

THE CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM VARIABLES

By

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1. Introduction.

Let (Ω, \mathcal{F}, P) be a probability space. For any two σ -fields \mathcal{A} and \mathcal{B} define the mixing coefficients ϕ and α and the maximal correlation coefficient ρ by

$$\phi(\mathcal{A}, \mathcal{B}) = \sup |P(B|A) - P(B)| \quad A \in \mathcal{A}, B \in \mathcal{B}, P(A) > 0;$$

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup |P(A \cap B) - P(A)P(B)| \quad A \in \mathcal{A}, B \in \mathcal{B};$$

$$\rho(\mathcal{A}, \mathcal{B}) = \sup |\text{Corr}(\xi, \eta)| \quad \xi \in L^2(\mathcal{A}), \eta \in L^2(\mathcal{B}).$$

Let $\{X_j: -\infty < j < \infty\}$ be a strictly stationary sequence of random variables on (Ω, \mathcal{F}, P) . For integers n let \mathcal{F}_n be the σ -field generated by $\{X_j: j \leq n\}$ and \mathcal{F}_n the σ -field generated by $\{X_j: j \geq n\}$. The sequence $\{X_j\}$ is said to be ϕ -mixing (or *uniformly mixing*) if

$$\phi(n) \equiv \phi(\mathcal{F}_0, \mathcal{F}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(see Ibragimov [9]), *strongly mixing* if

$$\alpha(n) \equiv \alpha(\mathcal{F}_0, \mathcal{F}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(see Rosenblatt [15]) and *completely regular* if

$$\rho(n) \equiv \rho(\mathcal{F}_0, \mathcal{F}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(see Kolmogorov-Rozanov [13]).

Among these coefficients, the following inequalities always hold:

$$4\alpha(n) \leq \rho(n) \leq 2\phi^{1/2}(n).$$

The left-hand inequality is an easy consequence of the definitions of the coefficients $\alpha(n)$ and $\rho(n)$, and the right-hand inequality is a consequence of the Ibragimov fundamental inequality for ϕ -mixing sequences (see [11, Theorem 17.2.3, p. 309]). Thus a ϕ -mixing sequence is completely regular (the converse

is false; see [11, pp. 310-314]), and a completely regular sequence is strongly mixing (the converse is false; see [16, pp. 206-209], but for Gaussian sequences complete regularity is equivalent to strong mixing; see [13]). Formulations of various mixing conditions are given by Ibragimov-Rozanov [12] for stationary Gaussian sequences in terms of the spectral density, and by Rosenblatt [16] for stationary Markov sequences in terms of the transition operator.

Let $\{X_j\}$ be a strictly stationary sequence with $EX_j=0$ and $EX_j^2<\infty$. Set

$$S_n = \sum_{j=1}^n X_j, \quad \sigma_n^2 = ES_n^2.$$

In numerous papers conditions are investigated which guarantee asymptotic normality of the distribution of the normed sum $\sigma_n^{-1}S_n$ (see, for example, [2, Chap. 4], [9], [10] and [11, Chap. 18]).

We are interested in knowing when the r th absolute moment of $\sigma_n^{-1}S_n$ ($r>2$) converges to that of the normal distribution. When X_j are independent (but not necessarily identically distributed) random variables, Bernstein [1] presented a necessary and sufficient condition (the r th Lindeberg condition) for the convergence of absolute moments in the central limit theorem. Brown [4, 5] gave an alternative proof of Bernstein's result. Hall [8] extended Bernstein's theorem in both the independence and the martingale cases. For stationary ϕ -mixing and strongly mixing sequences the author [17, 18] obtained some results on the convergence of moments. Recently, in the ϕ -mixing case, the following much broader result was proved; the proof is completely different from those in [17] and [18].

THEOREM A ([19]). *Let $\{X_j\}$ be a strictly stationary sequence with $EX_j=0$ and $E|X_j|^r<\infty$ for some $r>2$. If $\phi(n)\rightarrow 0$ and $\sigma_n^2\rightarrow\infty$ as $n\rightarrow\infty$, then*

$$\lim_{n\rightarrow\infty} E|S_n/\sigma_n|^r = \int_{-\infty}^{\infty} (2\pi)^{-1/2} |u|^r \exp(-u^2/2) du.$$

In Theorem A it is not assumed that $\phi(n)\rightarrow 0$ at a specific rate, while the series-type conditions on the mixing coefficients were imposed in all the theorems of [18] (cf. [9, Theorem 1.4]). The purpose of this paper is to generalize the above ϕ -mixing result to the complete regularity case. The basic idea, which was used in [19], is a martingale representation of the sum S_n , and the proof is based on Ibragimov's moment inequality (Lemma 2 below) and a martingale result of Hall [8].

