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A CLASSIFICATION OF ORTHOGONAL TRANSFORMATION GROUPS OF LOW COHOMOGENEITY

Dedicated to Professor Ichiro Yokota on his 60th birthday

By

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Contents

1. Introduction

A *Lie transformation group* on a smooth manifold M is a pair (G, M) of a Lie group G which acts smoothly on M. This paper is concerned with the *cohomogeneity* (abbrev. *coh*) of (G, M), which is defined by

$$\mathrm{coh}\ (G, M)=\dim M-\dim G+\min\ \{\dim G_x;\ x\in M\},$$

where G_x is the isotropy subgroup of G at x. Then

$$\mathrm{coh}\ (G, M)\geqq\dim M-\dim G\ (=: doh\ (G, M)),$$

$\{x\in M;\ \mathrm{coh}\ (G, M)=\mathrm{doh}\ (G, M)+\dim G_x\}$ is an open subset of M, and

$$\mathrm{coh}\ (G^o, M)=\mathrm{coh}\ (G, M)$$

where G^o is the identity connected component of G.

An *orthogonal transformation group* (abbrev. *o.t.g.*) on an N dimensional Euclidean space $\boldsymbol{E}^N$ is defined as a pair $(G, \boldsymbol{E}^N)$ of a connected Lie subgroup G of the full orthogonal group $O(N)$ on $\boldsymbol{E}^N$. $(G, \boldsymbol{E}^N)$ is said to be *contained in* another o.t.g. $(G', \boldsymbol{E}^N)$ on $\boldsymbol{E}^N$ if there is a real linear isometry $\iota:\boldsymbol{E}^N\to\boldsymbol{E}^N$ and a Lie group monomorshism $\tau:G\to G'$ such that

$$\tau(g)\iota=\iota g \text{ for all } g\in G.$$

If moreover τ is a Lie group isomorphism, $(G, \boldsymbol{E}^N)$ is said to be *equivalent* to $(G', \boldsymbol{E}^N)$.

Let ρ be a linear representation on $\boldsymbol{R}^N$ over the field $\boldsymbol{R}$ of all real numbers of a Lie group G. We say $(G, \rho, \boldsymbol{R}^N)$ an *orthogonal linear triple* and ρ an *orthogonal representation* of G if there is a positive definite inner product on $\boldsymbol{R}^N$ which is invariant under the action of

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ρ(G). Suppose ρ' is another orthogonal representation of G. We call $(G, \rho', \boldsymbol{R}^N)$ and $(G, \rho, \boldsymbol{R}^N)$ are *equivalent as real representation* if ρ' and ρ are equivalent as real representations of G.

An orthogonal linear triple $(G, \rho, \boldsymbol{R}^N)$ naturally induces an o.t.g. $(\rho(G^o), \boldsymbol{E}^N)$ which is well defined up to equivalences and denoted by $O(G, \rho, \boldsymbol{R}^N)$. We denote

$$\mathrm{coh}\,(G, \rho, \boldsymbol{R}^N)=\mathrm{coh}\,(O(G, \rho, \boldsymbol{R}^N)),$$
$$\mathrm{doh}\,(G, \rho, \boldsymbol{R}^N)=\mathrm{doh}\,(O(G, \rho, \boldsymbol{R}^N)).$$

If G is compact, then any real representation of G is an orthogonal linear representation, and the corresponding o.t.g. is called a *compact linear group*.

An o.t.g. is called *maximal* if it is not properly contained in an o.t.g. of the same cohomogeneity. Suppose $(G, \boldsymbol{E}^N)$ is a maximal o.t.g. If it contains a compact linear group $(K, \boldsymbol{E}^N)$ of the same cohomogeneity, then itself is a compact linear group. In fact, the closure $\bar{G}$ of G in $O(N)$ is compact and

$$\mathrm{coh}\,(\bar{G}, \boldsymbol{E}^N)=\mathrm{coh}\,(G, \boldsymbol{E}^N)$$

since $\{x\in\boldsymbol{E}^N;\ G(x)$ is closed (i.e., $\bar{G}(x)=\overline{G(x)}=G(x)$), $\mathrm{coh}\,(G, \boldsymbol{E}^N)=N-\dim G+\dim G_x\}$ contains an open dense subset $\{x\in\boldsymbol{E}^N;\ \mathrm{coh}\,(K, \boldsymbol{E}^N)=N-\dim K+\dim K_x\}$ of $\boldsymbol{E}^N$.

Hsiang-Lawson [11] studied a classification of all compact linear groups of cohomogeneity 2 or 3 and maximal by means of the classification of compact linear groups which has a non trivial isotropy subgroup at a point of a principal orbit (cf. Kramer [15], Hsiang [10] and Hsiang-Hsiang [9]). As a result, most of them can be induced from the linear isotropy representations of Riemannian symmetric pairs.

Conversely, the linear isotropy representation of each Riemannian symmetric pair of rank r induces a compact linear group of cohomogeneity r(cf. Takagi-Takahashi [19]). Any of its orbit in the representation space is an *R-space* in the meaning of Takeuchi [20] (cf. Takeuchi-Kobayashi [21]). A *principal R-space* denotes an R-space of the highest dimension among all R-spaces associated with a given Riemannian symmetric pair.

From tables of Takagi-Takahashi [19, Table I and II], it appears that two principal R-spaces associated with two distinct Riemannian symmetric rairs of rank 2 are not equivalent as Riemannian manifolds nor Riemannian submanifolds of a hypersphere of the representation space. Especially if two maximal o.t.g.'s of cohomogeneity 2 contain o.t.g.'s from two distinct Riemannian symmetric pairs of rank 2 respectively, then they are not equivalent (cf. Ozeki-Takeuchi [17; Theorem 1, Theorem 2]).

However it is well known that the o.t.g. from the Riemannian symmetric pair $(G_2, SO(4))$ of rank 2 is missed in a theorem of Hsiang-Lawson [11; Theorem 5] (cf. Takagi-Takahashi [19], Uchida [23]). More than before, Uchida [23] pointed out many examples of real *reducible* (i.e., non irreducible) compact linear groups of cohomogeneity 3 which shows that another theorem of Hsiang-Lawson[11; Theorem 6] shoud be properly modified. Uchida [23; Theorem] also gave a classification theorem of real *reducible* compact linear groups of cohomogeneity 3 and maximal in a correct form by the use of a classification of compact Lie groups which act transitively on spheres (cf. Montgomery-Samelson [16], Borel [3], [4]).

In this paper, we study the classification of real *irreducible* o.t.g.'s of cohomogeneity at most 3 by a direct method (cf. Sato-Kimura [18], Yokota [25]). We have the list of them in Section 4, which shows that the other theorem of Hsiang-Lawson [11; Theorem 7] should be properly modified and also gives a classification of real irreducible compact linear groups of cohomogeneity 3 in a correct form (cf. Theorem 4.8, Remark 4.10).

Our results also give a proof of the fact that a compact linear group of cohomogeneity 2 and maximal is equivalent to an o.t.g. which is induced from the linear isotropy representation of a Riemannian symmetric pair of rank 2. Topologically, Asoh [2] has already completed the classification of compact Lie groups acting on spheres with an orbit of codimension one, which properly modified the result of H.C. Wang [26] (cf. Hsiang-Hsiang [8]). Recently, Dadok [5] classified real irreducible compact linear groups with certain property, so-called 'polar', which is satisfied by each compact linear group of cohomogeneity 2.

2. Preliminaries

For each type of compact simple Lie algebra of dimension g and rank k, we shall investigate (cf. Adams [1], Goto-Grosshans [6])

(1) 'Real' complex irreducible representations of degree m such that

$$d_0 := m - g \leqq 3,$$

(2) Complex irreducible representations of degree m such that

$$d_1 = 2m - g \leqq 4,$$

(3) 'Quaternion' complex irreducible representations of degree $2m$ such that

$$d_2 := 4m - g \leqq 6.$$

We denote a compact simple Lie algebra of type X_k by X_k ($X = A, B, C, D, E, F$, or G) and the corresponding compact simply connected Lie group also by X_k. A complex irreducible representation of the highest weight Λ is denoted by Λ. Especially the trivial representation is denoted by 0. The fundamental weights with respect to the simple roots $\alpha_1, \alpha_2, \cdots, \alpha_k$ are denoted by

$$\Lambda_1, \Lambda_2, \cdots, \Lambda_k.$$

(A)

The simple roots of A_k are given by a Dynkin diagram:

$$\alpha_1 - \alpha_2 - \cdots - \alpha_k \quad (k\geqq 1).$$

(1) 'Real' complex irreducible representations of A_k are given by

$$\Lambda=2\lambda_1\Lambda_1(\text{if } k=1),\ \Sigma_{i=1}^{h+1}\lambda_i(\Lambda_i+\Lambda_{k-i+1})(\text{if } k=2h+2),$$
$$\lambda_{2h+2}\Lambda_{2h+2}+\sum_{i=1}^{2h+1}\lambda_i(\Lambda_i+\Lambda_{k-i+1})(\text{if } k=4h+3),$$

or

$$2\lambda_{2h+3}\Lambda_{2h+3}+\sum_{i=1}^{2h+2}\lambda_i(\Lambda_i+\Lambda_{k-i+1})(\text{if } k=4h+5),$$

where h and $\lambda_i(i=1, \cdots, [(k+1)/2])$ are non-negative integers, and [p] denotes the maximal integer at most p.

PROPOSITION 2.1 *If* $d_0:=\deg \Lambda-k^2-2k\leqq 3$, *then* Λ *is equivalent as a complex representation of* $A_k(k\geqq 1)$ *to one of the followings*:

$$d_0<0:\ \Lambda_2(k=3),\quad 0(k\geqq 1),$$
$$d_0=0:\ 2\Lambda_1(k=1),\quad \Lambda_1+\Lambda_k(k\geqq 2),$$
$$d_0=2:\ 4\Lambda_1(k=1).$$

PROOF: If $\lambda_i\geqq 1$ for some $i=4, \cdots,$ or $[(k+1)/2]$, then $k\geqq 7$ and $d_0\geqq\deg \Lambda_4-k^2-2k\geqq {}_{k+1}C_4-k^2-2k\geqq 7$. If $[(k+1)/2]\geqq 3$ and $\lambda_3\geqq 1$, then $k\geqq 5$ and $d_0\geqq\deg(\Lambda_3+\Lambda_{k-2})-k^2-2k=(k+2)(k+1)^2k^2(k-4)/36-k^2-2k\geqq 140$. If $\lambda_2\geqq 1$ and $k\geqq 4$, then $d_0\geqq\deg(\Lambda_2+\Lambda_{k-1})-k^2-2k=(k+1)^2(k^2-4)/4-k^2-2k\geqq 51$. Therefore $\Lambda=0(k\geqq 1)$, $2\lambda_1\Lambda_1(k=1)$, $\lambda_1(\Lambda_1+\Lambda_k)(k\geqq 2)$, or $\lambda_2\Lambda_2+\lambda_1(\Lambda_1+\Lambda_3)(k=3)$. If $k=1$ and $\lambda_1\geqq 3$, then $d_0\geqq\deg 6\Lambda_1-3=4$. If $k\geqq 2$ and $\lambda_1\geqq 2$, then $d_0\geqq\deg 2(\Lambda_1+\Lambda_k)-k^2-2k=k(k+1)^2(k+4)/4-k^2-2k\geqq 19$. If $k=3$ and $\lambda_2\geqq 2$, then $d_0\geqq\deg 2\Lambda_2-15=5$. If $k=3$ and $\lambda_1=\lambda_2=1$, then $d_0\geqq\deg(\Lambda_1+\Lambda_2+\Lambda_3)-15=49$. Q.E.D.

(2) Complex irreducible representations of $A_k(k\geqq 1)$ are given by $\Lambda=\Sigma_{i=1}^{k}\lambda_i\Lambda_i$ where $\lambda_i(i=1, \cdots, k)$ are non-negative integers.

PROPOSITION 2.2 *If* $d_1:=2\deg \Lambda-k^2-2k\leqq 4$, *then* Λ *is equivalent as a complex representation of* $A_k(k\geqq 1)$ *to one of the followings*:

$$0(k\geqq 1),\quad \Lambda_1(k\geqq 1),\quad 2\Lambda_1(k=1, 2),\quad \Lambda_2(k\geqq 2),$$
$$2\Lambda_2(k=2),\quad \Lambda_{k-1}(k\geqq 4),\quad \Lambda_k(k\geqq 3).$$

PROOF: If $k=1$ and $\lambda_1\geqq 3$, then $\deg \Lambda\geqq\deg 3\Lambda_1=4$ and $d_1\geqq 5$. If $k=2$ and λ_1(or λ_2) $\geqq 3$, then $\deg \Lambda\geqq\deg 3\Lambda_1=10$ and $d_1\geqq 12$. If $k\geqq 2$, $\lambda_1\geqq 1$ and $\lambda_k\geqq 1$, then $\deg \Lambda\geqq\deg(\Lambda_1+\Lambda_k)=k(k+2)$ and $d_1\geqq 8$. If $k\geqq 3$ and λ_1(or λ_k)$\geqq 2$, then $\deg \Lambda\geqq\deg 2\Lambda_1=(k+1)(k+2)/2$ and $d_1\geqq 5$. If $\lambda_i\geqq 1$ for some $i=3, \cdots, k-2$, then $\deg \Lambda\geqq\deg \Lambda_3=k(k^2-1)/6$, $k\geqq 5$ and $d_1\geqq 5$. If λ_2(or λ_{k-1})$\geqq 2$ and $2\leqq k-1$, then $\deg \Lambda\geqq\deg 2\Lambda_2=k(k+1)^2(k+2)/12$, $k\geqq 3$ and $d_1\geqq 25$. If $\lambda_2\geqq 1$, $\lambda_{k-1}\geqq 1$ and $2<k-1$, then $\deg \Lambda\geqq\deg(\Lambda_2+\Lambda_{k-1})=(k+1)^2(k^2-4)/4$,

$k \geqq 4$ and $d_1 \geqq 126$. If $\lambda_1 \geqq 1$, $\lambda_{k-1} \geqq 1$ and $1<k-1$, then $\deg \Lambda \geqq \deg(\Lambda_1+\Lambda_{k-1})=(k+2)(k^2-1)/2$, $k \geqq 3$ and $d_1 \geqq 15$. If $\lambda_2 \geqq 1$, $\lambda_k \geqq 1$ and $2<k$, then $d_1 \geqq 15$. If $\lambda_1 \geqq 1$, $\lambda_2 \geqq 1$ (or $\lambda_{k-1} \geqq 1$, $\lambda_k \geqq 1$) and $2<k-1$, then $\deg \Lambda \geqq \deg(\Lambda_1+\Lambda_2)=2k(k+1)(k+2)/3$, $d_1 \geqq 56$. Q.E.D.

REMARK 2.3 $2\Lambda_1(k=1)$, $\Lambda_2(k=3)$ are 'real'. $\Lambda_1(k=1)$ is 'quaternion'. Λ_1, $\Lambda_k(k \geqq 2)$ (resp. Λ_2, $\Lambda_{k-1}(k \geqq 4)$, resp. $2\Lambda_1$, $2\Lambda_2(k=2)$) are conjugate from each other.

(3) 'Quaternion' complex irreducible representations of $A_k(k \geqq 1)$ are given as $\Lambda=(2\lambda_{2h+1}+1)\Lambda_{2h+1}+\Sigma_{i=1}^{2h}\lambda_i(\Lambda_i+\Lambda_{k-i+1})$ where $k=4h+1$, λ_i and h are non-negative integers.

PROPOSITION 2.4 *If* $d_2:=2\deg \Lambda-k^2-2k \leqq 8$, *then* Λ *is equivalent as a complex representation of* $A_k(k \geqq 1)$ *to one of the followings*:

$$d_2=1: \quad \Lambda_1(k=1),$$
$$d_2=5: \quad 3\Lambda_1(k=1), \quad \Lambda_3(k=5).$$

PROOF: If $k=4h+1 \geqq 6$, then $k \geqq 9$ and $d_2 \geqq 2\deg \Lambda_{2h+1}-k^2-2k \geqq 2\deg \Lambda_5-k^2-2k \geqq 405$. So $k=1$ or 5. Suppose $k=1$. If $\lambda_1 \geqq 2$, then $d_2=2\deg(2\lambda_1+1)\Lambda_1-3 \geqq 2\deg 5\Lambda_1-3=9$. So $\Lambda=\Lambda_1$ or $3\Lambda_1$. Next suppose $k=5$. If $\lambda_2 \geqq 1$, then $d_2 \geqq 2\deg(\Lambda_2+\Lambda_4)-35=343$. If $\lambda_1 \geqq 1$, then $d_2 \geqq 2\deg(\Lambda_1+\Lambda_5)-35=35$. If $\lambda_3 \geqq 1$, then $d_2 \geqq 2\deg 3\Lambda_3-35=1925$. So $\Lambda=\Lambda_3$. Q.E.D.

(C)

The simple roots of C_k are given by a Dynkin diagram:

$$\alpha_1 - \alpha_2 - \cdots - \alpha_{k-1} \Leftarrow \alpha_k \quad (k \geqq 2).$$

(1) 'Real' complex irreducible representations of $C_k(k \geqq 2)$ are given by $\Lambda=\Sigma_{i=1}^{k}\lambda_i\Lambda_i$ where $\Sigma_{i:odd}\lambda_i$ is even and $\lambda_i(i=1, \cdots, k)$ are non-negative integers.

