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EXPANSIVENESS OF REAL FLOWS

Dedicated to Professor Yukihiro Kodama on his sixtieth birthday

By

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§ 1. Introduction.

Expansive transformations play important roles in topological dynamics. However there are several notions for expansiveness of real flows and the relationships between them have not been clarified enough. We investigate in this paper the relationship between some of expansive notions and show that the notions are unified into two kinds of expansiveness (Theorem and Theorem A). One is the expansiveness introduced by R. Bowen and P. Walters [2] and another is the weak expansiveness found in [5] in investigating the geometric Lorentz flow introduced by J. Guckenheimer [3].

Let X be a compact metric space with metric d and $\boldsymbol{R}$ denote the additive group of real numbers. A map $F: X\times\boldsymbol{R}\rightarrow X$ is called a *flow* on X if F is continuous and $f_{t+s}x=f_{t}(f_{s}x)$, $f_{0}x=x$ for every $t, s\in\boldsymbol{R}$ and $x\in X$, where $f_{t}x=F(x, t)$.

R. Bowen and P. Walters introduced in [2] the notion of expansiveness as follows: A flow F is *expansive* if for any $\varepsilon>0$ there exists $\delta>0$ such that if $x, y\in X$ satisfy $d(f_{t}x, f_{s(t)}y)<\delta$ $(t\in\boldsymbol{R})$ for some continuous map $s; \boldsymbol{R}\rightarrow\boldsymbol{R}$ with $s(0)=0$, then $y=f_{t}x$ for some $|t|<\varepsilon$.

$f_{I}(S)=\{f_{t}x; t\in I, x\in S\}$ for an interval I and $S\subset X$. A flow F on X is called *weakly expansive* if F satisfies the property that for any $\varepsilon>0$ there exists $\delta>0$ with the property that if there exist a pair of points $x, y\in X$ and a strictly increasing surjective homeomorphism $h: \boldsymbol{R}\rightarrow\boldsymbol{R}$ with $h(0)=0$ such that $d(f_{t}x, f_{h(t)}y)<\delta$ for every $t\in\boldsymbol{R}$, then $f_{h(t_{0})}y\in f_{(t_{0}-\varepsilon, t_{0}+\varepsilon)}(\{x\})$ for some $t_{0}\in\boldsymbol{R}$.

THEOREM (R. Bowen and P. Walters [2]). *The following are equivalent for a flow F.*

(i) *F is expansive.*

(ii) *For any $\varepsilon>0$ there exists $\delta>0$ such that if $x, y\in X$ satisfy $d(f_{t}x, f_{s(t)}y)$*

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$<\delta$ $(t\in \boldsymbol{R})$ *for some continuous map* $s: \boldsymbol{R}\rightarrow\boldsymbol{R}$ *with* $s(0)=0$, *then* y *is in the same orbit as* x *and the orbit from* x *to* y *lies inside* $B_{\varepsilon}(x)=\{y\in X;\ d(x, y)\leqq\varepsilon\}$.

(iii) *For any* $\varepsilon>0$ *there exists* $\delta>0$ *with the property: for* $\boldsymbol{t}=(t_i)_{i=-\infty}^{\infty}$ *and* $\boldsymbol{u}=(u_i)_{i=-\infty}^{\infty}$ *to which satify*

$$t_0=u_0=0, \qquad 0<t_{i+1}-t_i\leqq\delta, \qquad |u_{i+1}-u_i|\leqq\delta.$$

and

$$t_i \longrightarrow \infty, \qquad t_{-i} \longrightarrow -\infty \qquad \text{as} \quad i\rightarrow\infty$$

if $d(f_{t_i}x, f_{u_i}y)<\varepsilon$ $(i\in \boldsymbol{Z})$, *then* $y=f_t x$ *for some* $|t|<\varepsilon$.

(iv) *For any* $\varepsilon>0$ *there exists* $\delta>0$ *such that if* $x, y\in X$ *satisfy* $d(f_t x, f_{h(t)}y)$ $<\delta$ $(t\in \boldsymbol{R})$ *for some strictly increasing surjective homeomorphism* $h: \boldsymbol{R}\rightarrow\boldsymbol{R}$ *with* $h(0)=0$, *then* $y=f_t x$ *for some* $|t|<\varepsilon$.

Theorem was first proved by R. Bowen and P. Walters [2] for flows without fixed points. However Theorem is true for all real flows dropped the condition of fixed points. In § 2 we shall explain that reason.