2. Statement of a result.

First we state a result of Ibragimov [10, Theorem 2.1], which generalizes an earlier result of his own [9, Theorem 1.4].

THEOREM B (Ibragimov). *Let $\{X_j\}$ be a strictly stationary sequence with $EX_j=0$ and $EX_j^2<\infty$. (i) If $\lim_{n\rightarrow\infty}\rho(n)=0$ and $\limsup_{n\rightarrow\infty}\sigma_n^2=\infty$, then $\sigma_n^2=nh(n)$, where $h(n)$ is a slowly varying function in the sense of Karamata. (ii) If in addition $E|X_j|^r<\infty$ for some $r>2$, then*

$$\lim_{n\rightarrow\infty} P\{S_n/\sigma_n < x\} = \int_{-\infty}^x (2\pi)^{-1/2} \exp(-u^2/2) du .$$

REMARKS. Theorem B (ii) fails if its hypothesis $E|X_j|^r<\infty$ ($r>2$) is omitted ; a counterexample is constructed by Bradley [3]. Lifshits [14] proved some central limit theorems on Markov chains under $\rho(n)\rightarrow 0$ and other slightly weaker conditions.

In this article the conditions of Theorem B, without any additional conditions, will be shown to imply the convergence of the r th absolute moments in the central limit theorem. More precisely, we shall prove

THEOREM C. *Let $\{X_j\}$ be a strictly stationary sequence with $EX_j=0$ and $E|X_j|^r<\infty$ for some $r>2$. If $\rho(n)\rightarrow 0$ and $\sigma_n^2\rightarrow\infty$ as $n\rightarrow\infty$, then*

$$(1) \quad \lim_{n\rightarrow\infty} E|S_n/\sigma_n|^r = \int_{-\infty}^{\infty} (2\pi)^{-1/2} |u|^r \exp(-u^2/2) du .$$

As we have remarked in Sect. 1, the ϕ -mixing condition implies the complete regularity condition, thus Theorem C contains Theorem A as a special case. For strongly mixing sequences the relation (1) holds under the conditions $EX_j=0$, $E|X_j|^{r+\delta}<\infty$ for some $r>2$ and $\delta>0$, $EX_1^2+2\sum_{j=2}^{\infty} E(X_1X_j)>0$ and $\sum_{n=1}^{\infty} n^{r/2-1}(\alpha(n))^{\delta/(r+\delta)}<\infty$ (see [18]).

3. The proof.

In the proof, limits will be taken as $n\rightarrow\infty$. The symbol K denotes a generic constant, not necessarily the same at each appearance. β_r denotes the r th absolute moment of the standard normal distribution. $I(A)$ denotes the indicator function of the event A .

For the proof of Theorem C we need a few well-known inequalities.

LEMMA 1. Suppose that the random variables ξ and η , respectively, are measurable with respect to \mathcal{F}_k and \mathcal{F}_{k+n} ;

1) If $E\xi^2 < \infty$ and $E\eta^2 < \infty$, then

$$(2) \quad |E(\xi\eta) - E\xi \cdot E\eta| \leq (E\xi^2)^{1/2} (E\eta^2)^{1/2} \rho(n);$$

2) if $|\xi| \leq B$ a. s. and $E|\eta|^s < \infty$ for some $s > 1$, then

$$(3) \quad |E(\xi\eta) - E\xi \cdot E\eta| \leq 6B(E|\eta|^s)^{1/s} (\alpha(n))^{1-1/s}.$$

The inequality (2) is an immediate consequence of the definition of the coefficient $\rho(n)$. The inequality (3) is due to Davydov [7]. The following inequality, due to Ibragimov [10], is fundamental to our proof.

LEMMA 2. Under the assumptions of Theorem C, there exists a constant C such that

$$(4) \quad E|S_n|^r \leq C\sigma_n^r \quad \text{for all } n \geq 1.$$

We shall divide the sum S_n into three parts:

$$S_n = S'_n + S''_n = \sigma_n T_n + \sigma_n T'_n + S''_n,$$

and show that $\sigma_n^{-1} S''_n$ and T'_n are asymptotically negligible, while the r th absolute moment of T_n converges to β_r , where the variable T_n will be chosen to be a martingale.