PROPOSITION 2.5 *If* $d_0:=\deg \Lambda-k(2k+1) \leqq 3$, *then* Λ *is equivalent as a complex representation of* $C_k(k \geqq 2)$ *to one of the followings*:

$$d_0<0: \quad 0(k \geqq 2), \quad \Lambda_2(k \geqq 2),$$
$$d_0=0: \quad 2\Lambda_1(k \geqq 2).$$

PROOF: Suppose $k \geqq 5$. Then $\deg \Lambda_3<\deg \Lambda_i$ for $i=4, \cdots, k$ and $\deg \Lambda_3-\dim C_k=4k(k^2-3k-7) \geqq 20$. $\deg 3\Lambda_1-\dim C_k=k(2k+1)(4k-1)/3 \geqq 165$. $\deg(\Lambda_1+\Lambda_2)-\dim C_k=k(8k^2-6k-11)/3 \geqq 265$. $\deg 2\Lambda_2-\dim C_k=k^2(4k^2-13)/3 \geqq 725$. So $\Lambda=0$, Λ_2 or $2\Lambda_1$. Suppose $k=4$. Then the assertion holds since $\deg \Lambda_3-\dim C_4=12$, $\deg \Lambda_4-\dim C_4=6$, $\deg 2\Lambda_2-\dim C_4=272$, $\deg 3\Lambda_1-\dim C_4=84$ and $\deg(\Lambda_1+\Lambda_2)-\dim C_4=124$. Suppose

$k=3$. Then the assertion holds since $\deg 3\Lambda_1 - \dim C_3 = 35$, $\deg(\Lambda_1+\Lambda_2) - \dim C_3 = 43$, $\deg(\Lambda_1+\Lambda_3) - \dim C_3 = 49$, $\deg 2\Lambda_3 - \dim C_3 = 63$ and $\deg 2\Lambda_2 - \dim C_3 = 69$. Suppose $k=2$. Then the assertion holds since $\deg 4\Lambda_1 - \dim C_2 = 25$, $\deg 2\Lambda_2 - \dim C_2 = 4$ and $\deg(2\Lambda_1+\Lambda_2) - \dim C_2 = 25$. Q.E.D.

(2) Complex irreducible representations of $C_k (k \geqq 2)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i \Lambda_i$ where $\lambda_i (i=1, \cdots, k)$ are non-negative integers.

PROPOSITION 2.6 *If* $d_1 := 2\deg \Lambda - k(2k+1) \leqq 6$, *then* Λ *is equivalent as a complex representation of* $C_k (k \geqq 2)$ *to one of the followings*:

$$0 (k \geqq 2), \quad \Lambda_1 (k \geqq 2), \quad \Lambda_2 (k=2).$$

PROOF: Suppose $k \geqq 3$. If Λ is not equivalent to 0 nor Λ_1, then $\deg \Lambda \geqq \deg \Lambda_2$, so $d_1 \geqq 2\deg \Lambda_2 - \dim C_k = 2k^2 - 3k - 2 \geqq 7$. Suppose $k=2$. The the assertion holds since $2\deg 2\Lambda_1 - \dim C_2 = 10$, $2\deg(\Lambda_1+\Lambda_2) - \dim C_2 = 22$ and $2\deg 2\Lambda_2 - \dim C_2 = 18$. Q.E.D.

(3) 'Quaternion' complex irreducible representations of $C_k (k \geqq 2)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i \Lambda_i$ where $\sum_{i:odd} \lambda_i$ is odd and $\lambda_i (i=1, \cdots, k)$ are non-negative integers.

PROPOSITION 2.7 *If* $d_2 := 2\deg \Lambda - k(2k+1) \leqq 6$, *then* Λ *is equivalent as a complex representation of* $C_k (k \geqq 2)$ *to one of the followings*:

$$\Lambda_1 (k \geqq 2).$$

PROOF: Suppose $k \geqq 3$. If Λ is not equivalent to Λ_1, then $\deg \Lambda \geqq \deg \Lambda_2$, so $d_2 \geqq 2\deg \Lambda_2 - \dim C_k = 2k^2 - 3k - 2 \geqq 7$. Suppose $k=2$. If Λ is not equivalent to Λ_1, then $\deg \Lambda \geqq \deg(\Lambda_1+\Lambda_2) = 16$, so $d_2 \geqq 22$. Q.E.D.

(B)

The simple roots of B_k are given by a Dynkin diagram:

$$\alpha_1 — \alpha_2 — \cdots — \alpha_{k-1} \Longrightarrow \alpha_k \quad (k \geqq 3).$$

(1) 'Real' complex irreducible representations of $B_k (k \geqq 3)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i \Lambda_i$(if $k=4h+3$ or $4h+4$), $2\lambda_k \Lambda_k + \sum_{i=1}^{k-1} \lambda_i \Lambda_i$ (otherwise) where h and $\lambda_i (i=1, \cdots, k)$ are non-negative integers.

PROPOSITION 2.8 *If* $d_0 := \deg \Lambda - k(2k+1) \leqq 5$, *then* Λ *is equivalent as a complex representation of* $B_k (k \geqq 3)$ *to one of the followings*:

$$d_0 < 0: \quad \Lambda_1 (k \geqq 3), \quad \Lambda_k (k=3 \text{ or } 4), \quad 0 (k \geqq 3),$$
$$d_0 = 0: \quad \Lambda_2 (k \geqq 3).$$

PROOF: If $\lambda_i \geqq 1$ for some $i=3, \cdots, k-1$, then $k \geqq 4$ and $d_0 \geqq \deg \Lambda_3 - \dim B_k = k(2k+1)(2k-4)/3 \geqq 48$. If $\lambda_1 \geqq 2$, then $d_0 \geqq \deg 2\Lambda_1 - \dim B_k = 2k \geqq 6$. If $\lambda_2 \geqq 2$, then $d_0 \geqq \deg 2\Lambda_2 - \dim B_k = (2k+3)(2k+1)(k+1)(k-1)/3 - k(2k+1) \geqq 147$. If $\lambda_1 \geqq 1$ and $\lambda_2 \geqq 1$, then $d_0 \geqq \deg(\Lambda_1+\Lambda_2) - \dim B_k = (2k+1)(k+1)(4k-3) \geqq 84$. Then $\Lambda = \Lambda_1$, Λ_2, Λ_k, or

$\Lambda_2+\Lambda_k$(if $k=4h+3$ or $4h+4$), Λ_1 or Λ_2 (otherwise) since $\deg 2\Lambda_k-\dim B_k={}_{2k+1}C_{k+1}-k(2k+1)\geqq 14$ and $\deg(\Lambda_1+\Lambda_k)-\dim B_k=k2^{k+1}-k(2k+1)\geqq 27$. If $k=4h+3$ or $4h+4$, $k\geqq 5$ and $\lambda_k\geqq 1$, then $k\geqq 8$ and $d_0\geqq\deg\Lambda_k-\dim B_k=2^k-k(2k+1)\geqq 120$. If $k=3$(resp. 4), then $\deg(\Lambda_2+\Lambda_k)-\dim B_k=91$(resp. 396). Q.E.D.

(2) Complex irreducible representations of $B_k(k\geqq 3)$ are given by $\Lambda=\Sigma_{i=1}^k\lambda_i\Lambda_i$ where $\lambda_i(i=1,\cdots,k)$ are non-negative integers.

PROPOSITION 2.9 *If $d_1:=2\deg\Lambda-k(2k+1)\leqq 8$, then Λ is equivalent as a complex representation of $B_k(k\geqq 3)$ to one of the followings*:

$$d_1<0:\quad \Lambda_1(k\geqq 3),\quad \Lambda_k(k=3 \text{ or } 4),\quad 0(k\geqq 3).$$

PROOF: If $\lambda_i\geqq 1$ for some $i=2,\cdots,k-1$, then $d_1\geqq 2\deg\Lambda_2-k(2k+1)=k(2k+1)\geqq 21$. If $\lambda_1\geqq 2$, then $d_1\geqq 2\deg 2\Lambda_1-k(2k+1)=k(2k+5)\geqq 33$. If $\lambda_k\geqq 2$, then $d_1\geqq 2\deg 2\Lambda_k-k(2k+1)=2\,{}_{2k+1}C_{k+1}-k(2k+1)\geqq 49$. If $\lambda_1\geqq 1$ and $\lambda_k\geqq 1$, then $d_1\geqq 2\deg(\Lambda_1+\Lambda_k)-k(2k+1)=k2^{k+2}-k(2k+1)\geqq 75$. If $k\geqq 5$, then $2\deg\Lambda_k-k(2k+1)=2^{k+1}-k(2k+1)\geqq 9$. Q.E.D.

(3) 'Quaternion' complex irreducible representations of $B_k(k\geqq 3)$ are given by $\Lambda=\Sigma_{i=1}^{k-1}\lambda_i\Lambda_i+(2\lambda_k+1)\,\Lambda_k$ where $k=4h+5$ or $4h+6$, h and $\lambda_i(i=1,\cdots,k)$ are non-negative integers. Then $k\geqq 5$.

PROPOSITION 2.10 *There is no 'quaternion' complex irreducible representation of B_k* such that $d_2:=2\deg\Lambda-k(2k+1)\leqq 8$.

PROOF: Since $k\geqq 5$, $d_2\geqq 2\deg\Lambda_k-k(2k+1)=2^{k+1}-k(2k+1)\geqq 9$. Q.E.D.

(D)

The simple roots of D_k are given by a Dynkin diagram:

$$\begin{array}{c}\alpha_1 - \alpha_2 - \cdots - \alpha_{k-2} - \alpha_{k-1} \quad (k\geqq 4).\\ \qquad\qquad\quad | \qquad\qquad\qquad \\ \qquad\qquad\quad \alpha_k \qquad\qquad\qquad\end{array}$$

(1) 'Real' complex irreducible representations of $D_k(k\geqq 4)$ are given by $\Lambda=\Sigma_{i=1}^{k-2}\lambda_i\Lambda_i+\lambda_{k-1}(\Lambda_{k-1}+\Lambda_k)$(if $k=2h+5$), $\Sigma_{i=1}^{k}\lambda_i\Lambda_i$ (if $k=4h+4$), or $\Sigma_{i=1}^{k-2}\lambda_i\Lambda_i+\lambda_{k-1}^*\Lambda_{k-1}+\lambda_k^*\Lambda_k$ (if $k=4h+6$), where $\lambda_{k-1}^*+\lambda_k^*$ is even, h and $\lambda_i^{(*)}(i=1,\cdots,k)$ are non-negative integers.

PROPOSITION 2.11 *If $d_0:=\deg\Lambda-k(2k-1)\leqq 6$, then Λ is equivalent as a complex representation of $D_k(k\geqq 4)$* to one of the followings:

$$d_0<0:\quad 0(k\geqq 4),\quad \Lambda_1(k\geqq 4),\quad \Lambda_4(k=4),\quad \Lambda_3(k=4)$$
$$d_0=0:\quad \Lambda_2(k\geqq 4).$$

PROOF: If $\lambda_i\geqq 1$ for some $i=3,\cdots$, or $k-2$, then $k\geqq 5$ and $d_0\geqq\deg\Lambda_3-k(2k-1)=k(2k-1)(2k-5)/3\geqq 75$. So $\lambda_i=0$ for $i=3,\cdots,k-2$. Since $\deg 2\Lambda_1-k(2k-1)=2k-1\geqq 7$, $\deg 2\Lambda_2-k(2k-1)=k^2(4k^2-13)\geqq 272$ and $\deg(\Lambda_1+\Lambda_2)-k(2k-1)=k(4k-5)(2k+1)/3\geqq 132$, we have $\lambda_1+\lambda_2\leqq 1$. Suppose $\lambda_{k-1}^{(*)}$ or $\lambda_k^{(*)}\geqq 1$. If $k\geqq 8$, then $d_0\geqq 2^{k-1}-k(2k-1)\geqq 8$. If $k=7$, then $d_0\geqq\deg(\Lambda_6+\Lambda_7)-91=2912$. If $k=6$, then $d_0\geqq\deg(\Lambda_5+\Lambda_6)-66=726$ or $d_0\geqq\deg(2\Lambda_5)-66=\deg(2\Lambda_6)-66={}_{11}C_6-66=396$. If $k=5$, then $d_0\geqq\deg(\Lambda_4+\Lambda_5)-45=165$. If $k=4$ and $\lambda_1\geqq 1$, then $d_0\geqq\deg(\Lambda_1+\Lambda_4)-28=\deg(\Lambda_1+\Lambda_3)-28=28$. If $k=4$ and $\lambda_2\geqq 1$, then $d_0\geqq\deg(\Lambda_2+\Lambda_4)-28=\deg(\Lambda_2+\Lambda_3)-28=132$. So $k=4$ and $\Lambda=\Lambda_4$ or Λ_3. Q.E.D.

(2) Complex irreducible representations of $D_k(k\geqq 4)$ are given by $\Lambda=\Sigma_{i=1}^k\lambda_i\Lambda_i$ where $\lambda_i(i=1,\cdots,k)$ are non-negative integers.

PROPOSITION 2.12 *If $d_1:=2\deg\Lambda-k(2k-1)\leqq 36$, then Λ is equivalent as a complex representation of $D_k(k\geqq 4)$* to one of the followings:

$$d_1<0:\quad 0(k\geqq 4),\quad \Lambda_1(k\geqq 4),\quad \Lambda_3(k=4),\quad \Lambda_4(k=4),$$
$$\Lambda_4(k=5),\quad \Lambda_5(k=5),\quad \Lambda_5(k=6),\quad \Lambda_6(k=6).$$

PROOF: If $\lambda_i\geqq 1$ for some $i=2,\cdots,k-2$, then $d_1\geqq 2\deg\Lambda_2-k(2k-1)=k(2k-1)\geqq 28$. So that $\lambda_i=0$ for $i=2,\cdots,k-2$. Since $2\deg 2\Lambda_1-k(2k-1)=(k+2)(2k-1)\geqq 42$, we have $\lambda_1\leqq 1$. Suppose $\lambda_{k-1}+\lambda_k\geqq 1$. Then $k\leqq 6$ since $d_1\geqq 2\deg\Lambda_k-k(2k-1)=2\deg\Lambda_{k-1}-k(2k-1)=2^k-k(2k-1)\geqq 37$ if $k\geqq 7$. We have that $\lambda_1+\lambda_{k-1}+\lambda_k\leqq 1$ since $2\deg(\Lambda_1+\Lambda_k)-k(2k-1)=2\deg(\Lambda_1+\Lambda_{k-1})-k(2k-1)=(2^k-k)(2k-1)\geqq 84$, $2\deg(\Lambda_{k-1}+\Lambda_k)-k(2k-1)=k(2k-1)[4(2k-2)!/\{(k-1)!(k+1)!\}-1]\geqq 84$ and $2\deg 2\Lambda_k-k(2k-1)=2\deg 2\Lambda_{k-1}-k(2k-1)=k(2k-1)\{2(2k-2)!/(k!)^2-1\}\geqq 42$. Q.E.D.

REMARK 2.13 $\Lambda_4(k=5)$ and $\Lambda_5(k=5)$ are conjugate. $\Lambda_3(k=4)$ and $\Lambda_4(k=4)$ are 'real', and there are outer automorphisms $\tau_i(i=1,2)$ of D_4 such that $\Lambda_{30}\circ\tau_1$ and $\Lambda_{40}\circ\tau_2$ are equivalent as complex representations of D_4 to Λ_1. There is also an outer automorphism τ_3(resp. τ_4) of D_6(resp. D_5) such that $\Lambda_{50}\circ\tau_3$(resp. $\Lambda_{40}\circ\tau_4$) and Λ_6(resp. Λ_5) are equivalent as complex representations of D_6(resp. D_5).

(3) 'Quaternion' complex irreducible representations of $D_k(k\geqq 4)$ are given by $\Lambda=\Sigma_{i=1}^k\lambda_i\Lambda_i$ where $\lambda_{k-1}+\lambda_k$ is odd, $k=4h+6$, and h, $\lambda_i(i=1,\cdots,k)$ are non-negative integers.

PROPOSITION 2.14 *If $d_2:=2\deg\Lambda-k(2k-1)\leqq 36$, then Λ is equivalent as a complex representation of $D_k(k\geqq 4)$ to one of the followings*:

$$d_2=-2:\quad \Lambda_5(k=6),\quad \Lambda_6(k=6).$$

PROOF: The assertion follows from Proposition 2.12 and Remark 2.13. Q.E.D.