THEOREM A. *A weakly expansive flow without fixed points must be an expansive flow.*

The proof of Theorem A will be proceeded in § 3. By Theorem A it seems likely that some of properties obtained for expansive flow hold also for weakly expansive flows with fixed points. Concerning with topological entropy Theorem B below is a natural extension of Theorem 5 [2] for weakly expansive flows. $h(F)$ is the topological entropy of F and $\nu(t)$ denotes the number of closed orbits of F with a period $\tau\in[0, t]$. It is easily checked that a weakly expansive flow is h-expansive in the sense of [1] and so Theorem B is readily confirmed according as the proof of Theorem 5 [2].

THEOREM B. *Let* F *be a weakly expansive flow. Then*

$$\limsup_{t\rightarrow\infty}\,[(1/t)\log\nu(t)]\leqq h(F)<\infty.$$

§ 2. Proof for the case with fixed points.

A point $x\in X$ is called a *periodic point* if $f_t x=x$ for some $t>0$, and called a *fixed point* if $f_t x=x$ for any $t\in \boldsymbol{R}$. The smallest $t>0$ with $f_t x=x$ is called the *period* of a point x. Fix(F) denotes the set of all fixed point of F.

Each fixed point is an isolated point of X for both expansive flows and flows with the property (ii) of Theorem (c.f. [2, Lemma 1]). Since (i) and (ii) are equivalent for a flow F without fixed point by [2, Theorem 3], it is easily checked that (i) and (ii) are equivalent.

That (i)→(iii) and (iii)→(iv) has been essentially proved in the proof of [2, Theorem 3]. We show that (iv)→(i).

Assume that F satisfies the property (iv). Let $\delta>0$ be a constant of the property (iv) for 1. Put $\delta_0=\delta/3$. $U_r(x)$ is an open ball with radius r and center $x\in X$.

LEMMA 1. *If X is connected and* Card$(X)\geqq 2$, *then F has no fixed points.*

PROOF. Assume that F has a fixed point x_0. We have by the property (iv) that there is no fixed point in $U_\delta(x_0)\setminus\{x_0\}$ and that if $x\in U_{\delta_0}(x_0)\setminus\{x_0\}$, then $d(f_t x, x_0)\geqq 2\delta_0$ for some $t\in \boldsymbol{R}$.

For any $x\in X$, put

$$T(x)=\inf\{|t| : d(f_t x, x_0)\geqq\delta_0\}.$$

Obviously $T(x)>0$ for any $x\in U_{\delta_0/2}(x_0)$. By the facts that X is connected and *Card* $(X)\geqq 2$ and that x_0 is a fixed point of F, we have $y_0\in U_{\delta_0/2}(x_0)$ with $T(y_0)\geqq 2$.

Put

$$T_0=\inf\{t>0\,;\, d(f_t y_0, x_0)=\delta_0\}$$

and

$$S_0=\sup\{t<0\,;\, d(f_t y_0, x_0)=\delta_0\}.$$

Then $T_0\geqq 2$ and $S_0\leqq -2$, where $\inf\phi=\infty$ and $\sup\phi=-\infty$.

Let $y_1=f_1 y_0$. Take a strictly increasing surjective homeomorphism $h:\boldsymbol{R}\to\boldsymbol{R}$ with $h(0)=0$ such that $h(t)=t-1$ in $|t|\geqq 2$, $h(t)=t/2$ in $0\leqq t\leqq 2$ and $h(t)=3t/2$ in $-2\leqq t\leqq 0$. We have

$$\begin{aligned} d(f_t y_0, f_{h(t)} y_1) &= d(f_t y_0, f_{h(t)+1} y_0) \\ &\leqq d(f_t y_0, x_0)+d(x_0, f_{h(t)+1} y_0) \\ &\leqq \delta_0+\delta_0<\delta \end{aligned}$$

for $|t|\leqq 2$. Obviously

$$d(f_t y_0, f_{h(t)} y_1)=d(f_t y_0, f_{h(t)+1} y_0)=0$$

for $|t|\geqq 2$. Hence $d(f_t y_0, f_{h(t)} y_1)<\delta$ for any $t\in\boldsymbol{R}$. Therefore we have that $y_1=f_t y_0$ for some $|t|<1$ by the property (iv), From this fact we have that $f_{1-t} y_0=y_0$ and $1-t<2$. Hence y_0 is a periodic point of F with period $1-t$. Since $\{f_t y_0 : -2\leqq t\leqq 2\}\subset B_{\delta_0}(x_0)$, we have that $d(f_t y_0, x_0)\leqq\delta_0$ for any $t\in\boldsymbol{R}$.