The first step is to represent the sum S_n in the form

$$S_n = \sum_{j=1}^k y_j + \sum_{j=1}^{k+1} z_j = S'_n + S''_n,$$

where

$$y_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i, \quad 1 \leq j \leq k;$$

$$z_j = \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i, \quad 1 \leq j \leq k;$$

$$z_{k+1} = \sum_{i=k(p+q)+1}^n X_i,$$

$p = p(n)$ and $q = q(n) \in \{1, 2, \dots, n\}$ and satisfy the following conditions:

$$(5) \quad \begin{aligned} \text{a) } & p \rightarrow \infty, \quad q \rightarrow \infty, \quad n^{-1}p \rightarrow 0, \quad p^{-1}q \rightarrow 0, \\ \text{b) } & n^{1+\beta} q^{1-\beta} p^{-2} \rightarrow 0 \quad \text{for some } \beta > 0, \\ \text{c) } & np^{-1} \rho^{2/\tau}(q) \rightarrow 0, \end{aligned}$$

and $k = k(n) = [n/(p+q)]$. Here $[a]$ denotes the greatest integer $\leq a$. Such systems of p and q actually exist. In fact, if we set

$$\begin{aligned}\lambda(n) &= \max\{\rho^{1/r}(\lceil n^{1/4} \rceil), (\log n)^{-1}\}, \\ p &= \max\{\lceil n\rho^{2/r}(\lceil n^{1/4} \rceil)(\lambda(n))^{-1} \rceil, \lceil n^{3/4}(\lambda(n))^{-1} \rceil\}, \\ q &= \lceil n^{1/4} \rceil,\end{aligned}$$

then all the conditions in (5) are satisfied;

- a) $p \rightarrow \infty, q \rightarrow \infty, n^{-1}p \rightarrow 0, p^{-1}q \rightarrow 0,$
- b) $n^{1+\beta}q^{1-\beta}p^{-2} \leq n^{-(1-3\beta)/4}\lambda^2(n) \rightarrow 0,$ if $\beta \leq 1/3,$
- c) $np^{-1}\rho^{2/r}(q) \leq \lambda(n) \rightarrow 0.$

Now we break the sum S'_n into two parts. We denote by \mathcal{L}_{nj} the σ -fields $\mathcal{P}_{jp+(j-1)q}$, and define the random variables

$$Y_{nj} = y_{nj} - E\{y_{nj} | \mathcal{L}_{n,j-1}\}, \quad 1 \leq j \leq k,$$

where $y_{nj} = y_j / \sigma_n$. Then $\{Y_{nj}, \mathcal{L}_{nj} : 1 \leq j \leq k\}$ is trivially a martingale difference sequence for each $n \geq 1$. Let

$$T_n = \sum_{j=1}^k Y_{nj}, \quad T'_n = S'_n / \sigma_n - T_n = \sum_{j=1}^k E\{y_{nj} | \mathcal{L}_{n,j-1}\}.$$

Then $S_n = \sigma_n T_n + \sigma_n T'_n + S''_n$.

The theorem will be proved in three stages:

- (i) $E|S''_n / \sigma_n|^r \rightarrow 0,$
- (ii) $E|T'_n|^r \rightarrow 0,$
- (iii) $E|T_n|^r \rightarrow \beta_r.$

In view of (i), (ii), (iii) and the inequality:

$$|(E|S_n / \sigma_n|^r)^{1/r} - (E|T_n|^r)^{1/r}|^r \leq 2r^{-1}(E|T'_n|^r + E|S''_n / \sigma_n|^r),$$

the assertion of the theorem follows.

PROOF OF (i). Since $\sigma_n^2 = nh(n)$, where $h(n)$ is a slowly varying function (Theorem B), using Lemma 2, Minkowski's inequality and stationarity, and arguing as in [11, p. 337], we obtain

$$\begin{aligned}E|S''_n / \sigma_n|^r &\leq \sigma_n^{-r}(k(E|z_1|^r)^{1/r} + (E|z_{k+1}|^r)^{1/r})^r \\ &\leq K(k\sigma_q / \sigma_n + \sigma_{q'} / \sigma_n)^r \\ &= K\left(\left(\frac{k^2 q h(q)}{nh(n)}\right)^{1/2} + \left(\frac{q' h(q')}{nh(n)}\right)^{1/2}\right)^r \rightarrow 0,\end{aligned}$$

where $q' = n - k(p+q)$ is the number of terms in z_{k+1} , and (i) is proved.

