq s \leqq 1.$$

Then $\tilde{\phi}(\delta)$ is homotopic to $\tilde{\phi}_{(1)} = 1 + 1/r$ and so $\nabla \log\tilde{\phi}(\delta)$ is homotopic to $\nabla \log\tilde{\phi}_{(1)}$, which is self-dual in R^4. In the same way we can see that $\nabla \log\tilde{\phi}(\delta)$ is homotopic to $\nabla \log\hat{\phi}$ which is trivial on End$M$. Now we consider k-instantons. For this purpose we consider the functions

$$\tilde{\phi}_k(\delta) = 1 + \sum_{i=1}^{k} \frac{1}{r_i} \cdot \frac{\mathrm{sh}r_i}{\mathrm{ch}r_i - \cos\tau} \cdot e^{-\iota\delta}$$

$$\hat{\phi}_k = 1 + \sum_{i=1}^{k} \frac{1}{r_i} \cdot \frac{\operatorname{sh} r_i}{\operatorname{ch} r_i - \cos\tau} \cdot f(r_i)$$

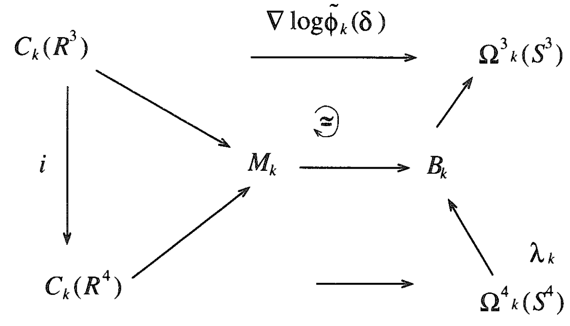
where r_i denotes the distance from a point to i-th base point in $R^3, i = 1, 2, \cdots, k$. A set of k-distinct base points can be regarded as an element of the configuration space $C_k(R^3)$. We denote by A the connection which is obtained from $\hat{\phi}_k$. The space R^3 is deformable onto the unit open disc by a homotopy

$$(1-s)x + \frac{2s\operatorname{Tan}^{-1}\|x\|}{\|x\|\pi} \cdot x \quad \text{for } 0 \leqq s \leqq 1 \text{ and } x \neq 0,$$

where the origin in R^3 is fixed. Thus we can assume that k-distinct points lies in the unit open disc in R^3. Then by the construction in Remark 2 in [1] we have a 1-form a such that A+a is self-dual where the connection A has a compact support. Therefore the 1-form a also has a compact support. For $g \in \mathscr{G}_k'(\delta)$, by making use of the homotopy $g^{-1}(A + (1-s)a)g + g^{-1}dg,\ 0 \leqq s \leqq 1$, we can see that the homotopy gives a homotopy in the space $\mathscr{B}_k'(\delta) = \mathscr{A}_k(\delta)/\mathscr{G}_k'(\delta)$. Then the class $[A+a]$ is homotopic to the class $[A]$. Thus the gauge potential $\nabla \log(\tilde{\phi}_k)$ gives an element of $\mathscr{M}_k'(\delta)$.

3. Proof of Main theorem

We prove the theorem by making use of a modified Atiyah-Jones diagram [2]. We denote by B_k and M_k the moduli space of connections and self-dual connections on an $SU(2)$ bundle over R^4 with topological charge k respectively. By the consideration in Section, $\log\tilde{\phi}(\delta)$ is homotopic to $\log\tilde{\phi}_{(1)}$. Then by the lemma (3, 6) in [2] we have a homotopy-commutative diagram



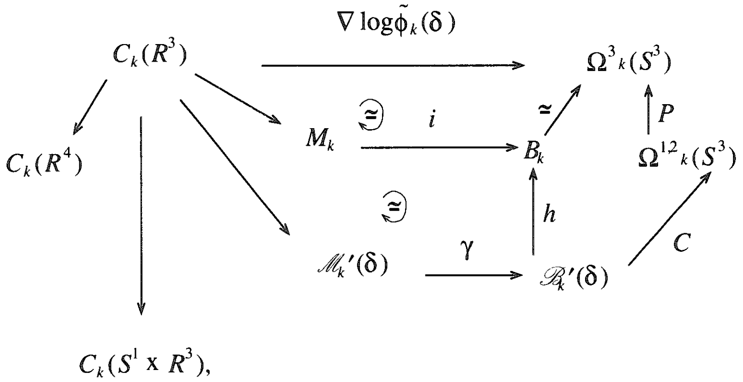
where λ_k is the map (3.4) in [2].