This contradicts that $d(f_t y_0, x_0) \geqq 2\delta_0$ for some $t \in \boldsymbol{R}$. Hence F has no fixed point.

LEMMA 2. *Each fixed point of F is an isolated point of X.*

PROOF. Assume that there exists $x_0 \in Fix(F)$ which is not isolated in X. Take $x_n \in B_{1/n}(x_0) \setminus \{x_0\}$ for each $n \in \boldsymbol{N}$, where $\boldsymbol{N}$ is the set of all positive integers. $\mathcal{C}(X)$ denotes the set of all non-empty closed subsets of X. Let $C(x_n)$ be the connected component of x_n in X. Obviously $C(x_n) \in \mathcal{C}(X)$.

Since $\mathcal{C}(X)$ is compact with respect to the Hausdorff metric, we can assume that $C(x_n)$ converges to some $C \in \mathcal{C}(X)$ when n goes to ∞. Then C is connected.

We claim that $\{x_0\} \subsetneqq C$. Indeed, if $C = \{x_0\}$, then $C(x_n) \subset B_\delta(x_0)$ for sufficiently large n, where $\delta > 0$ is a constant of the property (iv) for 1. Since $f_t(x_n) \in C(x_n)$ for any $t \in \boldsymbol{R}$, we have that $x_n = x_0$ by the property (iv), which contradicts that $x_n \neq x_0$.

Since $C(x_0)$ is the connected component of x_0 in X, $C(x_0) \supset C \supsetneqq \{x_0\}$. Hence $\mathrm{Card}(C(x_0)) \geqq 2$. We have that $f_t(C(x_0) = C(x_0)$ for any $t \in \boldsymbol{R}$. Hence F induces a flow on $C(x_0)$ with a fixed point x_0. Since $C(x_0)$ is compact and connected and $\mathrm{Card}(C(x_0)) \geqq 2$, this contradicts Lemma 1.

Since F satisfies the property (iv), F has finitely many fixed points. Put $\mathrm{Fix}(F) = \{x_1, \cdots, x_k\}$. By Lemma 2, $X \setminus \{x_1, \cdots, x_k\}$ is a compact invariant set of F. F is a flow on $X \setminus \{x_1, \cdots, x_k\}$ with the property (iv) and has no fixed points on it. Hence F is expansive by [2, Theorem 3]. Take $\varepsilon > 0$. Let α be an expansive constant of F on $X \setminus \{x_1, \cdots, x_k\}$ for ε.

Put $\delta_i = d(x_i, X \setminus \{x_i\})$ for $i = 1, \cdots, k$, where $d(x, X \setminus \{x\}) = \inf\{d(x, y); y \in X \setminus \{x\}\}$. Then $\delta_i > 0$ from Lemma 2. Put $\alpha_0 = \min\{\alpha, \delta_1, \cdots, \delta_k\}$. We show that α_0 is an expansive constant of F for ε.

Assume that there is a continuous map $s: \boldsymbol{R} \to \boldsymbol{R}$ with $s(0) = 0$ such that $d(f_t x, f_{s(t)} y) < \alpha_0$ for any $t \in \boldsymbol{R}$. If $x \in \mathrm{Fix}(F)$, then by the property (iv) we have that $y = x$. Hence $y = f_0 x$. Similarly $y = f_0 x$ when $y \in \mathrm{Fix}(F)$. If $x, y \in \mathrm{Fix}(F)$, then $y = f_t x$ for some $|t| < \varepsilon$ since $\alpha_0 \leqq \alpha$. Hence F must be an expansive flow on X. The proof is completed.

§ 3. Proof of Theorem A.

For a continuous flow F we define

$$\varepsilon_0(F) = \inf\{t > 0;\ F(t, x) = x \quad \text{for some } x \in X \setminus \mathrm{Fix}(F)\}.$$

Obviously $\varepsilon_0(F)=\infty$ if F has no periodic points except fixed points.