Before proving (ii) and (iii), we note that under the requirements imposed on p, q and k ,

$$(6) \quad k\sigma_p^2/\sigma_n^2 = 1 + o(1).$$

In fact,

$$E(S'_n/\sigma_n)^2 = k\sigma_p^2/\sigma_n^2 + 2 \sum_{j=2}^k (k-j+1)E(y_{n_1}y_{n_j})$$

by stationarity. Since y_{n_1} is measurable with respect to \mathcal{F}_p and y_{n_j} , $2 \leq j \leq k$, are measurable with respect to \mathcal{F}_{p+q} , applying the inequality (2),

$$\sum_{j=2}^k (k-j+1)|E(y_{n_1}y_{n_j})| \leq (k\sigma_p/\sigma_n)^2 \rho(q).$$

Moreover, by condition (5),

$$k\rho(q) \sim np^{-1}\rho(q) \leq np^{-1}\rho^{2/r}(q) \rightarrow 0.$$

Hence

$$(7) \quad E(S'_n/\sigma_n)^2 = (k\sigma_p^2/\sigma_n^2)(1 + o(1)).$$

On the other hand,

$$(8) \quad \begin{aligned} E(S'_n/\sigma_n)^2 &= E(S_n/\sigma_n)^2 + E(S''_n/\sigma_n)^2 - 2E(S_n S''_n/\sigma_n^2) \\ &= 1 + o(1) \end{aligned}$$

by (i). The equality (6) now follows from (7) and (8).

PROOF OF (ii). For simplicity we put

$$w_{n_j} = E\{y_{n_j} | \mathcal{L}_{n, j-1}\}, \quad 1 \leq j \leq k,$$

and because of the stationarity, we put

$$a_n = E|y_{n_j}|^r, \quad 1 \leq j \leq k.$$

By Hölder's inequality,

$$\begin{aligned} E|w_{n_j}|^r &= E(w_{n_j}w_{n_j} | w_{n_j}|^{r-2}) \\ &= E(E\{y_{n_j}w_{n_j} | w_{n_j}|^{r-2} | \mathcal{L}_{n, j-1}\}) \\ &= E(y_{n_j}w_{n_j} | w_{n_j}|^{r-2}) \\ &\leq (E|y_{n_j}w_{n_j}|^{r/2})^{2/r} (E|w_{n_j}|^r)^{1-2/r}, \end{aligned}$$

so that we have

$$E|w_{n_j}|^r \leq E|y_{n_j}w_{n_j}|^{r/2}.$$

Since w_{n_j} is measurable with respect to $\mathcal{F}_{(j-1)p+(j-2)q}$ and y_{n_j} is measurable with respect to $\mathcal{F}_{(j-1)(p+q)}$ for each $1 \leq j \leq k$, using (2) and Jensen's inequality,

$$\begin{aligned} E|y_{n_j}w_{n_j}|^{r/2} &\leq (E|y_{n_j}|^r)^{1/2} (E|w_{n_j}|^r)^{1/2} \rho(q) + E|y_{n_j}|^{r/2} E|w_{n_j}|^{r/2} \\ &\leq a_n \rho(q) + a_n^{1/2} E|w_{n_j}|^{r/2}. \end{aligned}$$

Using (2) and Jensen's inequality again,

$$\begin{aligned} E|w_{nj}|^{r/2} &= E(y_{nj}w_{nj}|w_{nj}|^{r/2-2}I(|w_{nj}|>0)) \\ &\leq (Ey_{nj}^2)^{1/2}(E|w_{nj}|^{r-2})^{1/2}\rho(q) \\ &\leq a_n^{1/2}\rho(q). \end{aligned}$$

Combining the estimates above, we find that

$$(9) \quad E|w_{nj}|^r \leq 2a_n\rho(q).$$

We obtain from Minkowski's inequality, (4)-(6) and (9) that

$$\begin{aligned} E|T'_n|^r &\leq \left(\sum_{j=1}^k (E|w_{nj}|^r)^{1/r}\right)^r \\ &\leq 2k^r a_n\rho(q) \\ &\leq K(k\sigma_p^2/\sigma_n^2)^{r/2} k^{r/2}\rho(q) \rightarrow 0, \end{aligned}$$

and hence (ii) is proved.