(E)

The simple roots of exceptional Lie algebras are given by Dynkin diagrams:

$$
\begin{array}{lccccccccccccc}
G_2: & \alpha_1 & \Longrightarrow & \alpha_2 & & & & & & & & & & \\
F_4: & \alpha_1 & - & \alpha_2 & \Longrightarrow & \alpha_3 & - & \alpha_4 & & & & & & \\
E_6: & \alpha_1 & - & \alpha_2 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & & & & \\
 & & & & & | & & & & & & & & \\
 & & & & & \alpha_6 & & & & & & & & \\
E_7: & \alpha_1 & - & \alpha_2 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & & \\
 & & & & & | & & & & & & & & \\
 & & & & & \alpha_7 & & & & & & & & \\
E_8: & \alpha_1 & - & \alpha_2 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7 \\
 & & & & & | & & & & & & & & \\
 & & & & & \alpha_8 & & & & & & & &
\end{array}
$$

PROPOSITION 2.15 *Suppose Λ is a complex irreducible representation of an exceptional Lie algebra of dimension g. If $d_0 := \deg \Lambda - g \leqq 12$, then Λ is equivalent as a complex representation to one of the followings*:

$$
\begin{aligned}
d_0<0:&\quad \Lambda_2(G_2),\quad \Lambda_4(F_4),\quad \Lambda_1(E_6),\quad \Lambda_5(E_6),\quad \Lambda_6(E_7),\\
d_0=0:&\quad \Lambda_1(G_2),\quad \Lambda_1(F_4),\quad \Lambda_6(E_6),\quad \Lambda_1(E_7),\quad \Lambda_7(E_8).
\end{aligned}
$$

PROOF: *Case G_2*) If Λ is not equivalent to Λ_1 nor Λ_2, then $d_0 \geqq 13$ since $\deg 2\Lambda_1=77$, $\deg 2\Lambda_2=27$ and $\deg(\Lambda_1+\Lambda_2)=64$. *Case F_4*) If Λ is not equivalent to Λ_1 nor Λ_2, then $d_0 \geqq 221$ since $\deg 2\Lambda_1=\deg(\Lambda_1+\Lambda_4)=1053$, $\deg 2\Lambda_4=324$, $\deg \Lambda_2=1274$ and $\deg \Lambda_3=273$. *Case E_6*) If Λ is not equivalent to Λ_1, Λ_5, nor Λ_6, then $d_0 \geqq 273$ since $\deg 2\Lambda_1=\deg 2\Lambda_5=\deg \Lambda_2=\deg \Lambda_4=351$, $\deg \Lambda_3=2925$, $\deg 2\Lambda_6=2430$, $\deg(\Lambda_1+\Lambda_5)=650$ and $\deg(\Lambda_1+\Lambda_6)=\deg(\Lambda_5+\Lambda_6)=1728$. *Case E_7*) In Λ is not equivalent to Λ_1 nor Λ_6, then $d_0 \geqq 779$ since $\deg \Lambda_2=8645$, $\deg \Lambda_3=365750$, $\deg \Lambda_4=27664$, $\deg \Lambda_5=1539$, $\deg \Lambda_7=912$, $\deg 2\Lambda_1=7371$, $\deg 2\Lambda_6=1463$ and $\deg(\Lambda_1+\Lambda_6)=3920$. *Case E_8*) If Λ is not equivalent to Λ_7, then $d_0 \geqq 3627$ since $\deg \Lambda_1=3825$, $\deg \Lambda_2=6696000$, $\deg \Lambda_3=6899079264$, $\deg \Lambda_4=146325270$, $\deg \Lambda_5=2450240$, $\deg \Lambda_6=30380$, $\deg \Lambda_8=147250$, and $\deg 2\Lambda_7=27000$. Q.E.D.

REMARK 2.16 $\Lambda_2(G_2)$ is 'real' of degree 7. $\Lambda_4(F_4)$ is 'real' of degree 26. $\Lambda_1(E_6)$ and $\Lambda_5(E_6)$ are conjugate from each other and of degree 27. $\Lambda_6(E_7)$ is 'quaternion' of degree 56. $\Lambda_1(G_2)$, $\Lambda_1(F_4)$, $\Lambda_6(E_6)$, $\Lambda_1(E_7)$ and $\Lambda_7(E_8)$ are the adjoint representations, especially 'real', of degree 14, 52, 78, 144, 248 respectively. Any Λ of d_1 or $d_2 \leqq 12$ is contained in the above list since $d_1=d_2>d_0$.

Next propositions are also useful in sections 3 and 4.

PROPOSITION 2.17 *Each non trivial 'real' complex irreducible representation of degree at most 3 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings*:

degree 3: $2\Lambda_1(A_1)$.

PROOF: The assertion follows from Prop. 2.1, 2,5, 2.8, 2.11 and 2.15 since d_0 is less than the degree which is at most 3. Q.E.D.

PROPOSITION 2.18 *Each non trivial complex irreducible representation of degree at most 3 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings*:

degree 2: $\Lambda_1(A_1)$,
degree 3: $2\Lambda_1(A_1)$, $\Lambda_1(A_2)$, $\Lambda_2(A_2)$.

PROOF: The assertion follows from Prop.'s 2.2, 2.6, 2.9, 2.12 and 2.15 since $d_1=2$ degree$-g\leqq 2.3-3=3$. Q.E.D.

REMARK 2.19 $\Lambda_2(A_2)$ is conjugate to $\Lambda_1(A_2)$.

PROPOSITION 2.20 *Each non trivial 'quaternion' complex irreducible representation of degree at most 6 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings*:

degree 2: $\Lambda_1(A_1)$,
degree 4: $3\Lambda_1(A_1)$, $\Lambda_1(C_2)$,
degree 6: $5\Lambda_1(A_1)$, $\Lambda_1(C_3)$.

PROOF: The assertion is trivial in the case of A_1. Otherwise, it follows from Prop.'s 2.4, 2.7, 2.10, 2.14 and 2.15 since $d_2=2$ degree$-g\leqq 2\cdot 6-8=4$. Q.E.D.

3. Basic Classification by cohomogeneity

Let (G, M) be a Lie transformation group. For $x\in M$, we denote $G(x)$ the orbit of G through x, and G_x the isotropy subgroup of G at x.

LEMMA 3.1 *Let (G, M), (G, N) be Lie transformation groups and f be a G-equivariant submersion from M onto N with the property*:

$$f^{-1}(f(x))=G_{f(x)}x$$

at a fixed $x\in M$. Then we have that

$$dim\ M-\dim G+\dim G_x=\dim N-\dim G+\dim G_{f(x)}.$$

PROOF: $\dim M=\dim N+\dim f^{-1}(f(x))=\dim N+\dim G_{f(x)}(x)=\dim N+\dim G_{f(x)}-\dim G_x$ since $(G_{f(x)})_x=G_x$. Q.E.D.

Let $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ be the set of real numbers, complex numbers and quaternions respectively. Naturally $\boldsymbol{H}$ contains $\boldsymbol{C}$, and $\boldsymbol{C}$ contains $\boldsymbol{R}$. The *conjugate* $\overline{u+\mathrm{j}v}$ of $u+\mathrm{j}v\in\boldsymbol{H}$ $(u, v\in\boldsymbol{C})$ is defined by

$$\overline{u+\mathrm{j}v}=\bar{u}-jv$$

where $\bar{u}$ is the complex conjugate of u. For $u+jv$, $u'+jv'\in\boldsymbol{H}$, the *product* of them is defined by

$$(u+\mathrm{j}v)(u'+jv')=(uu'-\bar{v}v')+j(vu'+\bar{u}v').$$

Let F be $\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$. The set of all (n_1, n_2) − matrixes with coefficients F is denoted by $F(n_1, n_2)$. For $X\in F(n_1, n_2)$, we denote the *conjugate* of X with respect to the coefficients by $\bar{X}$, and the *transposed* matrix of X by ${}^tX$. We write $F^n=F(n, 1)$, $F(n)=F(n, n)$, and denote the identity matrix of $F(n)$ by I_n. We denote $hF(n)=\{X\in F(n); {}^t\bar{X}=X\}$, $pF(n)=\{X\in hF(n);$ X is positive definite$\}$, and use the following notations for classical groups:

$$GF(n)=\{X\in F(n);\ {}^t\bar{X}X=X{}^t\bar{X}=I_n\}.$$

If $F=\boldsymbol{R}$ or $\boldsymbol{C}$, denote

$$SF(n)=\{X\in GF(n);\ \det X=1\}.$$

Then $G\boldsymbol{R}(n)=O(n)$, $G\boldsymbol{C}(n)=U(n)$, $G\boldsymbol{H}(n)=Sp(n)$, $S\boldsymbol{R}(n)=SO(n)$ and $S\boldsymbol{C}(n)=SU(n)$ in usual notations. Any subgroup of $GF(n)$ acts on F^n linearly over right multiplications of F by usual manner and acts on $hF(n)$(resp. $pF(n)$) by

$$A\cdot X=AX{}^t\bar{A} \tag{3.1}$$

for $A\in GF(n)$, $X\in hF(n)$(resp. $pF(n)$). Each matrix of $hF(n)$ can be transformed to a diagonal form by the action of $GF(n)$(resp. $SF(n)$). Similarly any subgroup of $GF(n_1)\times GF(n_2)$ acts on $F(n_1, n_2)$ by

$$(A, B)\cdot X=AX{}^t\bar{B} \tag{3.2}$$

for $(A, B)\in GF(n_1)\times GF(n_2)$, $X\in F(n_1, n_2)$.

We use mappings k, k': $\boldsymbol{H}(n_1, n_2)\longrightarrow \boldsymbol{C}(2n_1, 2n_2)$,

h: $\boldsymbol{H}(n_1, n_2)\longrightarrow\boldsymbol{C}(2n_1, n_2)$ and h': $\boldsymbol{H}(n_1, n_2)\longrightarrow\boldsymbol{C}(n_1, 2n_2)$ such that

$$k(U+jV)=\begin{pmatrix} U & -\bar{V} \\ V & \bar{U}\end{pmatrix},\quad k'(U+Vj)=\begin{pmatrix} U & V \\ -\bar{V} & \bar{U}\end{pmatrix},\quad h(U+jV)=\begin{pmatrix} U \\ V\end{pmatrix},$$

$h'(U+Vj)=(U, V)$ for U, $V\in\boldsymbol{C}(n_1, n_2)$.

Then k, k' are real linear injections such that

$$\overline{{}^tk(P)}=k({}^t\bar{P}),\quad \overline{{}^tk'(P)}=k'({}^t\bar{P}),\quad k(PQ)=k(P)k(Q),\quad k'(PQ)=k'(P)k'(Q)$$
$$\text{for } P\in\boldsymbol{H}(n_1, n_2),\ Q\in\boldsymbol{H}(n_2, n_3),$$

and h(resp. h') is a linear bijection over right (resp. left) multiplications of $\boldsymbol{C}$ such that

$h(PQ)=k(P)h(Q)$(resp. $h'(PQ)=h'(P)k(Q)$).

For $P\in \boldsymbol{H}(n_1, n_2)$, we see that column-rank$_{\boldsymbol{H}}(P):=n_2-\dim_{\boldsymbol{H}}\{Q\in \boldsymbol{H}^{n_2}; PQ=0\}=(2n_2-\dim_{\boldsymbol{C}}\{Q\in \boldsymbol{H}^{n_2}; PQ=0\})/2=(\mathrm{rank}_{\boldsymbol{C}}k(P))/2=(\mathrm{rank}_{\boldsymbol{C}}k'(P))/2=(2n_1-\dim_{\boldsymbol{C}}\{Q\in \boldsymbol{H}(1, n_1); QP=0\})/2=n_1-\dim_{\boldsymbol{H}}\{Q\in \boldsymbol{H}(1, n_1); QP=0\}=:$ row-rank$_{\boldsymbol{H}}(P)$. Note that the linear independence in $\boldsymbol{H}^{n_2}$, $\boldsymbol{H}(1, n_1)$ over right multiplications of $\boldsymbol{H}$ is equivalent to one over left multiplications of $\boldsymbol{H}$ respectively owing to $\overline{pq}=\bar{q}\cdot\bar{p}$ (p, q in $\boldsymbol{H}$). Therefore rank$_{\boldsymbol{H}}(P):=$column-rank$_{\boldsymbol{H}}(P)=$row-rank$_{\boldsymbol{H}}(P)$ is well-defined. Denote $MF(n_1, n_2)=\{X\in F(n_1, n_2); \mathrm{rank}_F(X)=\max(n_1, n_2)\}$. Then $k(MH(n_1, n_2))=MC(2n_1, 2n_2)\cap k(\boldsymbol{H}(n_1, n_2))$.

Assume $n_1\geqq n_2$. Denote f: $MF(n_1, n_2) \longrightarrow pF(n_2)$

such that $f(X)={}^t\bar{X}X$ for $X\in MF(n_1, n_2)$. Then f is $GF(n_1)\times GF(n_2)$-equivariant with respect to the action (3.2) on $MF(n_1, n_2)$ and the following action on $pF(n_2)$:

$$(A, B)\cdot Y=BY{}^t\bar{B} \tag{3.3}$$

for $(A, B)\in GF(n_1)\times GF(n_2)$, $Y\in pF(n_2)$.

LEMMA 3.2 (1) *f is a submersion.*

(2) *$f^{-1}(f(X))=(GF(n_1)\times \{I_{n_2}\})\cdot X$ for $X\in MF(n_1, n_2)$.*

(3) *If $n_1>n_2$, then $f^{-1}(f(X))=(SF(n_1)\times \{I_{n_2}\})\cdot X$ for $X\in MF(n_1, n_2)$ where $F=\boldsymbol{R}$ or $\boldsymbol{C}$.*

PROOF: (1) Since any diagonal matrix in $pF(n_2)$ is in the image of f, it follows that f is onto from the diagonalizability by the action (3.3). To prove df_{X_0}: $F(n_1, n_2)\to hF(n_2)$; $X\to{}^t\bar{X}X_0+{}^t\bar{X}_0X$ is onto at $X_0\in MF(n_1, n_2)$, if we use the action (3.2) of $GF(n_1)\times GF(n_2)$, we may assume that X_0 has the following form for some non-zero $x_i\in\boldsymbol{R}$ ($i=1, \cdots, n_2$): $X_0=\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_{n_2} \\ & & \end{bmatrix}$. In fact, the action (3.3) of $\{I_{n_1}\}\times GF(n_2)$ transforms ${}^t\bar{X}_0X_0$ to a diagonal form and the action (3.2) of $GF(n_1)\times\{I_{n_2}\}$ gives a required form. Then it is easy to show that df_{X_0} is onto. (2) Suppose $f(X)=f(Y)$. Denote $X=[x_1, \cdots, x_{n_2}]$, $Y=[y_1, \cdots, y_{n_2}]$ where $x_i, y_i\in F^{n_1}$, then ${}^t\bar{x}_ix_j={}^t\bar{y}_iy_j$ ($i, j=1, \cdots, n_2$). We can choose x_h, y_k ($h, k=n_2+1, \cdots, n_1$) such that ${}^t\bar{x}_ix_h={}^t\bar{y}_iy_h=0$ and ${}^t\bar{x}_hx_k={}^t\bar{y}_hy_k=\delta_{hk}$. Then $X'=[x_1, \cdots, x_{n_1}]$, $Y'=[y_1, \cdots, y_{n_1}]$ have the inverse matrices. For $A=Y'X'^{-1}$, A is in $GF(n_1)$ since ${}^t\bar{X}'X'={}^t\bar{Y}'Y'$. We have $(A, I_{n_2})\cdot X=Y$. (3) If $F=\boldsymbol{R}$ or $\boldsymbol{C}$, then $X''=X'\cdot\mathrm{diag}[1, \cdots, 1, \det X'^{-1}]$ and $Y''=Y'\cdot\mathrm{diag}[1, \cdots, 1, \det Y'^{-1}]$ are in $SL(n_1, F)$. Then $B=Y''X''^{-1}$ is in $SF(n_1)$ and $(B, I_{n_2})\cdot X=Y$ if $n_1>n_2$. Q.E.D.

The tensor product $F^{n_1}\underset{F}{\otimes}\cdots\underset{F}{\otimes}F^{n_s}$ over F of $F^{n_1}, \cdots, F^{n_s}$ is defined if $F=\boldsymbol{R}$ or $\boldsymbol{C}$. Naturally $\boldsymbol{R}^{n_1}\underset{\boldsymbol{R}}{\otimes}\cdots\underset{\boldsymbol{R}}{\otimes}\boldsymbol{R}^{n_s}=\{z \text{ in } \boldsymbol{C}^{n_1}\underset{\boldsymbol{C}}{\otimes}\cdots\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}^{n_s}; \bar{z}=z\}$ where denotes the complex conjugation extended naturally on $\boldsymbol{C}^{n_1}\underset{\boldsymbol{C}}{\otimes}\cdots\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}^{n_s}$. If $F=\boldsymbol{H}$, then we consider the real linear map $\tilde{J}$: $\boldsymbol{C}^{2n_1}\underset{\boldsymbol{C}}{\otimes}\cdots\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}^{2n_s}\to\boldsymbol{C}^{2n_1}\underset{\boldsymbol{C}}{\otimes}\cdots\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}^{2n_s}$; $\Sigma_i z_i(h(P_{i1})\otimes\cdots\otimes h(P_{is}))\to\Sigma_i \bar{z}_i(h(P_{i1}j)\otimes\cdots\otimes h(P_{is}j))$, where $z_i\in\boldsymbol{C}$ and $P_{it}\in\boldsymbol{H}^{n_t}$($t=1, \cdots, s$). Then $\tilde{J}^2=id$(if s is even), or -id (if s is odd). The *ten-*

sor product $H^{n_1}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_s}$ *over right* H of $H^{n_1},\cdots,H^{n_s}$ is defined by $H^{n_1}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_s}:=\{Z\in C^{2n_1}\underset{C}{\otimes}\cdots\underset{C}{\otimes}C^{2n_s};\tilde{J}Z=Z\}$ (if s is even), or $C^{2n_1}\underset{C}{\otimes}\cdots\underset{C}{\otimes}C^{2n_s}$ with the quaternion structure $\tilde{J}$ (if s is odd). If $s=1$, then $\tilde{J}$ is the standard quaternion structure on $C^{2n_1}=h(H^{n_1})$. If $s=2$, then $H^{n_1}\underset{H}{\otimes}H^{n_2}$ is a real form of $C^{2n_1}\underset{C}{\otimes}C^{2n_2}$ with respect to the real structure $\tilde{J}$ on $C^{2n_1}\underset{C}{\otimes}C^{2n_2}$. For an even s, $H^{n_1}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_s}$ is equivalent as real spaces to

$$(H^{n_1}\underset{H}{\otimes}H^{n_2})\underset{R}{\otimes}\cdots\underset{R}{\otimes}(H^{n_{s-1}}\underset{H}{\otimes}H^{n_s})$$

since the complexifications are isomorphic over C.