LEMMA 3. *The following* (i) *and* (ii) *are equivalent.*

(i) *F is weakly expansive.*

(ii) *For any $\varepsilon>0$ there exists $\alpha_0>0$ such that the following holds: for α with $0<\alpha\leqq\alpha_0$ and for $\boldsymbol{t}=(t_i)_{i=-\infty}^{\infty}$ and $\boldsymbol{u}=(u_i)_{i=-\infty}^{\infty}$ to which satisfy*

$$t_0=u_0=0, \qquad 0<t_{i+1}-t_i\leqq\alpha, \qquad 0<u_{i+1}-u_i\leqq\alpha$$

and

$$t_i,\ u_i \longrightarrow \infty, \qquad t_{-i},\ u_{-i} \longrightarrow -\infty \quad (i\to\infty),$$

if $d(f_{t_i}x, f_{u_i}y)<\alpha$ $(i\in\boldsymbol{Z})$ thex there exists $i\in\boldsymbol{Z}$ such that

$$f_{u_i}y\in f_{(-\varepsilon-\alpha+t_i,\, t_i+\alpha+\varepsilon)}(\{x\}).$$

PROOF. (i)→(ii). For any $\varepsilon>0$ take $\alpha_0>0$ such that

$$\alpha_0+2\sup\{d(z, f_u z);\ z\in X,\ |u|\leqq\alpha_0\}<\delta,$$

where δ is an expansive constant of F for ε. We assume that $\boldsymbol{t}=(t_i)_{i=-\infty}^{\infty}$ and $\boldsymbol{u}=(u_i)_{i=-\infty}^{\infty}$ satisfy the assumption of (ii). Let h be a strictly increasing surjective homeomorphism of $\boldsymbol{R}$ with $h(t_i)=u_i$ for any $i\in\boldsymbol{Z}$. Then we have

$$\begin{aligned} d(f_t x, f_{h(t)}y) &\leqq d(f_t x, f_{t_i}x)+d(f_{t_i}x, f_{u_i}y)+d(f_{u_i}y, f_{h(t)}y) \\ &\leqq \alpha+2\sup\{d(z, f_u z);\ z\in X,\ |u|\leqq\alpha\} \\ &<\delta \end{aligned}$$

for $t_i\leqq t\leqq t_{i+1}$. Since F is weakly expansive by (i), we have $f_{h(t_0)}y\in f_{(t_0-\varepsilon,\, t_0+\varepsilon)}(\{x\})$ for some $t_0\in\boldsymbol{R}$. By the definition of h there is $i\in\boldsymbol{Z}$ such that $t_i\leqq t_0\leqq t_{i+1}$ and $u_i\leqq h(t_0)\leqq u_{i+1}$. Hence $f_{u_i}y=f_{u_i-h(t_0)}\circ f_{h(t_0)}y=f_{u_i-h(t_0)+t}x$ for some $t\in(t_0-\varepsilon, t_0+\varepsilon)$. Put $s=u_i-h(t_0)+t$. Then

$$\begin{aligned} s &> u_i-h(t_0)+t_0-\varepsilon \geqq u_i-h(t_0)+t_i-\varepsilon \\ &\geqq t_i-\alpha-\varepsilon, \\ s &< u_i-h(t_0)+t_0+\varepsilon \leqq u_i-h(t_0)+t_{i+1}+\varepsilon \\ &\leqq u_i-h(t_0)+t_i+\alpha+\varepsilon \\ &\leqq t_i+\alpha+\varepsilon. \end{aligned}$$

Therefore there is $i\in\boldsymbol{Z}$ such that

$$f_{u_i}y\in f_{(-\varepsilon-\alpha+t_i,\, t_i+\alpha+\varepsilon)}(\{x\}).$$

(ii)→(i) Take and fix $\varepsilon>0$. Let $\alpha_0>0$ satisfy (ii) for $\varepsilon/2$ and let $0<\alpha\leqq$

$\min\{\alpha_0, \varepsilon/2\}$. Then it is enough to show that α is an expansive constant of F for ε.

Assume that h is a strictly increasing surjective homeomorphism of $\boldsymbol{R}$ with $h(0)=0$, and that $d(f_t x, f_{h(t)} y)<\alpha$ for any $t\in\boldsymbol{R}$. Choose $(t_i)_{i=-\infty}^{\infty}$ such that $t_0=0$, $0<t_{j+1}-t_i\leqq\alpha$ and $0<h(t_{i+1})-h(t_i)\leqq\alpha$ and that t_i, $h(t_i)\to\infty$ and t_{-i}, $h(t_{-i})\to-\infty$ as $i\to\infty$. Then we have $d(f_{t_i}x, f_{u_i}y)<\alpha$ where $u_i=h(t_i)$ for $i=\boldsymbol{Z}$. By (ii) there exists $i\in\boldsymbol{Z}$ such that $f_{u_i}y\in f_{(-\varepsilon/2-\alpha+t_i, t_i+\alpha+\varepsilon/2)}(\{x\})$. Thus $f_{h(t_i)}y\in f_{(t_i-\varepsilon, t_i+\varepsilon)}(\{x\})$. This implies that F is weakly expansive.