PROOF OF (iii). Define w_{nj} and a_n as before. For simplicity of notation we also define

$$\begin{aligned} u_{nj} &= y_{nj}^2 - Ey_{nj}^2, \quad 1 \leq j \leq k, \\ v_{nj} &= E\{u_{nj} | \mathcal{L}_{n,j-1}\}, \quad 1 \leq j \leq k \end{aligned}$$

and (because of the stationarity)

$$b_n = E|u_{nj}|^{r/2}, \quad 1 \leq j \leq k.$$

Now by stationarity,

$$\sum_{j=1}^k EY_{nj}^2 = k\sigma_p^2/\sigma_n^2 - \sum_{j=1}^k E(y_{nj}w_{nj}),$$

and using (2),

$$\sum_{j=1}^k E(y_{nj}w_{nj}) \leq (k\sigma_p^2/\sigma_n^2)\rho(q).$$

Thus, taking account of (6), we see that

$$\sum_{j=1}^k EY_{nj}^2 = 1 + o(1).$$

Therefore, according to Hall's [8] theorem, the proof of (iii) will be complete if we can show that

$$(10) \quad \max_{j \leq k} E\{Y_{nj}^2 | \mathcal{L}_{n,j-1}\} \rightarrow 0 \quad \text{in probability,}$$

$$(11) \quad \sum_{j=1}^k E|Y_{nj}|^r \rightarrow 0$$

and

$$(12) \quad E \left| \sum_{j=1}^k E \{ Y_{nj}^2 | \mathcal{L}_{n,j-1} \} - 1 \right|^{\tau/2} \rightarrow 0.$$

However, (11) immediately implies the conditional Lindeberg condition:

$$\text{for all } \varepsilon > 0, \quad \sum_{j=1}^k E \{ Y_{nj}^2 I(|Y_{nj}| > \varepsilon) | \mathcal{L}_{n,j-1} \} \rightarrow 0 \quad \text{in probability.}$$

Hence (10) is a consequence of (11) combined with (12) (see Brown [6, Theorem 1 and Lemma 1]). We have from Jensen's inequality, (4) and (6) that

$$\begin{aligned} \sum_{j=1}^k E |Y_{nj}|^r &\leq 2^{r-1} \sum_{j=1}^k (E |y_{nj}|^r + E |w_{nj}|^r) \\ &\leq 2^r k a_n \\ &\leq K(k \sigma_p^2 / \sigma_n^2)^{r/2} k^{-r/2+1} \rightarrow 0, \end{aligned}$$

and thus (11) holds.

Our goal is to show that (12) holds. Now,

$$\begin{aligned} &E \left| \sum_{j=1}^k E \{ Y_{nj}^2 | \mathcal{L}_{n,j-1} \} - 1 \right|^{\tau/2} \\ &= E \left| \sum_{j=1}^k E \{ y_{nj}^2 | \mathcal{L}_{n,j-1} \} - \sum_{j=1}^k w_{nj}^2 - 1 \right|^{\tau/2} \\ &\leq 2^{r/2-1} \left\{ E \left| \sum_{j=1}^k E \{ y_{nj}^2 | \mathcal{L}_{n,j-1} \} - 1 \right|^{\tau/2} + E \left(\sum_{j=1}^k w_{nj}^2 \right)^{\tau/2} \right\}. \end{aligned}$$

Making use of the inequality (9), and arguing just as in the proof of (ii), we get

$$\begin{aligned} E \left(\sum_{j=1}^k w_{nj}^2 \right)^{\tau/2} &\leq 2k^{\tau/2} a_n \rho(q) \\ &\leq K(k \sigma_p^2 / \sigma_n^2)^{\tau/2} \rho(q) \rightarrow 0. \end{aligned}$$

Moreover, we have from (6) and Minkowski's inequality that

$$\begin{aligned} &E \left| \sum_{j=1}^k E \{ y_{nj}^2 | \mathcal{L}_{n,j-1} \} - 1 \right|^{\tau/2} \\ &\sim E \left| \sum_{j=1}^k E \{ y_{nj}^2 | \mathcal{L}_{n,j-1} \} - k \sigma_p^2 / \sigma_n^2 \right|^{\tau/2} \\ &= E \left| \sum_{j=1}^k v_{nj} \right|^{\tau/2} \\ &\leq \left(\sum_{j=1}^k (E |v_{nj}|^{\tau/2})^{2/\tau} \right)^{\tau/2}. \end{aligned}$$

Consequently, to prove (12) it is sufficient to show that

$$(13) \quad \sum_{j=1}^k (E |v_{nj}|^{\tau/2})^{2/\tau} \rightarrow 0.$$

We shall separate the proof of (13) in three cases; $r > 4$, $r = 4$ and $2 < r < 4$.