Let $\rho_1,\cdots,\rho_s$ be linear representations of Lie groups $G_1,\cdots,G_s$ on $F^{n_1},\cdots,F^{n_s}$ over F respectively. If $F=R$ or C, then the *exterior tensor product* $\rho_1\underset{F}{\hat{\otimes}}\cdots\underset{F}{\hat{\otimes}}\rho_s$ *over* F is defined as the representation of the direct product group $G_1\times\cdots\times G_s$ on the tensor prodict space $F^{n_1}\underset{F}{\otimes}\cdots\underset{F}{\otimes}F^{n_s}$ over F such that

$$(\rho_1\underset{F}{\hat{\otimes}}\cdots\underset{F}{\hat{\otimes}}\rho_s)(g_1,\cdots,g_s):=\rho_1(g_1)\underset{F}{\otimes}\cdots\underset{F}{\otimes}\rho_s(g_s)$$

for $(g_1,\cdots,g_s)$ in $G_1\times\cdots\times G_s$, where the right hand side is the usual tensor product of linear transformations. If $F=H$, then note that $\tilde{J}$ commutes with the representation $(k\circ\rho_1)\underset{C}{\hat{\otimes}}\cdots\underset{C}{\hat{\otimes}}(k\circ\rho_s)$ of $G_1\times\cdots\times G_s$ on $h(H^{n_1})\underset{C}{\otimes}\cdots\underset{C}{\otimes}h(H^{n_s})$. The *exterior tensor product* $\rho_1\underset{H}{\hat{\otimes}}\cdots\underset{H}{\hat{\otimes}}\rho_s$ *over right* H is defined as the representation of $G_1\times\cdots\times G_s$ on $H^{n_1}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_s}$ such that

$$(\rho_1\underset{H}{\hat{\otimes}}\cdots\underset{H}{\hat{\otimes}}\rho_s)(g_1,\cdots,g_s):=((k\circ\rho_1)\underset{C}{\hat{\otimes}}\cdots\underset{C}{\hat{\otimes}}(k\circ\rho_s))(g_1,\cdots,g_s)\,|\,H^{n_1}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_s}.$$

If s is even, then it is equivalent as a real representation of $G_1\times\cdots\times G_s$ to $(\rho_1\underset{H}{\hat{\otimes}}\rho_2)\underset{R}{\hat{\otimes}}\cdots\underset{R}{\hat{\otimes}}(\rho_{s-1}\underset{H}{\hat{\otimes}}\rho_s)$. Next, we study the case of $s=2$ in more detail. The *identity representation* of a Lie subgroup K of $GF(n)$ is denoted by id. We consider the action (3.1) of K on $pF(n)$.

PROPOSITION 3.3 *If K is a Lie subgroup of $GF(n_2)$ and $n_1\geqq n_2$, then* (1) $\mathrm{coh}(GF(n_1)\times K,\mathrm{id}\underset{F}{\hat{\otimes}}\mathrm{id},F^{n_1}\underset{F}{\otimes}F^{n_2})=\mathrm{coh}(K,pF(n_2))$, (2) $\mathrm{coh}(SO(n_1)\times K,\mathrm{id}\underset{R}{\hat{\otimes}}\mathrm{id},R^{n_1}\underset{R}{\otimes}R^{n_2})=\mathrm{coh}(K,pR(n_2))$, (3) *If* $n_1>n_2$, *then* $\mathrm{coh}(SU(n_1)\times K,\mathrm{id}\underset{C}{\hat{\otimes}}\mathrm{id},C^{n_1}\underset{C}{\otimes}C^{n_2})=\mathrm{coh}(K,pC(n_2))$, (4) $\mathrm{coh}(K,pF(n_2))\geqq\mathrm{coh}(GF(n_2),pF(n_2))=n_2$ $(=\mathrm{coh}(SF(n_2),pF(n_2))$ *if* $F=R$ *or* $C)$.

PROOF: If $F=R$ or C, the representation space $F^{n_1}\underset{F}{\otimes}F^{n_2}$ is identified with $F(n_1,n_2)$ by the correspondence ι: $F^{n_1}\underset{F}{\otimes}F^{n_2}\to F(n_1,n_2)$ such that $\iota(e_i\otimes e_j)=E_{ij}$ $(i=1,\cdots,n_1;j=1,\cdots,n_2)$ with respect to the standard bases $\{e_i\}$, $\{e_j\}$, $\{E_{ij}\}$ of F^{n_1}, F^{n_2}, $F(n_1,n_2)$ respectively. Through ι, the action of $GF(n_1)\times K$ on $F(n_1,n_2)$ is induced as

$$(A,B)\cdot X=AX{}^tB$$

for $X\in F(n_1,n_2)$, $(A,B)\in GF(n_1)\times K$. The o.t.g. induced from this action is equivalent to one from the similar action of $GF(n_1)\times\bar{K}$ where $\bar{K}=\{\bar{B};B\in K\}$ is the conjugation of $K\in GF(n_2)$. Hence the o.t.g. induced from $\mathrm{id}\underset{F}{\hat{\otimes}}\mathrm{id}$ is equivalent to one from the action (3.2) of $GF(n_1)\times K$. When $F=H$, we consider ι: $C^{2n_1}\underset{C}{\otimes}C^{2n_2}\to C(2n_1,2n_2)$ for the standard basis $e_1=h(e_1'),\cdots,e_{n_i}=h(e_{n_i}'),e_{n_i+1}=h(e_1'j),\cdots,e_{2n_i}=h(e_{n_i}'j)$ of C^{2n_i} where $e_1',\cdots,e_{n_i}'$ is the standard basis of H^{n_i} $(i=1,2)$. Then we have

$$\iota(H^{n_1}\underset{H}{\otimes}H^{n_2})=k(H(n_1, n_2))$$

since $\tilde{J}Z_i=J_i\bar{Z}_i$ ($Z_i\in C^{2n_i}$), $\iota(\tilde{J}Z)=J_1\overline{\iota(Z)}{}^tJ_2$ ($Z\in C^{2n_1}\underset{C}{\otimes}C^{2n_2}$) and $k(H(n_1, n_2))=\{X\in C(2n_1, 2n_2); J_1\bar{X}{}^tJ_2=X\}$ where

$$J_i=\begin{pmatrix} 0_{n_i} & -I_{n_i} \\ I_{n_i} & 0_{n_i} \end{pmatrix} \qquad (i=1, 2).$$

Through ι, the action of $Sp(n_1)\times K$ on $k(H(n_1, n_2))$ is induced from the representation $\mathrm{id}\hat{\underset{H}{\otimes}}\mathrm{id}$ on $H^{n_1}\underset{H}{\otimes}H^{n_2}$ by $(A, B)\cdot k(X)=k(A)k(X){}^tk(B)$ for $X\in H(n_1, n_2)$, $(A, B)\in Sp(n_1)\times K$. The o.t.g. induced from this action is equivalent to the one which is induced from the action (3.2) of $Sp(n_1)\times K$ on $H(n_1, n_2)$, since ${}^t\overline{k(B)}=k({}^t\bar{B})$ and $k(A)k(X)k({}^t\bar{B})=k(AX{}^t\bar{B})$.

Then (1) follows from Lemma 3.1 and Lemma 3.2(0), (1), (2), since $MF(n_1, n_2)$ is open and dense in $F(n_1, n_2)$. (2) follows from (1) since $GR(n_1)^0=SO(n_1)$. (3) follows from Lemma 3.1 and Lemma 3.2(0), (1), (3). (4) follows from that $GF(n_2)$(resp. $SF(n_2)$ if $F=R$ or C) transforms any matrix in $pF(n_2)$ to a diagonal form. Q.E.D.

Denote $r(n_1, n_2, n_3)=\mathrm{coh}(SO(n_1)\times SO(n_2)\times SO(n_3), \mathrm{id}\hat{\underset{R}{\otimes}}\mathrm{id}\hat{\underset{R}{\otimes}}\mathrm{id}, R^{n_1}\underset{R}{\otimes}R^{n_2}\underset{R}{\otimes}R^{n_3})$, $c(n_1, n_2, n_3)=\mathrm{coh}(U(n_1)\times SU(n_2)\times SU(n_3), \mathrm{id}\hat{\underset{C}{\otimes}}\mathrm{id}\hat{\underset{C}{\otimes}}\mathrm{id}, C^{n_1}\underset{C}{\otimes}C^{n_2}\underset{C}{\otimes}C^{n_3})$, $q(n_1, n_2, n_3)=\mathrm{coh}((Sp(n_1)\times Sp(n_2))\times SO(n_3), (\mathrm{id}\hat{\underset{H}{\otimes}}\mathrm{id})\hat{\underset{R}{\otimes}}\mathrm{id}, (H^{n_1}\underset{H}{\otimes}H^{n_2})\underset{R}{\otimes}R^{n_3})$.

PROPOSITION 3.4

(1) $r(n_1, n_2, n_3)\geqq 18$ *if* $n_1\geqq n_2\geqq n_3\geqq 3$.

(2) $c(n_1, n_2, n_3)\geqq 6$ *if* $n_1\geqq n_2\geqq n_3\geqq 2$.

(3) $q(n_1, n_2, n_3)\geqq 3$ *if* $n_3\geqq 3$, $n_1\geqq n_2\geqq 1$.

(4) $q(n_1, n_2, n_3)\geqq 8$ *if* $n_3\geqq 3$, $n_1\geqq 2$, $n_1\geqq n_2\geqq 1$.

PROOF: Denote $\lambda(n_1, n_2, n_3)=\dim pB(n_2n_3)-\dim SO(n_2)\times SO(n_3)$ (if $n_1\geqq n_2n_3$) or $\dim R^{n_1}\underset{R}{\otimes}R^{n_2}\underset{R}{\otimes}R^{n_3}-\dim SO(n_1)\times SO(n_2)\times SO(n_3)$ (otherwise), $\kappa(n_1, n_2, n_3)=\dim pC(n_2n_3)-\dim SU(n_2)\times SU(n_3)$ (if $n_1\geqq n_2n_3$) or $\dim C^{n_1}\underset{C}{\otimes}C^{n_2}\underset{C}{\otimes}C^{n_3}-\dim U(n_1)\times SU(n_2)\times SU(n_3)$ (otherwise), and $\mu(n_1, n_2, n_3)=\dim pR(4n_1n_2)-\dim Sp(n_1)\times Sp(n_2)$ (if $n_3\geqq 4n_1n_2$), $\dim pH(n_2n_3)-\dim Sp(n_2)\times SO(n_3)$ (if $n_3\leqq 4n_1n_2$, $n_2n_3\leqq n_1$) or $\dim (H^{n_1}\underset{H}{\otimes}H^{n_3})\underset{R}{\otimes}R^{n_3}-\dim Sp(n_1)\times Sp(n_2)\times SO(n_3)$ (otherwise). Then $\lambda(n_1, n_2, n_3)\leqq r(n_1, n_2, n_3)$, $\kappa(n_1, n_2, n_3)\leqq c(n_1, n_2, n_3)$ and $\mu(n_1, n_2, n_3)\leqq q(n_1, n_2, n_3)$ by Prop. 3.3 since $(H^{n_1}\underset{H}{\otimes}H^{n_2})\underset{R}{\otimes}R^{n_3}$ is equivalent to $H^{n_1}\underset{H}{\otimes}(H^{n_2}\underset{R}{\otimes}R^{n_3})$ as $Sp(n_1)\times Sp(n_2)\times SO(n_3)$-spaces over R. Since $\lambda(x_1, x_2, x_3)=(x_2^2x_3^2+x_2x_3-x_2^2-x_3^2+x_2+x_3)/2$ (if $x_1\geqq x_2x_3$) or $x_1x_2x_3+(x_1+x_2+x_3-x_1^2-x_2^2-x_3^2)/2$ (otherwise), $\kappa(x_1, x_2, x_3)=x_2^2x_3^2-x_2^2-x_3^2+2$ (if $x_1\geqq x_2x_3$) or $2x_1x_2x_3-x_1^2-x_2^2-x_3^2+2$ (otherwise), and $\mu(x_1, x_2, x_3)=8x_1^2x_2^2+2x_1x_2-2x_1^2-2x_2^2-x_1-x_2$ (if $x_3\geqq 4x_1x_2$), $2x_3^2x_2^2-x_3x_2-2x_2^2-x_2-x_3^2/2+x_3/2$ (if $x_3\leqq 4x_1x_2$, $x_3x_2\leqq x_1$) or $4x_1x_2x_3-x_1(2x_1+1)-x_2(2x_2+1)-x_3^2/2+x_3/2$ (otherwise), they define continuous piecewise polynomial functions on R^3 if we take $x_i (i=1, 2, 3)$ as real numbers. (1) Since $\partial\lambda/\partial x_i(x_1, x_2, x_3)\geqq 0$ for $x_1\geqq x_2\geqq x_3\geqq 1$ $(i=1, 2, 3)$, we have $\lambda(n_1, n_2, n_3)\geqq\lambda(n_1, n_2, 3)\geqq\lambda(n_1,$

$3, 3)\geqq\lambda(3, 3, 3)=18$. (2) Similar to (1), $\kappa(n_1, n_2, n_3)\geqq\kappa(2, 2, 2)=6$. (3) Since $\partial\mu/\partial x_i(x_1, x_2, x_3)\geqq 0$ for $i=1, 2, 3$; $x_1, x_2, x_3\geqq 1$ (if $x_3\geqq 4x_1x_2$ or $x_3x_2\leqq x_1$), and $\partial\mu/\partial x_3(x_1, x_2, x_3)=(4x_1x_2-x_3)+1/2>1/2$, $\partial\mu/\partial x_2(x_1, x_2, x_3)=4(x_1x_3-x_2)-1\geqq 4x_1(x_3-1)-1\geqq 3$, $\partial\mu/\partial x_1(x_1, x_2, x_3)=4(x_2x_3-x_1)-1>-1$ for $x_1\geqq x_2\geqq 1$, $x_3\geqq 2$ (if $x_3<4x_1x_2$ and $x_3x_2>x_1$), we have $\mu(n_1, n_2, n_3)\geqq\mu(n_1, n_2, 3)\geqq\mu(n_1, 1, 3)=\mu(n_1-1, 1, 3)+\partial\mu/\partial x_1(n_1-\theta, 1, 3)$ $(0<\theta<1)\geqq\mu(n_1-1, 1, 3)$ (since $\mu(n_1, 1, 3)$ and $\mu(n_1-1, 1, 3)$ are integers, and $-1<\partial\mu/\partial x_1$ is also an integer, especially $\partial\mu/\partial x_1\geqq 0)\geqq\mu(1, 1, 3)=3$. (4) Similar to (3), $\mu(n_1, n_2, n_3)\geqq\mu(n_1, 1, 3)\geqq\mu(2, 1, 3)=8$. Q.E.D.

Let L be the Lie algebra of a connected Lie group G. We write the same letter for a linear representation of L and the corresponding representation of G. According to Iwahori [12], there is the following relation between real irreducible representations of L(resp. G) and complex irreducible representations of L(resp. G) (cf. Goto-Grosshans [6]). For a complex irreducible representation ρ on a complex vector space V, we denote the real restriction of ρ on the real restricted vector space $V_{\boldsymbol{R}}$(abbrev. V since $V=V_{\boldsymbol{R}}$ as a set) by $\rho_{\boldsymbol{R}}$(abbrev. ρ), which is not real irreducible if and only if ρ is 'real', and so we attach to ρ a real irreducible representation ρ^r as follows. $\rho^r=\sigma$ (if ρ is the complexification σ^C of a real representation σ on a real form W of V, i.e., ρ is 'real'.) or $\rho_{\boldsymbol{R}}$ (otherwise). Note that ρ_1^r and ρ_2^r are equivalent as real representations if and only if ρ_1 and ρ_2 are conjugate or equivalent as complex representations of L(resp. G). Conversely the complexification σ^C on W^C of a real irreducible representation σ on a real vector space W is not complex irreducible if and only if W has a L(resp. G)-invariant complex structure (then it is unique), and so we attach to σ a complex irreducible representation σ^c as follows. $\sigma^c=\sigma$ (if W has a L(resp. G)-invariant complex structure) or σ^C (otherwise). Note that ρ^{rc} and ρ(resp. σ^{cr} and σ) are equivalent as complex(resp. real) representations.