We denote by $\Omega^{1,2}{}_k(S^3)$ the set of based maps from the space $S^1 \times S^2$ to S^3 of degree k. For a map $p: S^1 \times S^2 \to S^3$ we define a map $\hat{p}: S^1 \times S^2 \to S^3$ by

$$\hat{p}(t,x) = p(t,x_0)^{-1} p(t,x) p(t_0,x)^{-1}$$

where x_0, t_0 are base points in S^1, S^2. Then the map $\hat{p}$ gives a map $\tilde{p}: S^3 \to S^3$. Thus we have a map $P: \Omega^{1,2}{}_k(S^3) \to \Omega^3{}_k(S^3)$. By making use of the natural projection $S^1 \times S^2 \to S^3$ we have a map $j: \Omega^3{}_k(S^3) \to \Omega^{1,2}{}_k(S^3)$ such that $P \cdot j$ = the identity map. By the proposition 2.3 in [2] we have a homotopy equivalence $B_k \to \Omega^3{}_k(S^3)$. By mimicking the proof of this proposition, we obtain a map $C: \mathscr{B}_k'(\delta) \to \Omega^{1,2}{}_k(S^3)$ which is compatible with the homotopy equivalence $B_k \to \Omega^3{}_k(S^3)$. Precisely the space $\mathscr{B}_k'(\delta)$ is deformable into the subspace $\mathscr{B}_k'(\delta)_\infty$ of the classes of connections which are flat outside a compact set in $M = S^1 \times R^3$ (this fact can be seen by making use of a cut off function and a homotopy as in the consideration in the section 2). For any such connection A there exists a flat section α of the principal bundle $P \to S^1 \times R^3$ with $\alpha | K_n^c = K_n^c \times g_0$, where K_n^c denotes the complement of the subspace $K_n = K \cup_N W_0 \cup_N W_1 \cup \cdots \cup_N W_n$ for a sufficiently large n. Pick any section β of P which agrees with α on $S^1 \times \ell, \ell$ is a line through the origin in $R^3, (S^1 \times R^3$ retracts onto $S^1 \times \ell$, therefore such β exists). For a sufficiently large n and a subspace $S^1 \times S^2 \times (t)$ in W_n, α and β differ by a map $g: S^1 \times S^2 \times (t) \to SU(2)$ with $g(S^1 \times \ell) = 1$. Then by assigning g to A we get the required map $\mathscr{B}_k'(\delta) \to \Omega^{1,2}{}_k(S^3)$.

Thus we obtain the following homotopy-commutative diagram:



where i and γ denote the inclusion maps and h denotes the composite map of $P \cdot C$ and a homotopy inverse $\Omega^3{}_k \to B_k$. The commutativity in the lower part follows from the consideration in the section 2. By the theorem due to G•Segal ([5]) the induced homomorphism

$$(\nabla \log \tilde{\phi}_k(\delta))_* : \mathrm{H}_q(C_k(R^3)) \to \mathrm{H}_q(\Omega^3{}_k(S^3))$$

is an isomorphism for $k >> q$. The homotopy type of $\Omega^3{}_k(S^3)$ is independent of k.

Then by the proposition (A.1) in [6], $H_q(C_k(R^3)) \to H_q(\Omega^3{}_k(S^3))$ is an isomorphism for $q \leqq [k/2]$. Therefore the homomorphism

$$(P \cdot C \cdot \gamma)_* : H_g(\mathscr{M}_k'(\delta)) \to H_g((\Omega^3{}_k(S^3))$$

is surjective for $q \leqq [k/2]$. Thus we have proved the theorem.

REMARK. By making use of a diffeomorphism

$$R^3 \times S^1 \ni (x, y, z, \theta) \to (x, y, e^z \cos\theta, e^z \sin\theta) \in R^4 - R^2 \cong S^4 - S^2,$$

we obtain a compactification of the space up to diffeomorphism. But I do not know a conformal compactification without singularities ([4]).

APPENDIX. Proof of the proposition in the section one.

Firstly I should remark that the manifold $M = S^1 \times R^3$ has been considered as an end-periodic manifold

$$M' = S^1 \times D^3{}_{3/2} \cup (S^1 \times S^2 \times (1,3)) \cup (S^1 \times S^2 \times (2,4)) \cup \ldots \qquad (2.[1])$$

The space $S^1 \times S^2 \times [1, \infty)$ admits the pull-back metric via π of the product metric on the space $S^1 \times S^2 \times S^1$. By making use of the cut off function f in the section 2, we connect the natural metric g_0 in the space $S^1 \times D^2{}_{3/2}$ with the metric g_1 on the End$M$, and we obtain a metric on the manifold M'

$$g = f(r) g_0 + (1 - f(r)) g_1 .$$

Then the restriction of the metric g over End$M$ is induced from the conformally flat metric g_1 on the manifold Y. Thus we obtain an end-periodic metric on the manifold M'.

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Department of Mathematics
Naruto University of Education
Takashima, Naruto 772 JAPAN