Suppose first that $r > 4$. By replacing y_{nj} , u_{nj} , a_n and r in the proof of (9) by u_{nj} , v_{nj} , b_n and $r/2$ respectively, we deduce that

$$E|v_{nj}|^{r/2} \leq 2b_n \rho(q).$$

Since

$$b_n \leq 2^{r/2-1} \{E|y_{nj}|^r + (E y_{nj}^2)^{r/2}\} \leq 2^{r/2} a_n,$$

then, by virtue of (4) and (6), we see that

$$(14) \quad \begin{aligned} \sum_{j=1}^k (E|v_{nj}|^{r/2})^{2/r} &\leq k(2b_n \rho(q))^{2/r} \\ &\leq k(2^{r/2+1} a_n \rho(q))^{2/r} \\ &\leq K(k \sigma_p^2 / \sigma_n^2) \rho^{2/r}(q) \rightarrow 0, \end{aligned}$$

and thus (13) is proved for the case $r > 4$.

When $r = 4$, using (2) and Jensen's inequality, we get

$$\begin{aligned} E|v_{nj}|^{r/2} &= E(u_{nj} v_{nj}) \\ &\leq (E u_{nj}^2)^{1/2} (E v_{nj}^2)^{1/2} \rho(q) \\ &\leq E u_{nj}^2 \rho(q). \end{aligned}$$

Hence (13) also holds for $r = 4$.

Finally, we assume that $2 < r < 4$. By Hölder's inequality,

$$(15) \quad \begin{aligned} E|v_{nj}|^{r/2} &= E(u_{nj} v_{nj} | v_{nj}|^{r/2-2} I(|v_{nj}| > 0)) \\ &\leq E(|u_{nj}|^{2-r/2} |u_{nj} v_{nj}|^{r/2-1}) \\ &\leq b_n^{4/r-1} (E|u_{nj} v_{nj}|^{r/4})^{2-4/r}. \end{aligned}$$

Using (2) and Jensen's inequality, and noting that $r/4 < 1$,

$$(16) \quad \begin{aligned} E|u_{nj} v_{nj}|^{r/4} &\leq (E|u_{nj}|^{r/2})^{1/2} (E|v_{nj}|^{r/2})^{1/2} \rho(q) + E|u_{nj}|^{r/4} E|v_{nj}|^{r/4} \\ &\leq b_n \rho(q) + b_n^{1/2} (E|v_{nj}|)^{r/4}. \end{aligned}$$

Since $\alpha(n) \leq \rho(n)$, applying the inequality (3) with $\xi = v_{nj} |v_{nj}|^{-1} I(|v_{nj}| > 0)$, $\eta = u_{nj}$ and $s = r/2$,

$$(17) \quad \begin{aligned} E|v_{nj}| &= E(u_{nj} v_{nj} | v_{nj}|^{-1} I(|v_{nj}| > 0)) \\ &\leq 6(E|u_{nj}|^{r/2})^{2/r} (\rho(q))^{1-2/r}. \end{aligned}$$

Inserting the inequalities (16) and (17) into (15), we have

$$\begin{aligned} E|v_{nj}|^{r/2} &\leq b_n^{4/r-1} \{b_n \rho(q) + 6^{r/4} b_n (\rho(q))^{r/4-1/2}\}^{2-4/r} \\ &\leq K b_n (\rho(q))^{(r-2)^2/2r}. \end{aligned}$$

Just as in (14), we obtain that for $2 < r < 4$,

$$\sum_{j=1}^k (E|v_{nj}|^{r/2})^{2/r} \leq K(k\sigma_p^2/\sigma_n^2)(\rho(q))^{(r-2)^2/r^2} \rightarrow 0,$$

and hence (13) follows as desired.

The proof of Theorem C is now complete.

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