For the rest of this section F is assumed to be a continuous flow on X without fixed points.

A subset $S\subset X$ is called a *local cross-section* of time $\zeta>0$ for a continous flow F if S is closed and $S\cap f_{[-\zeta,\zeta]}(\{x\})=\{x\}$ for all $x\in S$, where $\zeta<\varepsilon_0(F)/2$. If S is a local cross-section of time ζ, F maps $S\times[-\zeta, \zeta]$ homeomorphically onto $f_{[-\zeta,\zeta]}(S)$. By the interior S^* of S we mean the set $S\cap\operatorname{int} f_{[-\zeta,\zeta]}(S)$. Note that $f_{(-\varepsilon,\varepsilon)}(S^*)$ is open in X for any $\varepsilon>0$.

LEMMA 4 ([4]). *Under the above notations and assumptions, there is a* $0<\zeta<\varepsilon_0(F)/2$ *such that for each* $\alpha>0$ *we can find a finite family* $\mathcal{G}=\{S_1, S_2, \cdots, S_k\}$ *of pairwise disjoint local cross-sections of time* ζ *and diameter at most* α, *and a finite family of local cross-sections* $\mathcal{T}=\{T_1, T_2, \cdots, T_k\}$ *with* $T_i\subset S_i^*$ $(i=1,2,\cdots,k)$ *such that*

$$X=f_{[0,\alpha]}(T^+)=f_{[-\alpha,0]}(T^+)=f_{[0,\alpha]}(S^+)=f_{[-\alpha,0]}(S^+).$$

where $T^+=\bigcup_{i=1}^{k} T_i$ *and* $S^+=\bigcup_{i=1}^{k} S_i$.

Hereafter let $0<3\alpha<\zeta$ and β be the minimum time between sections of $\mathcal{G}$ i.e.

$$\beta=\sup\{\delta>0\,;\ f_{(0,\delta)}(\{x\})\cap S^+=\phi \text{ for } x\in S^+\}.$$

Obviously $0<\beta\leqq\alpha$. Let ρ satisfy $0<2\rho<\beta$.

For $x\in T^+$ let t be the smallest positive number such that $f_t(x)\in T^+$. Then we can define a *first return map* φ by $\varphi(x)=f_t x$. It is easily checked that $\varphi: T^+\to T^+$ is bijective but not continuous. Note that $\beta\leqq t\leqq\alpha$.

For $S_i\in\mathcal{G}$, let $D_\rho^i=f_{[-\rho,\rho]}(S_i)$ and define a projective map $P_\rho^i: D_\rho^i\to S_i$ by $P_\rho^i(x)=f_t x$, where $f_t x\in S_i$ and $|t|\leqq\rho$. Since $2\rho<\zeta$, P_ρ^i is well defined and onto continuous. We write $D_\rho=D_\rho^i$ and $P_\rho=P_\rho^i$ if there is no confusion.

LEMMA 5. *There is an* $0<a<\beta/2$ *such that for* $x, y\in S_i$ *if* $d(x, y)\leqq a$ *and* $f_t x\in T_j$ $(|t|\leqq 3\alpha)$ *for some* T_j, *then* $f_t y\in D_\rho^j$.

Proof is clear.

Using Lemma 5 we can set up a shadowing orbit of y relative to a φ-orbit of $x\in T^+$ as follows. If y is sufficiently close to x, the orbit of y will cross S_i at a time near the time when the orbit of x crosses T_i. For $x\in T_i$ and $y\in S_i$ with $d(x, y)\leqq a$, we can define a set of points $\{y_i\}$ where $y_0=y$ and $y_i=P_\rho(f_t y_{i-1})$, where t is the smallest positive time such that $\varphi^i(x)=f_t(\varphi^{i-1}(x))$, and we can continue this construction as long as $d(\varphi^i(x), y_i)\leqq a$. Then we obtain a time delated y shadow orbit along a piece of the orbit of x. We can also proceed the same construction as the above for negative powers of φ. For simplicity we write T, S instead of T_i, S_i respectively. Let $a>0$ be as in Lemma 5 and let $0<\eta<a$.