Let $(G, \boldsymbol{E}^N)$ be an o.t.g. Then the Lie algebra L of G is a real reductive Lie algebra and has a form:

$$L=L_0\oplus L_1\oplus\cdots\oplus L_s \tag{3.4}$$

where L_0 is the center of L, and $L_i(i=1,\cdots, s)$ are simple ideals of L. Let G_0, G_i be connected Lie subgroups of G corresponding to L_0, L_i respectively and $\tilde{G}_0$, $\tilde{G}_i$ be the universal covering groups of G_0, G_i respectively, then $\tilde{G}_i$ $(i=1,\cdots, s)$ are compact. Let $id{:}G\to SO(N)$ be the identity representation and $\tilde{id}$ be the corresponding representation of $\tilde{G}:=\tilde{G}_0\times\tilde{G}_1\times\cdots\times\tilde{G}_s$.

In this paper, we consider $(G, \boldsymbol{E}^N)$ in case that id is a real irreducible representation of G. Then G is compact (cf. Kobayashi-Nomizu[14]), and so $G_0\simeq U(1)$ or the trivial group 1. For $t\in\boldsymbol{R}^\times:=\boldsymbol{R}-\{0\}$, we denote $\hat{t}{:}\boldsymbol{R}\to U(1)$ the complex irreducible representation of $\boldsymbol{R}$ such that $\hat{t}(x)=e^{2\pi itx}$ for $x\in R$. We shall decompose $\tilde{id}^{\,c}$ of $\tilde{G}$ into an exterior tensor product of complex irreducible representations of $\tilde{G}_i(i=0,\cdots, s)$.

Case i) $\tilde{id}^{\,c}=\tilde{id}^{\,C}$: Then G_0 is trivial, and $(\tilde{G}, \tilde{id}^{\,c}, \boldsymbol{C}^N)$ is equivalent as complex representations to some

$$(\tilde{G}_1 \times \cdots \times \tilde{G}_s, \rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s, C^{n_1} \underset{C}{\otimes} \cdots \underset{C}{\otimes} C^{n_s})$$

where ρ_i is a self-conjugate complex irreducible representation of $\tilde{G}_i$ on C^{n_i}, $n_i \geqq 2 (i=1, \cdots, s)$, $\Pi_{i=1}^s n_i = N$, and $\# \{i; \rho_i$ is 'quaternion'$\}$ is even. We may assume $\rho_j (j=1, \cdots, 2r)$ are 'quaternion' and $\rho_k (k=2r+1, \cdots, 2r+q; s=2r+q)$ are 'real', and σ_i denotes a real representation of $\tilde{G}_i$ on R^{n_i} whose complexification is $\rho_{2r+i} (i=1, \cdots, q)$; where r and q are non-negative integers. Then $n_{2r+i} \geqq 3 (i=1, \cdots, q)$, and $(\tilde{G}, \widetilde{id}, R^N)$ is equivalent as real representation to

$$(\tilde{G}_1 \times \cdots \times \tilde{G}_{2r} \times \tilde{G}_{2r+1} \times \cdots \times \tilde{G}_{2r+q}, (\rho_1 \hat{\underset{H}{\otimes}} \rho_2) \hat{\underset{R}{\otimes}} \cdots \hat{\underset{R}{\otimes}} (\rho_{2r-1} \hat{\underset{H}{\otimes}} \rho_{2r}) \hat{\underset{R}{\otimes}} \sigma_1 \hat{\underset{R}{\otimes}} \cdots \hat{\underset{R}{\otimes}} \sigma_q,$$
$$(H^{n_1/2} \underset{H}{\otimes} H^{n_2/2}) \underset{R}{\otimes} \cdots \underset{R}{\otimes} (H^{n_{2r-1}/2} \underset{H}{\otimes} H^{n_{2r}/2}) \underset{R}{\otimes} H^{n_{2r+1}} \underset{R}{\otimes} \cdots \underset{R}{\otimes} R^{n_{2r+q}}) \quad (3.5)$$

Case ii) $\widetilde{id}^c = \widetilde{id}$, $G_0 \simeq U(1)$: Then $(\tilde{G}, \widetilde{id}^c, C^{N/2})$ is equivalent as complex representations to some

$$(R \times \tilde{G}_1 \times \cdots \times \tilde{G}_s, \hat{t} \hat{\underset{C}{\otimes}} \rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s, C \underset{C}{\otimes} C^{n_1} \underset{C}{\otimes} \cdots \underset{C}{\otimes} C^{n_s})$$

where $t \in R$, ρ_i is a complex irreducible representation of $\tilde{G}_i$ on C^{n_i}, $n_i \geqq 2 (i=1, \cdots, s)$ and $\Pi_{i=1}^s n_i = N/2$. So $(\tilde{G}, \widetilde{id}, R^N)$ is equivalent as real representation to

$$(Rx\tilde{G}_1 x \cdots x\tilde{G}_s, (\hat{t} \hat{\underset{C}{\otimes}} \rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s)_R, (C \underset{C}{\otimes} C^{n_1} \underset{C}{\otimes} \cdots \underset{C}{\otimes} C^{n_s})_R) \quad (3.6)$$

Case iii) $\widetilde{id}^c = \widetilde{id}$, $G_0 \simeq 1$: Then $(\tilde{G}, \widetilde{id}^c, C^{N/2})$ is equivalent as complex representations to some

$$(\tilde{G}_1 \times \cdots \times \tilde{G}_s, \rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s, C^{n_1} \underset{C}{\otimes} \cdots \underset{C}{\otimes} C^{n_s})$$

where ρ_i is a complex irreducible representation of $\tilde{G}_i$ on C^{n_i}, $n_i \geqq 2$ $(i=1, \cdots, s)$ and $\Pi_{i=1}^s n_i = N/2$. So $(\tilde{G}, \widetilde{id}, R^N)$ is equivalent as real representation to

$$(\tilde{G}_1 \times \cdots \times \tilde{G}_s, (\rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s)_R, C^{n_1} \underset{C}{\otimes} \cdots \underset{C}{\otimes} C^{n_s})_R) \quad (3.7)$$

where $\rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s$ is not 'real' since $(\rho_1 \hat{\underset{C}{\otimes}} \cdots \hat{\underset{C}{\otimes}} \rho_s)_R$ is real irreducible.

THEOREM 3.5 *Let* (G, E^N) *be an* o.t.g. *of cohomogeneity at most* 3. *If* $id: G \to SO(N)$ *is real irreducible and* $s \geqq 3$ (cf. (3.4)), then $(\tilde{G}, \widetilde{id}, R^N)$ *is equivalent as real representation to*

$$(\tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1, (\Lambda_1 \hat{\underset{H}{\otimes}} \Lambda_1) \hat{\underset{R}{\otimes}} (2\Lambda_1)^r, (H \underset{H}{\otimes} H) \underset{R}{\otimes} R^3) \quad (3.8)$$

Especially coh $(G, E^N) = 3$.

PROOF: Suppose id is real irreducible and $s \geqq 3$. Then $O(G, id, R^N)$ is contained in (1) $O((Sp(n_1/2) \times Sp(n_2/2)) \times SO(n_3), (id \hat{\underset{H}{\otimes}} id) \hat{\underset{R}{\otimes}} id, (H^{n_1/2} \underset{H}{\otimes} H^{n_2/2}) \underset{R}{\otimes} R^{n_3})$ for some $n_1, n_2 \geqq 2$, $n_3 \geqq 3$; $N = n_1 n_2 n_3$, (2) $O(SO(n_1) \times SO(n_2) \times SO(n_3), id \hat{\underset{R}{\otimes}} id \hat{\underset{R}{\otimes}} id, R^{n_1} \underset{R}{\otimes} R^{n_2} \underset{R}{\otimes} R^{n_3})$ for some n_1,

$n_2, n_3 \geqq 3$; $N=n_1n_2n_3$, or (3) $O(U(n_1)\times SU(n_2)\times SU(n_3), (id\underset{C}{\hat{\otimes}}id\underset{C}{\hat{\otimes}}id)_{R}, (C^{n_1}\underset{C}{\otimes}C^{n_2}\underset{C}{\otimes}C^{n_3})_{R})$ for some $n_1, n_2, n_3 \geqq 2$; $N=2n_1n_2n_3$ owing to (3.5), (3.6) and (3.7). On the other hand, coh (2) $\geqq 18$, coh (3)$\geqq 6$, coh ((1)(max $(n_1, n_2)\geqq 4$))$\geqq 8$ by Prop. 3.4(1)(2)(4). There G_0 is trivial, and $O(G, id, R^N)$ is contained in $O((Sp(1)\times Sp(1)\times SO(n_3), (id\underset{H}{\hat{\otimes}}id)\underset{R}{\hat{\otimes}}id, (H\otimes H)\otimes R^{n_3})$ which is equivalent to $O(SO(4)\times SO(n_3), id\underset{R}{\hat{\otimes}}id, R^4\underset{R}{\otimes}R^{n_3})$. Then $n_3=3$ since coh (G, E^N) $\leqq 3$. So $O(G, id, R^N)$ is contained in $O(\tilde{A}_1\times\tilde{A}_1\times\tilde{A}_1, (\Lambda_1\underset{H}{\hat{\otimes}}\Lambda_1)\underset{R}{\hat{\otimes}}(2\Lambda_1)^r, (H\underset{H}{\otimes}H)\underset{R}{\otimes}R^3)$. Since $s\geqq 3$, $\tilde{G}$ is isomorphic to $\tilde{A}_1\times\tilde{A}_1\times\tilde{A}_1$, and $O(\tilde{G}, \tilde{id}, R^N)=O(A_1\times A_1\times A_1, (\Lambda_1\underset{H}{\hat{\otimes}}\Lambda_1)\underset{R}{\hat{\otimes}}(2\Lambda_1)^r, (H\underset{H}{\otimes}H)\underset{R}{\otimes}R^3)$. Then (G, id, R^N) and (3.8) are equivalent as real representation since Λ_1, $2\Lambda_1$ are characterized by degrees of complex irreducible representations of $\tilde{A}_1$, and $12=2^2\cdot 3$(cf. Section 2). And coh $(G, E^N)=3$ by Prop. 3.3. Q.E.D.

Suppose $s=2$: $L=L_0\oplus L_1\oplus L_2$ (cf. (3.4)). Then $(\tilde{G}, \tilde{id}, R^N)$ is equivalent as real representation to one of the followings:

TYPE I) $(\tilde{G}_1\times\tilde{G}_2, \rho_1^r\underset{R}{\hat{\otimes}}\rho_2^r, R^{n_1}\underset{R}{\otimes}R^{n_2})$; $n_1\geqq n_2\geqq 3$, $N=n_1n_2$, ρ_i is a 'real' complex irreducible representation of $\tilde{G}_i$ on C^{n_i}, R^{n_i} is a $\tilde{G}_i$-invariant real form of $C^{n_i}(i=1, 2)$.

TYPE II) $(\tilde{G}_1\times\tilde{G}_2, \rho_1\underset{H}{\hat{\otimes}}\rho_2, H^{n_1}\underset{H}{\otimes}H^{n_2})$; $n_1\geqq n_2\geqq 1$, $N=4n_1n_2$, ρ_i is a 'quaternion' complex irreducible representation of $\tilde{G}_i$ on C^{2n_i}, and H^{n_i} is C^{2n_i} with the $\tilde{G}_i$-invariant quaternionic structure (i.e., the right multiplication of j)($i=1, 2$).

TYPE III) $(R\times\tilde{G}_1\times\tilde{G}_2, (t\underset{C}{\hat{\otimes}}\rho_1\underset{C}{\hat{\otimes}}\rho_2)_{R}, (C\underset{C}{\otimes}C^{n_1}\underset{C}{\otimes}C^{n_2})_{R})$; $n_1\geqq n_2\geqq 2$, $N=2n_1n_2$, ρ_i is a complex irreducible representation of $\tilde{G}_i(i=1, 2)$, $t\in R^{\times}$.

TYPE IV) $(\tilde{G}_1\times\tilde{G}_2, (\rho_1\underset{C}{\hat{\otimes}}\rho_2)_{R}, (C^{n_1}\underset{C}{\otimes}C^{n_2})_{R})$; $n_1\geqq n_2\geqq 2$, $N=2n_1n_2$, ρ_i is a complex irreducible representation of $\tilde{G}_i$ on $C^{n_i}(i=1, 2)$, and $\rho_1\otimes\rho_2$ is not 'real'.

LEMMA 3.6 *Let ρ_i be a linear representation on F^{m_i} of a compact Lie group K_i, and denote $d_i=2^im_i-\dim K_i$ where $i=0$(if $F=R$), 1(if $F=C$), or 2(if $F=H$). Then*

(1) *If $1\leqq n<m_i$, then* doh $(K_i\times GF(n), \rho_i\underset{F}{\hat{\otimes}}id, F^{m_i}\underset{F}{\otimes}F^n)\geqq d_i+n\{2^{i-1}(n-3))+1\}$ $(\geqq d_i+3$ *if moreover* $n\geqq 3)$.

(2) *If* $1\leqq n<m_i$, *then* doh $(K_i\times GF(n), \rho_i\underset{F}{\hat{\otimes}}id, F^{m_i}\underset{F}{\otimes}F^n)\geqq d_i+2^{i-1}\{n(n-1)-2\}+n$ $(\geqq d_i+2$ *if moreover* $n\geqq 2$ *and* $i\geqq 1)$.

PROOF: doh $(K_i\times GF(n), \rho_i\underset{F}{\hat{\otimes}}id, F^{m_i}\underset{F}{\otimes}F^n)\geqq\dim F^{m_i}\underset{F}{\otimes}F^n-\dim K_i\times GF(n)=d_i+2^i(n-1)\cdot m_i-(2^i-1)n-2^{i-1}n(n-1)$. Replacing m_i by n(resp. $n+1$), we have (1)(resp. (2)). Q.E.D.

Suppose $s=1$: $L=L_0\oplus L_1$ (cf. (3.4)). Then $(\tilde{G}, \tilde{id}, R^N)$ is equivalent as real representation to one of the followings:

TYPE V) $(\tilde{G}_1, \rho_1^r, R^{n_1})$; $n_1\geqq 3$, $N=n_1$, ρ_1 is a 'real' complex irreducible representation of $\tilde{G}_1$ on C^{n_1}, and R^{n_1} is a $\tilde{G}_1$-invariant real form of C^{n_1}.

TYPE VI) $(R\times\tilde{G}_1, (\hat{t}\underset{C}{\hat{\otimes}}\rho_1)_{R}, (C\underset{C}{\otimes}C^{n_1})_{R})$; $n_1\geqq 2$, $N=2n_1$, and ρ_1 is a complex irreducible representation of $\tilde{G}_1$ on C^{n_1}.

TYPE VII) $(\tilde{G}_1, \rho_1, C^{n_1})$; $n_1\geqq 2$, $N=2n_1$, ρ_1 is a complex irreducible representation of G_1 on C^{n_1}, and ρ_1 is not 'real'.

LEMMA 3.7 *If* $n_1 \leqq n_2$, *then* $GF(n_1)(\simeq GF(n_1)\times\{I_{n_2}\}$ *in* $GF(n_1)\times GF(n_2))$ *transforms any matrix* $X={}^t[x_1, \cdots, x_{n_1}]\in F(n_1, n_2)(x_i \in F^{n_2}$ *for* $i=1, \cdots, n_1)$ *to a form* $Y={}^t[y_1, \cdots, y_n]\in F(n_1, n_2)({}^t y_i \in F(1, n_2)$ *for* $i=1, \cdots, n_1)$ *such that* ${}^t y_i \bar{y}_j = c_i\delta_{ij}$ *for some* $c_i \in R$ $(i, j=1, \cdots, n_1)$ *by the action*(3.2).

PROOF: There is $A\in GF(n_1)$ such that A transforms $X^t\bar{X}\in pF(n_1)$ to a diagonal form $\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_{n_1}\end{pmatrix}$ by the action(3.1).

Then $Y=AX$ satisfied the desired property. Q.E.D.

Suppose $s=0$: $L=L_0$ (cf. (3.4)). Then $(G, id, \boldsymbol{R}^N)$ is equivalent as real representation to one of the followings:

TYPE VIII) $(\boldsymbol{R}, \hat{t}_{\boldsymbol{R}}, \boldsymbol{C}_{\boldsymbol{R}})$; $t\in \boldsymbol{R}^{\times}$.

TYPE IX) $(1, 0, \boldsymbol{R})$; 1 is the trivial group, and 0 is the trivial representation on $\boldsymbol{R}$.

Note that the o.t.g. of type VIII is equivalent to $O(SO(2), id, \boldsymbol{R}^2)$.

For general $s\geqq 0$, the estimate of coh $(G, \boldsymbol{E}^N)$ is given in each cases i), ii), iii), if *id*: $G\rightarrow SO(N)$ is real irreducible, by the following theorem. If moreover $s\geqq 3$, especially we have coh $(G, \boldsymbol{E}^N)\geqq s$.

THEOREM 3.8
(1) *In case* i), coh $(G, \boldsymbol{E}^N)=$coh *of* $(3.5)\geqq 4^r\cdot 3^q-6r-3q$,
(2) *In case* ii), coh $(G, \boldsymbol{E}^N)=$coh *of* $(3.6)\geqq 2^{s+1}-3s-1$,
(3) *In case* iii), coh $(G, \boldsymbol{E}^N)=$coh *of* $(3.7)\geqq 2^{s+1}-3s-1$.