For $x\in T$ the η-stable set of x is

$$W^s_\eta(x)=\{y\in S\,;\ d(\varphi^i(x), y_i)<\eta \text{ for all } i\geqq 0\}$$

and the η-unstable set of x is

$$W^u_\eta(x)=\{y\in S\,;\ d(\varphi^i(x), y_i)<\eta \text{ for all } i\leqq 0\}.$$

LEMMA 6 ([4]). *F is expansive if and only if given collections of local cross-sections $\mathcal{G}$ and $\mathcal{T}$ (with ζ and $3\alpha<\zeta$) and $\rho>0$ (with $2\rho<\beta$), there is $\eta>0$ such that $W^s_\eta(x)\cap W^u_\eta(x)=\{x\}$ for any $x\in T^+$.*

PROOF OF THEOREM A. Since F has no fixed points, take $\zeta>0$ as in Lemma 4. Put $\varepsilon=\zeta/3$ and let $\alpha_0>0$ be as in Lemma 3 (ii). Now take α with $0<2\alpha<\min\{\zeta/3, \alpha_0\}$. For this $\alpha>0$, we can find local cross-sections $\mathcal{G}$ and $\mathcal{T}$ by Lemma 4.

Let β be as the above. Take $\rho>0$ such that $2\rho<\beta$. Let $0<a<\beta/2$ be as in Lemma 5 and let $0<\eta<\min\{a, 2\alpha\}$. To obtain the conclusion, it is enough to show that $W^s_\eta(x)\cap W^u_\eta(x)=\{x\}$ for any $x\in T^+$.

Let $y\in W^s_\eta(x)\cap W^u_\eta(x)$ for $x\in T^+$. Then we have $d(\varphi^i(x), y_i)>\eta$ for all $i\in \boldsymbol{Z}$. Let τ_i be the smallest positive number such that $\varphi^i(x)=f_{\tau_i}(\varphi^{i-1}(x))$ for $i\in\boldsymbol{Z}$. Since $\varphi^i(x)=\varphi(\varphi^{i-1}(x))=f_{\tau_i}(\varphi^{i-1}(x))$, we have $\beta\leqq\tau_i\leqq\alpha$. Since $d(\varphi^i(x), y_i)<\eta$ for $i\in\boldsymbol{Z}$, the time difference λ_i between y_{i+1} and y_i satisfies $|\lambda_i-\tau_i|\leqq\rho$. Thus

$$\beta/2=\beta-\beta/2<\tau_i-\rho\leqq\lambda_i\leqq\tau_i+\rho<\tau_i+\alpha\leqq 2\alpha\,. \tag{1}$$

Now define doubly infinite sequences $\boldsymbol{t}=(t_i)_{i=-\infty}^{\infty}$ and $\boldsymbol{u}=(u_i)_{i=-\infty}^{\infty}$ where

$$t_i(u_i)=\begin{cases} \sum_{k=0}^{i}\tau_k(\lambda_k) & \text{if } i\geqq 0 \\ 0 & \text{if } i=0 \\ -\sum_{k=0}^{i}\tau_k(\lambda_k) & \text{if } i\leqq 0\,. \end{cases}$$

Then we have $\beta\leqq t_{i+1}-t_1\leqq\alpha\leqq 2\alpha$ for $i\in\boldsymbol{Z}$ and from (1), $\beta/2\leqq u_{i+1}-u_i\leqq 2\alpha$. Hence t_i, $u_i\to\infty$ and t_{-i}, $u_{-i}\to-\infty$ as $i\to\infty$. Since

$$d(f_{t_i}x,\ f_{u_i}y)=d(\varphi^i(x),\ y_i)<\eta<2\alpha\quad(i\in\boldsymbol{Z}),$$

there is $i\in\boldsymbol{Z}$ such that $f_{u_i}y\in f_{(-\varepsilon-2\alpha+t_i,\,t_i+2\alpha+\varepsilon)}(\{x\})$ (by Lemma 3) and $1\leqq l\leqq k$ such that $f_{t_i}x\in T_l$. Since $\varepsilon+2\alpha=\zeta/3+2\alpha<2\zeta/3<3<\zeta$ and since S_l is a local cross-section of time ζ, we obtain $f_{u_i}y=f_{t_i}x$. By using induction on i we see that $x=y$ holds. Hence F is an expansive flow on X by Lemma 6.

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