PROOF: (3) follows from (2). For (2), we may assume $n_1\geqq\cdots\geqq n_s\geqq 2$. If $s<3$, then (2) is trivial. Suppose $s\geqq 3$. If $n_1\geqq n_2\cdots n_s$, then we denote $f(n_1, \cdots, n_s)=\dim pC(n_2\cdots n_s)-\dim SU(n_2)\times\cdots\times SU(n_s)=n_2^2\cdots n_s^2-n_2^2-\cdots-n_s^2+s-1$. Then $\partial f/\partial n_i=2n_i(n_2^2\cdots\hat{n}_i^2\cdots n_s^2-1)$ or $0\geqq 0$. If $n_1\leqq n_2\cdots n_s$, then we denote $f(n_1, \cdots, n_s)=\dim \boldsymbol{C}^{n_1}\underset{\boldsymbol{C}}{\otimes}\cdots\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}^{n_s}-\dim U(n_1)\times SU(n_2)\times\cdots\times SU(n_s)=2n_1\cdots n_s-n_1^2-\cdots-n_s^2+s-1$. Then $\partial f/\partial n_i=2(n_1\cdots\hat{n}_i\cdots n_s-n_i)\geqq 2(n_2\cdots n_s-n_1)\geqq 0$. Therefore coh $(3.6)\geqq f(n_1, \cdots, n_s)\geqq f(2, \cdots, 2)=2^{s+1}-3s-1$.

(1) Suppose $s=2r+q\leqq 2$. If $r, q\leqq 1$, then (1) is trivial. If $r=0$, $q=s=2$, then (1) follows from Prop. 3.3. If $s=3$, then (1) follows from Prop. 3.4. Assume $s\geqq 4$. Suppose $r=0$: Then we may assume $n_1\geqq\cdots\geqq n_s\geqq 3$. If $n_1\geqq n_2\cdots n_s$, then denote $f(n_1, \cdots, n_s)=\dim p\boldsymbol{R}(n_2\cdots n_s)-\dim SO(n_2)\times\cdots\times SO(n_s)=(n_2^2\cdots n_s^2+n_2\cdots n_s-n_2^2-\cdots-n_s^2+n_2+\cdots+n_s)/2$. Then $\partial f/\partial n_i=n_i(n_2^2\cdots\hat{n}_i^2\cdots n_s^2-1)+(n_2\cdots\hat{n}_i\cdots n_s+1)/2$ or $0\geqq 0$. If $n_1\leqq n_2\cdots n_s$, then denote $f(n_1, \cdots, n_s)=\dim \boldsymbol{R}^{n_1}\underset{\boldsymbol{R}}{\otimes}\cdots\underset{\boldsymbol{R}}{\otimes}\boldsymbol{R}^{n_s}-\dim SO(n_1)\times\cdots\times SO(n_s)=n_1\cdots n_s-(n_1^2+\cdots+n_s^2)/2+(n_1+\cdots+n_s)/2$. Then $\partial f/\partial n_i=n_1\cdots\hat{n}_i\cdots n_s-n_i+1/2\geqq n_2\cdots n_s-n_1+1/2\geqq 1/2$. Therefore coh $(3.5)\geqq f(n_1, \cdots, n_s)\geqq f(3, \cdots, 3)=3^s-3s=3^q-3q$. Suppose $q=0$: Then we may assume $n_1\geqq\cdots\geqq n_s\geqq 2$. If $n_1n_2\geqq n_3\cdots n_s$, then denote $g(n_1, \cdots, n_s)=\dim$

$pR(n_3\cdots n_s)-\dim\ \ Sp(n_3/2)\times\cdots\times Sp(n_s/2)=(n_3^2\cdots n_s^2+n_3\cdots n_s-n_3^2-\cdots-n_s^2-n_3-\cdots-n_s)/2$. Since $\partial g/\partial n_i\geqq 0$ $(i=1,\cdots,s)$, coh (3.5)$\geqq g(n_1,n_2,n_3,\cdots,n_s)\geqq g(n_1,n_2,2,\cdots 2)=2^{2s-5}+2^{s-3}-3(s-2)=2^{2r}(2^{2r-5}+2^{-3})-6r+6\geqq 4^r-6r$. If $n_1n_2\leqq n_3\cdots n_s$, then denote $h(n_1,\cdots,n_s)=\dim\ \ H^{n_1/2}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_s/2}-\dim Sp(n_1/2)\times\cdots\times Sp(n_s/2)=n_1\cdots n_s-(n_1^2+\cdots+n_s^2+n_1+\cdots+n_s)/2$. Since $\partial h/\partial n_i=n_i\cdots\hat{n_i}\cdots n_s-n_i-1/2\geqq n_2\cdots n_s-n_1-1/2\geqq n_1n_2^2-n_1-1/2\geqq 2\cdot 4-2-1/2>0$ $(i=1,\cdots,s)$, coh (3.5)$\geqq h(n_1,\cdots,n_s)\geqq h(n_3,n_3,n_3,n_4,\cdots,n_s)\geqq h(n_4,n_4,n_4,n_4,n_5,\cdots,n_s)\geqq h(2,\cdots,2)=2^s-3s=4^r-6r$. Finally suppose $r,q\geqq 1$: Then we may assume $n_1\geqq\cdots\geqq n_{2r}\geqq 2$ and $n_{2r+1}\geqq\cdots\geqq n_{2r+q}\geqq 3$. If $n_1n_2\geqq n_3\cdots n_s$, then denote $g(n_1,\cdots,n_s)=\dim pR(n_3\cdots n_s)-\dim Sp(n_3/2)\times\cdots\times Sp(n_{2r}/2)\times SO(n_{2r+1})\times\cdots\times SO(n_{2r+q})=(n_3^2\cdots n_s^2+n_3\cdots n_s-n_3^2-\cdots-n_s^2-n_3-\cdots-n_{2r}+n_{2r+1}+\cdots+n_{2r+q})/2$. Since $\partial g/\partial n_i\geqq 0$ $(i=1,\cdots,s)$, coh (3.5)$\geqq g(n_1,\cdots,n_s)\geqq g(n_1,n_2,2,\cdots,2,\underbrace{3,\cdots,3}_{q})=2^{2r}\cdot 3^q(2^{2r-5}\cdot 3^q+2^{-5})+6-6r-3q\geqq 4^r\cdot 3^q-6r-3q$. If $n_1n_2\leqq n_3\cdots n_s$, then denote $h(n_1,\cdots,n_s)=\dim(H^{n_1/2}\underset{H}{\otimes}\cdots\underset{H}{\otimes}H^{n_{2r}/2})\underset{R}{\otimes}R^{n_{2r+1}}\underset{R}{\otimes}\cdots\underset{R}{\otimes}R^{n_{2r+q}}-\dim Sp(n_1/2)\times\cdots\times Sp(n_{2r}/2)\times SO(n_{2r+1})\times\cdots\times SO(n_{2r+q})=n_1\cdots n_s-(n_1^2+\cdots+n_s^2+n_1+\cdots+n_{2r}-n_{2r+1}-\cdots-n_{2r+q})/2$. Since $\partial h/\partial n_i\geqq n_2\cdots n_s-n_1-1/2\geqq n_1(n_2^2-1)-1/2\geqq 2(2^2-1)-1/2>0$, coh (3.5)$\geqq h(n_1,\cdots,n_s)\geqq h(n_3,n_3,n_3,n_4,\cdots,n_s)\geqq h(n_4,n_4,n_4,n_4,n_5,\cdots,n_s)\geqq h(2,\cdots,2,\underbrace{3,\cdots,3}_{q})=4^r\cdot 3^q-6r-3q$. Q.E.D.

4. Orthogonal transformation groups of cohomogeneity at most 3

(I) Let (G, E^N) be a real irreducible o.t.g. of type I.

PROPOSITION 4.1 coh $(G, E^N)\leqq 3$ *if and only if* $(\tilde{G}, \tilde{id}, R^N)$ *is equivalent as real representation to one of the followings*:

coh=1: none,

coh=2: none,

coh=3:
(1) $(A_1\times A_1, (2\Lambda_1)^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^3\underset{R}{\otimes}R^3)$,
(2) $(A_3\times A_1, \Lambda_2^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^6\underset{R}{\otimes}R^3)$,
(3) $(C_2\times A_1, \Lambda_2^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^5\underset{R}{\otimes}R^3)$,
(4) $(B_k\times A_1, \Lambda_1^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^{2k+1}\underset{R}{\otimes}R^3)$; $k\geqq 3$,
(5) $(D_k\times A_1, \Lambda_1^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^{2k}\underset{R}{\otimes}R^3)$: $k\geqq 4$,
(6) $(B_3\times A_1, \Lambda_3^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^8\underset{R}{\otimes}R^3)$,
(7) $(D_4\times A_1, \Lambda_i^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^8\underset{R}{\otimes}R^3)$; $i=3,4$.

PROOF: Suppose coh $(G, E^N)\leqq 3$. Then $(\tilde{G}, \tilde{id}, R^N)$ is equivalent as real representation to (1), $\cdots$, (6), or (7) owing to Prop. 3.3(2)(4), Prop. 2.17, Lemma 3.6(1)($F=R$, $i=0$, $n=3$), $3\geqq$ doh $(G, E^N)\geqq d_0+3$, Prop. 2.1 ($d_0\leqq 3$), doh $(A_k\times A_1, (\Lambda_1+\Lambda_k)^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^{\dim A_k}\underset{R}{\otimes}R^3)=2\dim A_k-3\geqq 13$ $(k\geqq 2)$, Prop. 2.5, doh $(C_k\times A_1, (2\Lambda_1)^r\hat{\underset{R}{\otimes}}(2\Lambda_1)^r, R^{\dim C_k}\underset{R}{\otimes}R^3)=$

$2\dim C_k-3\geqq 17$ $(k\geqq 2)$, doh $(C_k\times A_1, \Lambda_2^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^{k(2k-1)-1}\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)=4k(k-1)-6\geqq 18(k\geqq 3)$, Prop. 2.8, doh $(B_k\times A_1, \Lambda_2^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^{\dim B_k}\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)=2\dim B_k-3\geqq 39(k\geqq 3)$, doh $(B_4\times A_1, \Lambda_4^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^{16}\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)=9$, Prop. 2.11, doh $(D_k\times A_1, \Lambda_2^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^{\dim D_k}\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)=2\dim D_k-3\geqq 53(k\geqq 4)$, the equivalence of o.t.g.'s $O(D_4\times A_1, \Lambda_i^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^8\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)$ for $i=1, 3, 4$ (cf. Remark 2.13), Prop. 2.15, Remark 2.16, $2\dim E_8-3\geqq 2\dim E_7-3\geqq 2\dim E_6-3\geqq 2\dim F_4-3\geqq 2\dim G_2-3\geqq 25$, doh $(F_4\times A_1, \Lambda_4^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^{26}\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)=23$, doh $(G_2\times A_1, \Lambda_2^r\hat{\otimes}_{\boldsymbol{R}}(2\Lambda_1)^r, \boldsymbol{R}^7\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)=4$.

Conversely if $(G, \boldsymbol{E}^N)$ is induced from (1), $\cdots$, (5), or (7), then $(G, \boldsymbol{E}^N)$ can also be induced from $(SO(n_1)\times SO(3), id\hat{\otimes}_{\boldsymbol{R}}id, \boldsymbol{R}^{n_1}\otimes_{\boldsymbol{R}}\boldsymbol{R}^3)$ for some $n_1\neq 4$. So coh $(G, \boldsymbol{E}^N)=3$(cf. Prop. 3.3(2)(4)). An o.t.g. induced from (6) is of coh 3. In fact Spin $(7)\times SO(3)$ acts on $\boldsymbol{R}(8, 3)$ through ι by the action (3.2)(cf. Prop. 3.3 Proof), and the isotropy subgroup at

$$\begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \\ & & \\ & & \end{pmatrix},$$ where $|x_i|$ $(i=1, 2, 3)$ are non-zero distinct real numbers, is locally isomorphic to $SU(2)$(cf. Yokota [24, Theorem 5.27, Theorem 5.2]). Q.E.D.

(II) Let (G, E^N) be a real irreducible o.t.g. of type II.

PROPOSITION 4.2 coh $(G, \boldsymbol{E}^N)\leqq 3$ *if and only if* $(\tilde{G}, \tilde{id}, \boldsymbol{R}^N)$ *is equivalent as real representation to one of the followings*:

coh=1: (8) $(A_1\times A_1, \Lambda_1\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}\otimes_{\boldsymbol{H}}\boldsymbol{H})$,
(9) $(C_k\times A_1, \Lambda_1\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}^k\otimes_{\boldsymbol{H}}\boldsymbol{H})$; $k\geqq 2$,
coh=2: (10) $(C_k\times C_2, \Lambda_1\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}^k\otimes_{\boldsymbol{H}}\boldsymbol{H}^2)$; $k\geqq 2$,
(11) $(A_1\times A_1, 3\Lambda_1\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}^2\otimes_{\boldsymbol{H}}\boldsymbol{H})$,
coh=3: (12) $(C_k\times C_3, \Lambda_1\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}^k\otimes_{\boldsymbol{H}}\boldsymbol{H}^3)$; $k\geqq 3$,
(13) $(C_k\times A_1, \Lambda_1\hat{\otimes}_{\boldsymbol{H}}3\Lambda_1, \boldsymbol{H}^k\otimes_{\boldsymbol{H}}\boldsymbol{H}^2)$; $k\geqq 2$.

PROOF: Suppose coh $(G, \boldsymbol{E}^N)\leqq 3$. Then $n_2\leqq 3$(cf. Prop. 3.3(1)(4)). Assume $n_2=3$. Then $(\tilde{G}_2, \rho_2, \boldsymbol{H}^{n_2})$ is equivalent as complex representation to $(C_3, \Lambda_1, \boldsymbol{H}^3)$ owing to Prop. 2.20 and coh $(Sp(n_1)\times A_1, id\hat{\otimes}_{\boldsymbol{H}}5\Lambda_1, \boldsymbol{H}_1^n\otimes_{\boldsymbol{H}}\boldsymbol{H}^3)\geqq$doh $(A_1, p\boldsymbol{H}(3))=12$(cf. Prop. 3.3(1)). So $(\tilde{G}, \tilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to (12) owing to Lemma 3.6(1)($F=\boldsymbol{H}$, $i=2$, $m_2=n_1$, $n=n_2=3$, $d_i+k\{2^{i-1}(k-3)+1\}=d_2+3$), $3\geqq$doh $(G, \boldsymbol{E}^N)\geqq d_2+3$, Prop.'s 2.4, 2.7, 2.10, 2.14, doh $(D_6\times C_3, \Lambda_i\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}^{16}\otimes_{\boldsymbol{H}}\boldsymbol{H}^3)=105(i=5, 6)$, Prop. 2.15, Remark 2.16, doh $(E_7\times C_3, \Lambda_6\hat{\otimes}_{\boldsymbol{H}}\Lambda_1, \boldsymbol{H}^{28}\otimes_{\boldsymbol{H}}\boldsymbol{H}^3)=171$.

Assume $n_2=2$. Then $(\tilde{G}_2, \rho_2, \boldsymbol{H}^{n_2})$ is equivalent as complex representation to $(C_2, \Lambda_1, \boldsymbol{H}^2)$ or $(A_1, 3\Lambda_1, \boldsymbol{H}^2)$ owing to Prop. 2.20, deg $\rho_1=2n_1>4$(cf. Prop. 2.20 and doh $(A_1\times A_1,$

$3\Lambda_1 \underset{\boldsymbol{H}}{\hat{\otimes}} 3\Lambda_1, \boldsymbol{H}^2 \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H}^2)=10$). So $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to (10) or (13) owing to Lemma 3.6(2)($F=\boldsymbol{H}$, $i=2$, $m_2=n_1>n=n_2=2$), $3 \geqq \mathrm{doh}\,(G, \boldsymbol{E}^N) \geqq d_2+2$, deg $\rho_1>4$, Prop.'s 2.4, 2.7, 2.10, 2.14, doh $(D_6 \times A_1, \Lambda_i \underset{\boldsymbol{H}}{\hat{\otimes}} 3\Lambda_1, \boldsymbol{H}^{16} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H}^2) \geqq$ doh $(D_6 \times C_2, \Lambda_i \underset{\boldsymbol{H}}{\hat{\otimes}} \Lambda_1, \boldsymbol{H}^{16} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H}^2)=52$($i=5$, 6), Prop. 2.15, Remark 2.16, doh $(E_7 \times A_1, \Lambda_6 \underset{\boldsymbol{H}}{\hat{\otimes}} 3\Lambda_1, \boldsymbol{H}^{28} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H}^2) \geqq$ doh $(E_7 \times C_2, \Lambda_6 \underset{\boldsymbol{H}}{\hat{\otimes}} \Lambda_1, \boldsymbol{H}^{28} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H}^2)=59$.

Assume $n_2=1$. Then $(\tilde{G}_2, \rho_2, \boldsymbol{H}^{n_2})$ is equivalent as complex representation to $(A_1, \Lambda_1, \boldsymbol{H})$ by Prop. 2.20. So $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to (8), (9) or (11) owing to Lemma 3.6(1)($F=\boldsymbol{H}$, $i=2$, $m_2=n_1$, $n=1$, $d_i+n\{2^{i-1}(n-3)+1\}=d_2-3$), $3 \geqq \mathrm{doh}\,(G, \boldsymbol{E}^N) \geqq d_2-3$, Prop. 2.4, coh $(A_5 \times A_1, \Lambda_3 \underset{\boldsymbol{H}}{\hat{\otimes}} \Lambda_1, \boldsymbol{H}^{10} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H})=4$(cf. The linear isotropy representation of the symmetric pair $(E_6, SU(6) \cdot SU(1))$ of rank 4 is characterized as a real 40 dimensional irreducible almost faithful representation of $A_5 \times A_1$ owing to Section 2), Prop.'s 2.7, 2.10, 2.14, Remark 2.13, coh $(D_6 \times A_1, \Lambda_i \underset{\boldsymbol{H}}{\hat{\otimes}} \Lambda_1, \boldsymbol{H}^{16} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H})=4$($i=5$, 6)(cf. The linear isotropy representation of the symmetric pair (E_7, Spin $(12) \cdot Sp(1)$) of rank 4 is characterized as a real 64 dimensional irreducible almost faithful representation of $D_6 \times A_1$ owing to Section 2), Prop. 2.15, Remark 2.16, coh $(E_7 \times A_1, \Lambda_6 \underset{\boldsymbol{H}}{\hat{\otimes}} \Lambda_1, \boldsymbol{H}^{28} \underset{\boldsymbol{H}}{\otimes} \boldsymbol{H})=4$(cf. The linear isotropy representation of the symmetric pair $(E_8, E_7 \cdot Sp(1))$ of rank 4 is characterized as a real 112 dimensional irreducible almost faithful representation of $E_7 \times A_1$ owing to Section 2).

The linear isotropy representation of the symmetric pair (E_7, Spin(12)$\cdot$Sp(1)) of rank 4 is characterized as a real 64 dimensional irreducible almost faithful representation of $D_6 \times A_1$ owing to Section 2), Prop. 2.15, Remank 2.16, coh $(E_7 \times A_1, \Lambda_6 \underset{\boldsymbol{H}}{\hat{\otimes}} \Lambda_1, \boldsymbol{H}^{28} \underset{\boldsymbol{H}}{\hat{\otimes}} \boldsymbol{H})=4$(cf. The linear isotropy representation of the symmetric pair $(E_8, E_7 \cdot Sp(1))$ of rank 4 is characterized as a real 112 dimensional irreducible almost faithful representation of $E_7 \times A_1$ owing to Section 2).

Conversely an o.t.g. induced from (8) or (9) is of coh 1 by Prof. 3.3(1) (4) ($F=\boldsymbol{H}$, $n_2=1$, $K=Sp(1)$). An o.t.g. induced from (10) is of coh 2 by Prop. 3.3 (1) (4) ($F=\boldsymbol{H}$, $n_2=2$, $K=Sp(2)$). An o.t.g. induced from (12) is of coh 3 by Prop. 3.3 (1) (4) ($F=\boldsymbol{H}$, $n_2=3$, $K=Sp(3)$). An o.t.g. induced from (11) is of coh 2(cf. The linear isotropy representation of the symmetric pair $(G_2, SO(4))$ of rank 2 is characterized as a real 8 dimensional irreducible almost faithful representation of $A_1 \times A_1$ owing to Prop.'s 2.1, 2.2, 2.4). If $(G, \boldsymbol{E}^N)$ is induced from (13), then coh $(G, \boldsymbol{E}^N)=$ coh $(A_1, p\boldsymbol{H}(2)) \geqq$ doh $(A_1, p\boldsymbol{H}(2))=3$(cf. Prop. 3.3) and coh $(G, \boldsymbol{E}^N) \leqq$ coh $(A_1, h\boldsymbol{H}(2))=$ coh $(A_1, 0^r \oplus (4\Lambda_1)^r, \boldsymbol{R} \oplus \boldsymbol{R}^5)=1+$ coh $(A_1, (4\Lambda_1)^r, \boldsymbol{R}^5)$ $=3$(cf. The linear isotropy representation of the symmetric pair $(SU(3), SO(3))$ of rank 2 is characterized as a real 5 dimensional irreducible representation of A_1 owing to Prop.'s 2.1, 2.2, 2.4), where the action of A_1 on $p\boldsymbol{H}(2)$ is given as Prop. 3.3 and Lemma 3.2. Q.E.D.

(III) Let $(G, \boldsymbol{E}^N)$ be a real irreducible o.t.g. of type III.

PROPOSITION 4.3 coh $(G, \boldsymbol{E}^N) \leqq 3$ *if and only if* $(\tilde{G}, \tilde{id}, \boldsymbol{R}^N)$ *is equivalent as real representation to one of the followings*:

coh$=1$: *none,*

coh$=2$: (14) $(\boldsymbol{R}\times A_k\times A_1,\ \hat{t}\hat{\otimes}_{C}\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{k+1}\otimes_{C}\boldsymbol{C}^2)$; $k\geqq 1$, $t\in\boldsymbol{R}^\times$.

coh$=3$: (15) $(\boldsymbol{R}\times A_k\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{k+1}\otimes_{C}\boldsymbol{C}^3)$; $k\geqq 2$, $t\in\boldsymbol{R}^\times$.

(16) $(\boldsymbol{R}\times C_k\times A_1,\ \hat{t}\hat{\otimes}_{C}\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{2k}\otimes_{C}\boldsymbol{C}^2)$; $k\geqq 2$, $t\in R^\times$.

PROOF: Suppose coh $(G, \boldsymbol{E}^N)\leqq 3$. Then $n_2\leqq 3$(cf. Prop. 3.3(1)(4)).

Assume $n_2=3$. Then $(\tilde{G}_2, \rho_2, \boldsymbol{C}^{n_2})$ is equivalent as complex representation to $(A_2, \Lambda_1, \boldsymbol{C}^3)$ owing to Prop. 2.18, Remark 2.19 and coh $(U(n_1)\times A_1,\ id\hat{\otimes}_{C}2\Lambda_1,\ \boldsymbol{C}^{n_1}\otimes_{C}\boldsymbol{C}^3)\geqq$ doh $(A_1, p\boldsymbol{C}(3))=6$. If ρ_1 is 'real' and $n_1\geqq 6$, then coh $(G, \boldsymbol{E}^N)=$ coh $(U(1)\times\tilde{G}_1\times A_2,\ id\hat{\otimes}_{C}\rho_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{n_1}\otimes_{C}\boldsymbol{C}^3)=$ coh $(\tilde{G}_1\times(U(1)\times A_2),\ \rho_1'\hat{\otimes}_{R}(id\hat{\otimes}_{C}\Lambda_1)_{\boldsymbol{R}},\ \boldsymbol{R}^{n_1}\otimes_{R}(\boldsymbol{C}\otimes_{C}\boldsymbol{C}^3)_{\boldsymbol{R}})\geqq$ coh $(SO(n_1)\times U(3),\ id\hat{\otimes}_{R}id_{\boldsymbol{R}},\ \boldsymbol{R}^{n_1}\otimes_{R}\boldsymbol{C}^3_{\boldsymbol{R}})=$ coh $(U(3), p\boldsymbol{R}(6))\geqq$ doh $(U(3), p\boldsymbol{R}(6))=12$(cf. Prop. 3.3). So $(\tilde{G}_1, \rho_1, \boldsymbol{C}^{n_1})$ is not 'real' or $n_1\leqq 5$. Then $(\tilde{G}, \tilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to (15) owing to Lemma 3.6(1)($F=\boldsymbol{C}$, $i=1$, $m_1=n_1$, $n=n_2=3$), $3\geqq$ coh $(G, \boldsymbol{E}^N)\geqq d_1+3$, Prop. 2.2($\Lambda_2(k=3)$ is 'real' of degree 6), Remark 2.3, doh $(\boldsymbol{R}\times A_k\times A_2,\ \hat{t}\hat{\otimes}_{C}2\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{(k+2)(k=1)/2}\otimes_{C}\boldsymbol{C}^3)$ $=(k+1)(2k-1)-8\geqq 27(k\geqq 4)$, Prop. 2.6($\Lambda_2(k=2)$ is 'real' of degree 11), doh $(\boldsymbol{R}\times C_2\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^4\otimes_{C}\boldsymbol{C}^3)=5$, coh $(\boldsymbol{R}\times C_k\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{2k}\otimes_{C}\boldsymbol{C}^3)\geqq$ dim $\boldsymbol{C}\otimes_{C}\boldsymbol{C}^{2k}\otimes_{C}\boldsymbol{C}^3-$ dim $\boldsymbol{R}\times C_k\times A_2+$ dim $C_{k-3}=6$(cf. Any isotropy subgroup contains C_{k-3}), Prop. 2.9 ($\Lambda_1(k\geqq 3)$ is 'real' of degree$\geqq 7$), doh $(\boldsymbol{R}\times B_k\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_k\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{2^k}\otimes_{C}\boldsymbol{C}^3)$ $=3\cdot 2^{k+1}-2k^2-k-9\geqq 18(k\geqq 3)$, Prop. 2.12 ($\Lambda_1(k\geqq 4)$ is 'real' of degree$\geqq 8$), doh $(\boldsymbol{R}\times D_k\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_i\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{2^{k-1}}\otimes_{C}\boldsymbol{C}^3)=3\cdot 2^k-k(2k-1)-9\geqq 11$ for$\cdot i=k$, $k-1$ (if $k\geqq 4$), Prop. 2.15, Remark 2.16, doh $(\boldsymbol{R}\times E_6\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{27}\otimes_{C}\boldsymbol{C}^3)=75$, doh $(\boldsymbol{R}\times E_7\times A_2,\ \hat{t}\hat{\otimes}_{C}\Lambda_6\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{56}\otimes_{C}\boldsymbol{C}^3)=194$.

Assume $n_2=2$. Then $(\tilde{G}_2, \rho_2, \boldsymbol{C}^{n_2})$ is equivalent as complex representation to $(A_1, \Lambda_1, \boldsymbol{C}^2)$ by Prop. 2.18. If $(\tilde{G}_1, \rho_1, \boldsymbol{C}^{n_1})$ is 'real' of degree $n_1\geqq 4$, then coh $(G, \boldsymbol{E}^N)=$ coh $(U(1)\times\tilde{G}_1\times A_1,\ id\hat{\otimes}_{C}\rho_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{n_1}\otimes_{C}\boldsymbol{C}^2)=$ coh $(\tilde{G}_1\times(U(1)\times A_1),\ \rho_1'\hat{\otimes}_{R}(id\hat{\otimes}_{C}\Lambda_1)_{\boldsymbol{R}},\ \boldsymbol{R}^{n_1}\otimes_{R}(\boldsymbol{C}\otimes_{C}\boldsymbol{C}^2)_{\boldsymbol{R}})\geqq$ coh $(SO(n_1)\times U(2),\ id\hat{\otimes}_{R}id_{\boldsymbol{R}},\ \boldsymbol{R}^{n_1}\hat{\otimes}_{R}\boldsymbol{C}^2_{\boldsymbol{R}})=$ coh $(U(2), p\boldsymbol{R}(4))\geqq$ doh $(U(2), p\boldsymbol{R}(4))=6$. So $(\tilde{G}_1, \rho_1, \boldsymbol{C}^{n_1})$ is not 'real' or $n_1\leqq 3$. Then $(\tilde{G}, \tilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to (14) or (16) owing to Prop. 2.18, Lemma 3.6(2)($F=\boldsymbol{C}$, $i=1$, $m_1=n_1>n=n_2=2$), $3\geqq$ coh $(G, \boldsymbol{E}^N)$ $\geqq d_1+2$, Prop. 2.2($\Lambda_2(k=3)$ is 'real' of degree 6), Remark 2.3, doh $(\boldsymbol{R}\times A_k\times A_1,\ \hat{t}\hat{\otimes}_{C}\Lambda_2\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{k(k+1)}\otimes_{C}\boldsymbol{C}^2)=k^2-4\geqq 12(k\geqq 4)$, doh $(\boldsymbol{R}\times A_k\times A_1,\ \hat{t}\hat{\otimes}_{C}2\Lambda_1\hat{\otimes}_{C}\Lambda_1,\ \boldsymbol{C}\otimes_{C}\boldsymbol{C}^{(k+2)(k+1)/2}\otimes_{C}\boldsymbol{C}^2)$ $=(k+1)(k+3)-3\geqq 5(k\geqq 1)$, Prop. 2.6($\Lambda_2(k=2)$ is 'real' of degree 11), Prop. 2.9($\Lambda_1(k\geqq 3)$

is 'real' of degree$\geqq 7$), doh $(\boldsymbol{R}\times B_k\times A_1,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_k\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{2^k}\underset{C}{\otimes}\boldsymbol{C}^2)=2^{k+2}-k(2k+1)-4\geqq 7(k\geqq 3)$, Prop. 2.12($\Lambda_1(k\geqq 4)$ is 'real' of degree$\geqq 8$, $\Lambda_i(k=4)$ for $i=3, 4$ are 'real' of degree 8), doh $(\boldsymbol{R}\times D_k\times A_1,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_i\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{2^{k-1}}\underset{C}{\otimes}\boldsymbol{C}^2)=2^{k+1}-k(2k-1)-4\geqq 15$ for $i=k-1, k$(if $k\geqq 5$), Prop. 2.15, Remark 2.16, doh $(\boldsymbol{R}\times E_6\times A_1,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{27}\underset{C}{\otimes}\boldsymbol{C}^2)=26$, doh $(\boldsymbol{R}\times E_7\times A_1,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_6\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{56}\underset{C}{\otimes}\boldsymbol{C}^2)=87$.

Conversely an o.t.g. induced from (14)(resp. (15)) is of coh 2(resp. 3)(cf. Prop. 3.3(1)(4)). If $(G, \boldsymbol{E}^N)$ is induced from (16), then coh $(G, \boldsymbol{E}^N)=$coh $(U(1)\times C_k\times A_1,\ id\underset{C}{\hat{\otimes}}\Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{2k}\underset{C}{\otimes}\boldsymbol{C}^2)=$coh $(U(1)\times(C_k\times A_1),\ id\underset{C}{\hat{\otimes}}(\Lambda_1\underset{H}{\hat{\otimes}}\Lambda_1)^C,\ \boldsymbol{C}\underset{C}{\otimes}(\boldsymbol{H}^k\underset{H}{\otimes}\boldsymbol{H})^C)=$coh $(SO(2)\times(C_k\times A_1),\ id\underset{R}{\hat{\otimes}}(\Lambda_1\underset{H}{\hat{\otimes}}\Lambda_1),\ \boldsymbol{R}^2\underset{R}{\otimes}(\boldsymbol{H}^k\underset{H}{\otimes}\boldsymbol{H})=$coh $(C_k\times(SO(2)\times A_1),\ \Lambda_1\underset{H}{\hat{\otimes}}(id\underset{R}{\hat{\otimes}}\Lambda_1),\ \boldsymbol{H}^k\underset{H}{\otimes}(\boldsymbol{R}^2\underset{R}{\otimes}\boldsymbol{H}))=$coh $(SO(2)\times A_1,\ p\boldsymbol{H}(2))=$coh $(SO(2),\ p\boldsymbol{R}(2))+$coh $(A_1,\ (2\Lambda_1)^r,\ \boldsymbol{R}^3)=2+1=3$(cf. Prop. 3.3). Q.E.D.

(IV) Let $(G, \boldsymbol{E}^N)$ be a real irreducible o.t.g. of type IV.

PROPOSITION 4.4 coh $(G, \boldsymbol{E}^N)\leqq 3$ *if and only if* $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ *is equivalent as real representation to one of the followings*:

coh$=1$: *none*,

coh$=2$: (17) $(A_k\times A_1,\ \Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}^{k+1}\underset{C}{\otimes}\boldsymbol{C}^2)$; $k\geqq 2$,

coh$=3$: (18) $(A_k\times A_2,\ \Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}^{k+1}\underset{C}{\otimes}\boldsymbol{C}^3)$; $k\geqq 3$.

PROOF: Suppose coh $(G, \boldsymbol{E}^N)\leqq 3$. Then $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to (17) or (18) owing to Prop. 4.3. In fact, $(C_k\times A_1,\ \Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}^{2k}\underset{C}{\otimes}\boldsymbol{C}^2)(k\geqq 2)$ and $(A_1\times A_1,\ \Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}^2\underset{C}{\otimes}\boldsymbol{C}^2)$ are 'real', so they are not real irreducible, and coh $(A_2\times A_2,\ \Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}^3\underset{C}{\otimes}\boldsymbol{C}^3)=4$ since $(U(1)\times A_2\times A_2,\ id\underset{C}{\hat{\otimes}}\Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^3\underset{C}{\otimes}\boldsymbol{C}^3)$ is equivalent to the linear isotropy representation of the Hermitian symmetric pair $(SU(6), S(U(3)\times U(3)))$ of rank 3 whose restricted root system is of type C(cf. Tasaki–Yasukura[22], Helgason[7]).

Conversely an o.t.g. induced from (17)(resp. (18)) is of coh 2(resp. 3) since $(U(1)\times A_k\times A_1,\ id\underset{C}{\hat{\otimes}}\Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{k+1}\underset{C}{\otimes}\boldsymbol{C}^2)$ of $k\geqq 2$(resp. $(U(1)\times A_h\times A_2,\ id\underset{C}{\hat{\otimes}}\Lambda_1\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{h+1}\underset{C}{\otimes}\boldsymbol{C}^3)$ of $h\geqq 3$) is equivalent to the linear isotropy representation of the Hermitian symmetric pair $(SU(k+3), S(U(k+1)\times U(2)))$ of rank 2(resp. $(SU(h+4), S(U(h+1)\times U(3)))$ of rank 3) whose restricted root system is of type BC(cf. [22], [7]). Q.E.D.

(V) Let $(G, \boldsymbol{E}^N)$ be a real irreducible o.t.g. of type V.

PROPOSITION 4.5 coh $(G, \boldsymbol{E}^N)\leqq 3$ *if and only if* $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ *is equivalent as real representation to one of the followings*:

coh$=1$: (19) $(A_1,\ (2\Lambda_1)^r,\ \boldsymbol{R}^3)$, (20) $(A_3,\ \Lambda_2^r,\ \boldsymbol{R}^6)$,

(21) $(C_2,\ \Lambda_2^r,\ \boldsymbol{R}^5)$, (22) $(B_k,\ \Lambda_1^r,\ \boldsymbol{R}^{2k+1})$; $k\geqq 3$,

(23) $(D_k, \Lambda_1^r, \boldsymbol{R}^{2k})$; $k\geqq 4$, (24) $(D_4, \Lambda_i^r, \boldsymbol{R}^8)$; $i=3, 4$,
(25) $(B_3, \Lambda_3^r, \boldsymbol{R}^8)$ (26) $(B_4, \Lambda_4^r, \boldsymbol{R}^{16})$,
(27) $(G_2, \Lambda_2^r, \boldsymbol{R}^7)$,

coh=2: (28) $(A_2, (\Lambda_1+\Lambda_2)^r, \boldsymbol{R}^8)$, (29) $(A_1, (4\Lambda_1)^r, \boldsymbol{R}^5)$,
(30) $(C_3, \Lambda_2^r, \boldsymbol{R}^{14})$, (31) $(C_2, (2\Lambda_1)^r, \boldsymbol{R}^{10})$,
(32) $(G_2, \Lambda_1^r, \boldsymbol{R}^{14})$, (33) $(F_4, \Lambda_4^r, \boldsymbol{R}^{26})$,

coh=3: (34) $(A_3, (\Lambda_1+\Lambda_3)^r, \boldsymbol{R}^{15})$, (35) $(C_3, (2\Lambda_1)^r, \boldsymbol{R}^{21})$,
(36) $(C_4, \Lambda_2^r, \boldsymbol{R}^{27})$, (37) $(B_3, \Lambda_2^r, \boldsymbol{R}^{21})$.

PROOF: Suppose coh $(G, \boldsymbol{E}^N)\leqq 3$. Then $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to one of (19)~(37) owing to Prop. 2.1, coh $(A_k, (\Lambda_1+\Lambda_k)^r, \boldsymbol{R}^{\dim A_k})=k$, Prop. 2.5, coh $(C_k, (2\Lambda_1)^r, \boldsymbol{R}^{\dim C_k})=k$, coh $(C_k, \Lambda_2^r, \boldsymbol{R}^{(k-1)(2k+1)})=k-1$(cf. $O(C_k, \Lambda_2^r, \boldsymbol{R}^{(k-1)(2k+1)})$ is equivalent to the linear isotropy representation of the symmetric pair $(SU(2k), Sp(k))$ of rank $k-1$), Prop. 2.8, coh $(B_k, \Lambda_2^r, \boldsymbol{R}^{\dim B_k})=k$, Prop. 2.11, coh $(D_k, \Lambda_2^r, \boldsymbol{R}^{\dim D_k})=k$, the equivalence of $O(D_4, \Lambda_i^r, \boldsymbol{R}^8)$ for $i=1, 4, 3$, Prop. 2.15, coh $(F_4, \Lambda_1^r, \boldsymbol{R}^{52})=4$, coh $(E_6, \Lambda_6^r, \boldsymbol{R}^{78})=6$, coh $(E_7, \Lambda_1^r, \boldsymbol{R}^{144})=7$, coh $(E_8, \Lambda_7^r, \boldsymbol{R}^{248})=8$.

Conversely an o.t.g. induced from one of (19)~(24) is equivalent to $(SO(n), id, \boldsymbol{R}^n)$ for some $n\neq 4$, which is of coh 1. An o.t.g. induced from (25), (26) or (27) is of coh 1(cf. Yokota [24, Theorems 5.27, 5.50, 5.3]. O.t.g.'s (28)~(33) are equivalent to the linear isotropy representation of the symmetric pairs $(SU(3)\times SU(3), SU(3))$, $(SU(3), SU(2))$, $(SU(6), Sp(3))$, $(Sp(2)\times Sp(2), Sp(2))$, $(G_2\times G_2, G_2)$, (E_6, F_4) of rank 2 respectively (cf. Prop.'s 2.1, 2.5, 2.15). O.t.g.'s induced from (34)~(37) are equivalent to the linear isotropy representations of the symmetric pairs $(SU(4)\times SU(4), SU(4))$, $(Sp(3)\times Sp(3), Sp(3))$, $(SU(8), Sp(4))$, $(SO(7)\times SO(7), SO(7))$ of rank 3 respectively (cf. Prop.'s 2.1, 2.5, 2.8). They are also characterized by their degrees among 'real' complex irreducible representations. Q.E.D.

(VI) Let $(G, \boldsymbol{E}^N)$ be a real irreducible o.t.g. of type VI.

PROPOSITION 4.6 coh $(G, \boldsymbol{E}^N)\leqq 3$ if and only if $(\tilde{G}, \widetilde{id}, \boldsymbol{R}^N)$ is equivalent as real representation to one of the followings:

coh=1: (38) $(\boldsymbol{R}\times A_k, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_1, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{k+1})$; $k\geqq 1$, $t\in \boldsymbol{R}^\times$,
(39) $(\boldsymbol{R}\times C_k, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_1, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{2k})$; $k\geqq 2$, $t\in \boldsymbol{R}^\times$,

coh=2: (40) $(\boldsymbol{R}\times B_k, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_1, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{2k+1})$; $k\geqq 3$, $t\in \boldsymbol{R}^\times$,
(41) $(\boldsymbol{R}\times D_k, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_1, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{2k})$; $k\geqq 4$, $t\in \boldsymbol{R}^\times$,
(42) $(\boldsymbol{R}\times D_4, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_i, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{8})$; $i=3, 4$, $t\in \boldsymbol{R}^\times$,
(43) $(\boldsymbol{R}\times A_1, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}2\Lambda_1, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{3})$; $t\in \boldsymbol{R}^\times$,
(44) $(\boldsymbol{R}\times A_3, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_2, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{6})$; $t\in \boldsymbol{R}^\times$,
(45) $(\boldsymbol{R}\times C_2, \hat{t}\,\hat{\otimes}_{\boldsymbol{C}}\Lambda_2, \boldsymbol{C}\otimes_{\boldsymbol{C}}\boldsymbol{C}^{5})$; $t\in \boldsymbol{R}^\times$,

(46) $(\boldsymbol{R}\times G_2,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_2,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^7);\ t\in R^{\times}$,
(47) $(\boldsymbol{R}\times B_3,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_3,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^8);\ t\in R^{\times}$,
(48) $(\boldsymbol{R}\times D_5,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_5,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{16});\ t\in R^{\times}$,
(49) $(\boldsymbol{R}\times A_4,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_2,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{10});\ t\in R^{\times}$,
coh=3: (50) $(\boldsymbol{R}\times A_2,\ \hat{t}\underset{C}{\hat{\otimes}}2\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^6);\ t\in R^{\times}$,
(51) $(\boldsymbol{R}\times A_5,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_2,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{15});\ t\in R^{\times}$,
(52) $(\boldsymbol{R}\times A_6,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_2,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{21});\ t\in R^{\times}$,
(53) $(\boldsymbol{R}\times B_4,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_4,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{16});\ t\in R^{\times}$,
(54) $(\boldsymbol{R}\times E_6,\ \hat{t}\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{27});\ t\in R^{\times}$.

PROOF: Suppose coh $(G, \boldsymbol{E}^N)\leqq 3$. Then $(G, id, \boldsymbol{R}^N)$ is equivalent as real representation to one of (38)~(54) owing to Lemma 3.6(1)($F=C$, $i=1$, $n=1$), Prop. 2.2, Remark 2.3, coh $(U(1)\times A_k,\ id\underset{C}{\hat{\otimes}}\Lambda_2,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{(k+1)k})=[(k+1)/2]$ (cf. $(U(1)\times A_k,\ id\underset{C}{\hat{\otimes}}\Lambda_2,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{(k+1)k})$ is equivalent to the linear isotropy representation of the symmetric pair $(SO(2k+2), U(k+1))$ of rank $[(k+1)/2]$), $[(k+1)/2]\geqq 4(k\geqq 7)$, Prop. 2.6, Prop. 2.9, Prop. 2.12, Remark 2.13, coh $(U(1)\times D_6,\ id\underset{C}{\hat{\otimes}}\Lambda_6,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{32})\geqq 4$(cf. $(U(1)\times D_6,\ id\underset{C}{\hat{\otimes}}\Lambda_6,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{32})$ is contained in the linear isotropy representation of the symmetric pair $(E_7, Sp(1)\cdot Spin(12))$ of rank 4), Prop. 2.15, Prop. 2.16, coh $(U(1)\times F_4,\ id\underset{C}{\hat{\otimes}}\Lambda_4,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{26})\geqq 7$(cf. Each isotropy subgroup contains a group which is isomorphic to $SU(3)\subset G_2\subset \mathrm{Spin}(7)\subset \mathrm{Spin}(8)\subset F_4$ by Yokota[24, Prop.'s 5.45, 5.48, Thm's 5.33, 5.27, 5.2]), coh $(U(1)\times E_7,\ id\underset{C}{\hat{\otimes}}\Lambda_6,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{56})\geqq 4$(cf. $(U(1)\times E_7,\ id\underset{C}{\hat{\otimes}}\Lambda_6,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{56})$ is contained in the linear isotropy representation of the symmetric pair $(E_8, Sp(1)\cdot E_7)$ of rank 4), doh $(U(1)\times G_2,\ id\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{14})=13$, doh $(U(1)\times F_4,\ id\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{52})=51$, doh $(U(1)\times E_6,\ id\underset{C}{\hat{\otimes}}\Lambda_6,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{78})=77$, doh $(U(1)\times E_7,\ id\underset{C}{\hat{\otimes}}\Lambda_1,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{133})=132$, doh $(U(1)\times E_8,\ id\underset{C}{\hat{\otimes}}\Lambda_7,\ \boldsymbol{C}\underset{C}{\otimes}\boldsymbol{C}^{248})=247$.

Conversely coh (38)=coh (39)=1 since $SU(k+1)$ and $Sp(k)$ are transitive on hyperspheres in the representation spaces. (40)~(45) are equivalent to $(SO(2)\times SO(n),\ id\underset{R}{\hat{\otimes}}id,\ \boldsymbol{R}^2\underset{R}{\otimes}\boldsymbol{R}^n)$ for some $n\neq 4$ of coh 2. The o.t.g. induced from (46) is equivalent to $O(SO(2)\times G_2,\ id\underset{R}{\hat{\otimes}}\Lambda_2^r,\ \boldsymbol{R}^2\underset{R}{\otimes}R^7)$ and the isotropy subgroup at $\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}$ in $\boldsymbol{R}(2, 7)\simeq\boldsymbol{R}^2\underset{R}{\otimes}\boldsymbol{R}^7$ $(\alpha>\beta>0)$ is isomorphic to $SU(2)$ by Yokota [24, Example 5.1], so coh (46)=2 (cf. Prop. 3.3(1)(4)). The o.t.g. induced from (48) is equivalent to the linear isotropy representation of the symmetric pair $(E_6, U(1)\cdot \mathrm{Spin}\,(10))$ of rank 2 by Prop. 2.12 and Remark 2.13 since it is characterized by its degree up to equivalence. Since $[(k+1)/2]=2$ for $k=4$, coh (49)=2. The o.t.g. induced from (50) is equivalent to the linear isotropy representation of the symmetric pair $(Sp(3), U(3))$ of rank 3 by Prop. 2.2 and Remark 2.3. Since

$[(k+1)/2]=3$ for $k=5$ or 6, coh (51) = coh (52) = 3. The o.t.g. induced from (53) is equivalent to $O(SO(2)\times Spin(9), id\hat{\otimes}_{R}\Delta_4', R^2\otimes_{R} R^{16})$. Any element of $R(2, 16)\simeq R^2\otimes_{R} R^{16}$ to the form $\begin{pmatrix} \alpha & 0 & 0\cdots 0 & 0 & 0 & 0 & 0\cdots 0 \\ 0 & \beta & 0\cdots 0 & \gamma & \delta & \varepsilon & 0\cdots 0 \end{pmatrix}$, and the isotropy subgroup is isomorphic to $SU(3)$ if $\alpha^2\neq\beta^2+\gamma^2+\delta^2+\varepsilon^2$ owing to the use of the mapping f in Lemma 3.2 and Yokota [24, Theorems 5.51, 5.27, 5.2]. So coh (53) = 3. The o.t.g. induced from (54) is equivalent to the linear isotropy representation of the symmetric pair $(E_7, U(1)\cdot E_6)$ of rank 3 by Prop. 2.15 and Remark 2.16. So coh (54) = 3. Q.E.D.

(VII) Let (G, E^N) be a real irreducible o.t.g. of type VII.

PROPOSITION 4.7 coh $(G, E^N)\leqq 3$ *if and only if* $(\tilde{G}, \tilde{id}, R^N)$ *is equivalent as real representation to one of the followings*:

coh = 1: (55) $(A_k, \Lambda_1, C^{k+1})$; $k\geqq 1$,
(56) (C_k, Λ_1, C^{2k}); $k\geqq 2$,
coh = 2: (57) (D_5, Λ_5, C^{16}),
(58) (A_4, Λ_2, C^{10}),
coh = 3: (59) (A_6, Λ_2, C^{21}).

PROOF: Suppose coh $(G, E^N)\leqq 3$. Then $(\tilde{G}, \tilde{id}, R^N)$ is equivalent as real representation to (55)~(58) or (59) by Prop. 4.6. In fact, $(B_k, \Lambda_1, C^{2k+1})$, (D_k, Λ_1, C^{2k}), $(A_1, 2\Lambda_1, C^3)$, (A_3, Λ_2, C^6), (C_2, Λ_2, C^5), (G_2, Λ_2, C^7), (B_3, Λ_3, C^8), (B_4, Λ_4, C^{16}) are 'real' and not real irreducible, so they are not of type VII, and coh $(A_2, 2\Lambda_1, C^6)$ = coh (A_5, Λ_2, C^{15}) = coh $(E_6, \Lambda_1, C^{27})=4$ since the restricted root systems of $(Sp(3), U(3))$, $(SO(12), U(6))$, $(E_7, U(1)\cdot E_6)$ are of type BC (cf. [7], [22]).

Conversely coh (55) = coh (56) = 1 is evident. O.t.g.'s induced from (57), (58) are of coh 2 since the restricted root systems of $(E_6, U(1)\cdot Spin(10))$ and $(SO(10), U(5))$ are of type BC. The o.t.g. induced from (59) is of coh 3 since the restricted root system of $(SO(14), U(7))$ is of type BC (cf. [7] and [22]). Q.E.D.

Now we have the following result.

THEOREM 4.8 *Let* (G, E^N) *be an* o.t.g. *such that the identity representation id*: $G\rightarrow SO(N)$ *is real irreducible. Then coh* $(G, E^N)\leqq 3$ *if and only if* $(\tilde{G}, \tilde{id}, R^N)$ *is equivalent as real representation to one of the followings*:

coh = 1: (IX), (VIII), (8), (9), (19), (20), (21), (22), (23),
(24), (25), (26), (27), (38), (39), (55), (56).
coh = 2: (10), (11), (14), (17), (28), (29), (30), (31), (32),
(33), (40), (41), (42), (43), (44), (45), (46), (47),
(48), (49), (57), (58).

coh$=3$: (3.7), (1), (2), (3), (4), (5), (6), (7), (12), (13),
(15), (16), (18), (34), (35), (36), (37), (50), (51),
(52), (53), (54), (59).

PROOF: Unifying (3.7) of Theorem 3.5, Propositions 4.1~4.7 and type VIII, IX in Section 3, we have the result. Q.E.D.

REMARK 4.9 O.t.g.'s induced from (25), (26), (27), (39), (55), (56), (17), (46), (47), (57), (58), (6), (18), or (59) are not maximal. O.t.g.'s induced from (13), (16), or (53) are not obtained from the linear isotropy representations of any Riemannian symmetric pairs. Others are equivalent to the linear isotropy representations of some Riemannian symmetric pairs of rank at most 3 if they are maximal. (26) is obtained from the linear isotropy representation of (F_4, Spin(9)). The o.t.g. induced from (24) (resp. (42), (7)) is equivalent to one from (23)(resp. (41), (5)) of $k=4$.

REMARK 4.10 O.t.g.'s induced from (13) or (16) are missed in the Theorem 7 of Hsiang-Lawson [11] if k and 3 are relatively prime and $k\geqq 4$, since the dimension of the representation spaces of (13) or (16) is $8k$ and the others of cohomogeneity 3 are of dimension $3m$ for some integer m except (53) of dimension 16.

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